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Contents

1 Introduction

The object of this thesis is to establish an important step towards a global metaplectic correspondence. The origin of this correspondence lies in the theory of modular forms. Let k be an odd, positive integer, and let N be a positive integer divisible by 4. The space $S_{k/2}(\Gamma_0(N),\chi)$, of cusp forms of weight $k/2$ and character χ for the congruence subgroup $\Gamma_0(N)$, is less manageable than the space of cusp forms of integral weight. As a partial remedy to this difficulty, Shimura ([28]) constructed a map

$$S_{k/2}(\Gamma_0(N),\chi) \to S_{k-1}(\Gamma_0(N/2),\chi^2),$$

for $k \geq 5$, which behaves nicely with respect to the L-functions of the forms in the above spaces. Shimura suggested that representation-theoretic methods be used to further clarify the nature of this map.

Gelbart and Piatetski-Shapiro conceived the spaces $S_{k-1}(\Gamma_0(N),\chi^2)$ and $S_{k/2}(\Gamma_0(N),\chi)$ as automorphic representations of GL(2) and $\widetilde{\mathrm{GL}}(2)$ respectively. Here $\widetilde{\mathrm{GL}}(2)$ is a metaplectic covering of GL(2). The first global metaplectic correspondence was established by Flicker ([13]). He proved that for each genuine automorphic representation $\tilde{\pi}$ of $\widetilde{\mathrm{GL}}(2)$, there exists a unique automorphic representation π of GL(2) such that

$$\mathrm{tr}\tilde{\pi}(\tilde{f}) = \mathrm{tr}\pi(f),$$

for certain related functions $\tilde{f}$ and f in the respective Hecke algebras. His proof was accomplished using the Selberg trace formula and followed Langlands' proof of cyclic base change for GL(2) ([25]).

Arthur and Clozel gave a proof of cyclic base change for GL(r) by using the invariant trace formula of Arthur ([10], [6], [7]). More specifically, they effected a global correspondence by proving a term-by-term identity between the invariant trace formulas of GL(r) and its restriction from a cyclic extension. This thesis proves such a term-by-term identity between GL(r) and its n-fold metaplectic covering under the assumption that n is relatively prime to all positive integers less than or equal to $r \geq 2$.

We now give a slightly more detailed overview of the results. The following two sections are essentially paraphrases of the local metaplectic results of Flicker and Kazhdan ([15]).

Flicker and Kazhdan prove a local metaplectic correspondence by using the "simple" trace formula. The novelties which appear in sections two and three arise from the assumption made on n and r. The local and global n-fold metaplectic coverings, $\tilde{G}(F_S)$ and $\tilde{G}(\mathbb{A})$ respectively, of $G = \mathrm{GL}(r)$ are defined in section two; as are the maps

$$G(F_S) \xrightarrow{*} \tilde{G}(F_S),\ G(\mathbb{A}) \xrightarrow{*} \tilde{G}(\mathbb{A}),$$

which preserve conjugacy classes. These maps are referred to as orbit maps. They are our means of comparing objects defined from G and $\tilde{G}$.

In the third section we define the sets of representations relevant to the metaplectic correspondence, namely the genuine representations. We then describe the function spaces pertinent to the trace formula. These are the Hecke and Paley-Wiener spaces. We will assume the nonArchimedean trace Paley-Wiener theorem to hold for metaplectic coverings of $\mathrm{GL}(r)$. The notion of matching functions, that is functions with matching orbital integrals, is given. The local metaplectic correspondence is exploited to define a map of local Hecke algebras

$$\mathcal{H}^{\mathrm{met}}(G(F_S)) \xrightarrow{*} \mathcal{H}(\tilde{G}(F_S)),$$

which maps a function in the domain to a matching function in the image. The local metaplectic correspondence may then be described as a map of representations

$$\pi \mapsto \pi^*$$

such that

$$\mathrm{tr}\pi(f) = \mathrm{tr}\pi^*(f^*),\ f \in \mathcal{H}^{\mathrm{met}}(G(F_S)).$$

The fourth section is rather more technical and is concerned with the normalization of intertwining operators of induced representations. This normalization is essential for the definition of the invariant trace formula. Our method of normalization follows the ideas of [10] and is obtained through the comparison of matching functions and the Plancherel formula. The Plancherel formula of Harish-Chandra is not proved for metaplectic coverings. To compensate for this, we list the properties of $\tilde{G}$ requisite for the proof of the Plancherel formula.

We enter the heart of the matter in section five. This is where we introduce the expected form of the invariant trace formula for $\tilde{G}$,

$$n\sum_{M\in\mathcal{L}}|W_0^M||W_0^G|^{-1}\sum_{\tilde{\gamma}\in(\mathrm{s}_0(M(F)))_{\tilde{M},S}}a^{\tilde{M}}(S,\tilde{\gamma})I_{\tilde{M}}(\tilde{\gamma},\tilde{f}),$$
$$=\sum_t\sum_{M\in\mathcal{L}}|W_0^M||W_0^G|^{-1}\int_{\Pi(\tilde{M},t)}a^{\tilde{M}}(\tilde{\pi})I_{\tilde{M}}(\tilde{\pi},\tilde{f})d\tilde{\pi}.$$

Due to the great number of details that must be verified, this formula remains conjectural and is assumed to be true in this thesis. In other words, we assume that most of the results of the papers of Arthur listed in the bibliography, with the exception of [3], hold for metaplectic coverings of the general linear group. This formula is rewritten in a more suggestive form as

$$n\sum_{M\in\mathcal{L}}|W_0^M||W_0^G|^{-1}\sum_{\gamma\in(M(F))_{M,S}/\mu_n^M}a^{\tilde{M}}(S,\mathrm{s}_0(\gamma^n))I_{\tilde{M}}(\gamma^*,f^*),$$
$$=\sum_t\sum_{M\in\mathcal{L}}|W_0^M||W_0^G|^{-1}\int_{\Pi^{\mathcal{M}}(M,t)}a^{M,\mathcal{M}}(\pi)I_{\tilde{M}}(\pi^*,f^*)d\tilde{\pi},$$

(*cf.* Proposition 9.1 and Proposition 15.1), before it is compared to the invariant trace formula of G. The two main theorems of this thesis are Theorem 9.1 and Theorem 12.1. Their statement and proof is the focus of the remaining sections. For a synopsis of these sections the reader should turn to section five. Very loosely stated, Theorem 9.1 and Theorem 12.1 posit the following equalities between the terms of the two trace formulas of $\tilde{G}$ and G.

$I_{\tilde{M}}(\gamma^*,f^*)=\sum_{\eta\in\mu_n^M/\mu_n^G}I_M(\eta\gamma,f),\ \gamma\in M(F_S)$,

$a^{\tilde{M}}(S,\mathrm{s}_0(\gamma^n))=a^M(S,\gamma),\ \gamma\in M(F)$,

$I_{\tilde{M}}(\pi^*,f^*)=I_M(\pi,f),\ \pi\in\Pi_{\mathrm{unit}}^{\mathrm{met}}(M(\mathbf{A})^1)$,

$a^{M,\mathcal{M}}(\pi)=a^M(\pi)$.

It is the final equality which bears the information we seek for a global metaplectic correspondence for $\mathrm{GL}(r)$. If one considers the specific case $M=G$, one obtains an identity of traces (*cf.* §4 [7]),

$$\sum a^{G}_{\mathrm{disc}}(\pi)\mathrm{tr}\pi(f) = \sum a^{\tilde{G}}_{\mathrm{disc}}(\tilde{\pi})\mathrm{tr}\tilde{\pi}(f^{*}),$$

where the sums range over certain global unitary representations of $\mathrm{GL}(r)$ and $\widetilde{\mathrm{GL}}(r)$ respectively, and $a^{G}_{\mathrm{disc}}(\pi)$ and $a^{\tilde{G}}_{\mathrm{disc}}(\tilde{\pi})$ are constants. One should be able to isolate the trace of cuspidal representations occurring in the above sums by choosing f appropriately, using strong multiplicity one for $\mathrm{GL}(r)$, using the local metaplectic correspondence and using the linear independence of characters. For a more detailed discussion of such matters the reader is referred to §28 [15], where a global metaplectic correspondence is achieved for a smaller set of representations.

For the reader familiar with cyclic base change, it may be helpful to bear in mind that the structure of the proofs of the main theorems follow chapter two of [10] very closely. The results of this thesis which have counterparts in chapter two of [10] have references to these counterparts in parentheses immediately following their own numbering. All citations from [10] will be from chapter two unless otherwise stated.

2 The Groups

In this section we establish some notation and describe the metaplectic coverings of the general linear group. Many definitions for the metaplectic group apply for the general linear group as well, if one considers the case of a trivial covering. If this happens to be the case, the definitions for the general linear group are not made separately. Haar measures are fixed after the above groups are defined so that we may perform harmonic analysis on them without any ambiguity. We also introduce some vector spaces which play an important role in the representation theory of these groups.

Unless otherwise stated, we assume r to be an integer greater than one and n to be a positive integer such that

$$\gcd(n,i) = 1,\ 1 \leq i \leq r.$$

Let F denote a number field containing the group μ_n of nth roots of unity. Since n is greater than two, F is totally imaginary. The completion of F at a valuation v is denoted by F_v and its absolute value (determined by Haar measure) is denoted by $|\cdot|_v$. If v is nonArchimedean, we let R_v be the ring of integers of F_v. We write $\mathbf{A}$ for the adèle ring of F.

The general linear group of rank r, $\mathrm{GL}(r)$, is denoted by G. Thus, for instance, $G(F)$ stands for $\mathrm{GL}(r, F)$, and $G(\mathbf{A})$ stands for $\mathrm{GL}(r, \mathbf{A})$. Throughout, S signifies a finite set of valuations of F. We define $G(F_S)$ to be $\prod_{v \in S} \mathrm{GL}(r, F_v)$.

We recall the description of the metaplectic coverings of G by following §2 [15]. For each valuation v there are 2-cocycles

$$\tau_{mv} : G(F_v) \times G(F_v) \to \mu_n,$$

for $0 \leq m \leq n$, which yield central extensions

$$1 \to \mu_n \to \tilde{G}_m(F_v) \to G(F_v) \to 1$$

called n-fold metaplectic coverings of $G(F_v)$. We only consider the case $m = 0$ and set $\tilde{G}(F_v) = \tilde{G}_0(F_v)$. One can define the n-fold metaplectic covering of $G(F_S)$ by way of the cocycle $\tau_S = \prod_{v \in S} \tau_{0v}$. It is denoted by $\tilde{G}(F_S)$. Similarly, the n-fold metaplectic covering of $G(\mathbf{A})$ is defined by way of the cocycle $\tau = \prod_v \tau_{0v}$ and is written as $\tilde{G}(\mathbf{A})$. The aforementioned coverings are also equipped with maps

$$1 \to \mu_n \xrightarrow{\mathbf{i}} \tilde{G}(F_S) \underset{\mathbf{s}}{\overset{\mathbf{p}}{\rightleftarrows}} G(F_S) \to 1,$$

$$1 \to \mu_n \xrightarrow{\mathbf{i}} \tilde{G}(\mathbf{A}) \underset{\mathbf{s}}{\overset{\mathbf{p}}{\rightleftarrows}} G(\mathbf{A}) \to 1.$$

Elements of $\tilde{G}(F_S))$ are of the form (γ, ζ), where $\gamma \in G(F_S)$ and $\zeta \in \mu_n$. The maps in the above sequence may be expressed more concretely by $\mathbf{i}(\zeta) = (1, \zeta)$, $\mathbf{p}((\gamma, \zeta)) = \gamma$ and $\mathbf{s}(\gamma) = (\gamma, 1)$. Multiplication in $\tilde{G}(F_S)$ may be described by the equality

$$(g_1, \zeta_1)(g_2, \zeta_2) = (g_1 g_2, \tau(g_1, g_2)\zeta_1\zeta_2), \ (g_1, \zeta_1), (g_2, \zeta_2) \in \tilde{G}(F_S).$$

Parallel statements are true for $\tilde{G}(\mathbf{A})$.

Given a subgroup H of G, we write $\tilde{H}$ for $\mathbf{p}^{-1}(H)$. We say that $\tilde{G}$ splits over a subgroup H of G if $\tilde{H}$ is group-isomorphic to $H \times \mu_n$. The group $\tilde{G}(F_v)$ splits over $G(F_v)$

if v is a complex valuation of F. The splitting homomorphism is s. Consequently the representation theory of $\tilde{G}(F_v)$ for complex valuations v of F essentially reduces to the representation theory of $G(\mathbf{C})$. The upshot of this is that all of the local Archimedean assertions made in this paper will be simple to prove.

It is also shown in §2 [15] that $\tilde{G}(\mathbf{A})$ splits over $G(F)$. The splitting homomorphism of $\tilde{G}(\mathbf{A})$ over $G(F)$ is denoted by s_0.

There is a map

(1) $$G(F_v) \xrightarrow{*} \tilde{G}(F_v)$$

which preserves conjugacy classes (§8 [15]). This map will be referred to as the orbit map. Since n is assumed to be odd, this map is given by

$$\gamma^* = (\gamma, 1)^n, \ \gamma \in G(F_v).$$

This map extends to maps $G(F_S) \xrightarrow{*} \tilde{G}(F_S)$ and $G(\mathbf{A}) \xrightarrow{*} \tilde{G}(\mathbf{A})$ in an obvious manner. The orbit map allows us to compare conjugacy classes, and ultimately the trace formulas, of G and $\tilde{G}$.

For the remainder of this thesis $\mathcal{L}$ will denote the set of Levi subgroups of G containing a fixed minimal Levi subgroup M_0 of G. Without loss of generality, we take M_0 to be the diagonal subgroup. Let $P_0 = M_0U_0$ be the upper triangular subgroup and U_0 be its unipotent radical. We will denote a generic element of $\mathcal{L}$ by M, until it is fixed in §16. Note that

(2) $$M = \prod_{i=1}^{\ell} M(i),$$

where $M(i) \cong \mathrm{GL}(r_i)$ and $\sum_{i=1}^{\ell} r_i = r$. The set of Levi subgroups of M containing M_0 is denoted by $\mathcal{L}^M$. Set K_v to be $\mathrm{GL}(r, R_v)$ if v is nonArchimedean, and $U(r, \mathbf{C})$ otherwise. In addition set $K_S = \prod_{v \in S} K_v$. Fix a Haar measure on $M(F_v)$ so that the measure of the compact group $M(F_v) \cap K_v$ is one. This fixes a Haar measure on $\tilde{M}(F_v)\backslash\tilde{G}(F_v)$, and consequently also on $\tilde{M}(F_v)$, by way of the map

$$\tilde{M}(F_v)\backslash\tilde{G}(F_v) \to M(F_v)\backslash G(F_v).$$

Unless otherwise specified, given subgroups $H \subset H'$ of G with fixed Haar measures, we define the measure on the quotient space $\tilde{H}\backslash\widetilde{H'}$ via the pull-back of the apparent homeomorphism

$$\tilde{H}\backslash\widetilde{H'} \to H\backslash H'.$$

Let $A_G(F_v)$ denote the centre of $G(F_v)$. The centre of $\tilde{G}(F_v)$ is $\widetilde{A^n_G}(F_v)$ (Proposition 0.1.1 [19]), where

$$A^n_G(F_v) = \{\gamma^n : \gamma \in A_G(F_v)\}.$$

For nonArchimedean valuations v, we fix Haar measures on $A_G(F_v)$ and $\widetilde{A^n_G}(F_v)$ so that the quotient measures on $G(F_v)/A_G(F_v)$ and $\tilde{G}(F_v)/\widetilde{A^n_G}(F_v)$ are related as prescribed in §24 [15].

Since F embeds diagonally into $F_S = \prod_{v\in S} F_v$ and $\mathbf{A}$, we may define the groups of rational characters $X(M(F_S))_F$ and $X(M(\mathbf{A}))_F$ of $M(F_S)$ and $M(\mathbf{A})$ respectively. Let us agree to suppress the notation F_S and $\mathbf{A}$ for the rest of the section. This will allow us to define objects over both of these rings at the same time. We define the real vector space $\mathfrak{a}_M$ as $\mathrm{Hom}(X(M)_F, \mathbf{R})$. We define the group $X(\tilde{M})_F$ as $\{\xi \circ \mathbf{p} : \xi \in X(M)_F\}$, and the real vector space $\mathfrak{a}_{\tilde{M}}$ as $\mathrm{Hom}(X(\tilde{M})_F, \mathbf{R})$. The obvious isomorphism between $X(M)_F$ and $X(\tilde{M})_F$ leads to an isomorphism between $\mathfrak{a}_{\tilde{M}}$ and $\mathfrak{a}_M$. On occasion, we identify $\mathfrak{a}_{\tilde{M}}$ with $\mathfrak{a}_M$ by means of this isomorphism.

The map $G \xrightarrow{*} \tilde{G}$ induces a homomorphism $X(\tilde{M})_F \xrightarrow{*} X(M)_F$ such that

$$\tilde{\xi}^*(\gamma) = \tilde{\xi}(\gamma^*) = \tilde{\xi}(\mathrm{s}(\gamma^n)) = \tilde{\xi}^n(\mathrm{s}(\gamma)),$$

for all $\gamma \in M$ and $\tilde{\xi} \in X(\tilde{M})_F$. This map in turn induces an isomorphism,

$$\mathfrak{a}_M \xrightarrow{*} \mathfrak{a}_{\tilde{M}},$$

such that $X \mapsto nX$, for all $X \in \mathfrak{a}_M$. Define the adjoint map

$$\mathfrak{a}^*_{M,\mathbf{C}} = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}_M, \mathbf{C}) \xrightarrow{*} \mathfrak{a}^*_{\tilde{M},\mathbf{C}}$$

by $\lambda^* = n^{-1}\lambda$, for $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$ (with apologies for the double usage of the symbol $*$). Note that $X(\tilde{M})_F$ embeds in $\mathfrak{a}^*_{\tilde{M},\mathbf{C}}$ as a lattice.

As customary, A_M denotes the centre of M. Fix a Euclidean norm on $\mathfrak{a}_{M_0}$ which is invariant under W_0^G, the Weyl group of (G, A_{M_0}). We endow $\mathfrak{a}_M \subset \mathfrak{a}_{M_0}$ with the Euclidean measure obtained from $\mathfrak{a}_{M_0}$ by restriction. The measure on $\mathfrak{a}_{\tilde{M}}$ is taken to be the measure on $\mathfrak{a}_M$.

The maps $H_M : M \to \mathfrak{a}_M$ and $H_{\tilde{M}} : \tilde{M} \to \mathfrak{a}_{\tilde{M}}$ are defined by the respective relations

$$e^{\langle H_M(\gamma),\xi\rangle} = \prod_v |\xi_v(\gamma_v)|_v, \ \gamma_v \in M(F_v), \ \xi = \prod_v \xi_v \in X(M)_F$$

and

$$e^{\langle H_{\tilde{M}}(\tilde{\gamma}),\tilde{\xi}\rangle} = \prod_v |\tilde{\xi}_v(\tilde{\gamma}_v)|_v, \ \tilde{\gamma}_v \in \tilde{M}(F_v), \ \tilde{\xi} = \prod_v \tilde{\xi}_v \in X(\tilde{M})_F.$$

In the adèlic context, these maps produce Haar measures on $M(\mathbb{A})^1 = \ker(H_M(\mathbb{A}))$ and $\tilde{M}(\mathbb{A})^1 = \ker(H_{\tilde{M}}(\mathbb{A}))$.

3 The Local Metaplectic Correspondence

The purpose of this section is to describe the local metaplectic correspondence of [15]. We must first define suitable representations of the groups discussed in the previous section. Thereafter we define function spaces of these groups and their representations. The reader will be assumed to be familiar with the notation of §1 [8].

Fix a unitary character $\tilde{\omega}$ of $A_G^n(F_S)$. We set ω to be the unitary character of $A_G(F_S)$ defined by $\omega(\gamma) = \tilde{\omega}(\gamma^n)$ for all $\gamma \in A_G(F_S)$. An representation $\tilde{\pi}$ of $\tilde{M}(F_S)$ is admissible if

$$\{\tilde{\gamma} \in \tilde{M}((F_S) : \tilde{\pi}(\tilde{\gamma})v = v\}$$

is open for all v in the vector space V of $\tilde{\pi}$, and the subspace formed by the elements of V fixed by $\tilde{\pi}(K_S \cap \tilde{M}(F_S))$ is finite dimensional. Let $\Pi(M(F_S))$ be the set of (equivalence classes of) irreducible admissible representations π of $M(F_S)$ such that $\pi(\gamma) = \omega(\gamma)$ for all $\gamma \in A_G(F_S)$. Let $\Pi_{\text{temp}}(M(F_S))$ and $\Pi_{\text{unit}}(M(F_S))$ be the subsets of $\Pi(M(F_S))$ which are respectively tempered and unitary.

A representation $\tilde{\pi}$ of $\tilde{M}(F_S)$ is said to be genuine if

$$\tilde{\pi}((\gamma,\zeta)) = \zeta\tilde{\pi}((\gamma,1)), \ (\gamma,\zeta) \in \tilde{M}(F_S).$$

Let $\Pi(\tilde{M}(F_S))$ be the set of (equivalence classes of) genuine irreducible admissible representations $\tilde{\pi}$ of $\tilde{M}(F_S)$ such that

$$\tilde{\pi}((\gamma^n, 1)) = \tilde{\omega}(\gamma^n),\ \gamma \in A_G(F_S).$$

Again, $\Pi_{\text{temp}}(\tilde{M}(F_S))$ and $\Pi_{\text{unit}}(\tilde{M}(F_S))$ denote the subsets of $\Pi(\tilde{M}(F_S))$ which are respectively tempered and unitary.

Let $M_1 \in \mathcal{L}^M$ and let P_1 be the unique parabolic subgroup of M containing $M \cap P_0$. Given $\tilde{\pi} \in \Pi(\tilde{M}_1(F_S))$, the unitarily induced representation $\mathrm{Ind}_{\tilde{P}_1}^{\tilde{M}}\tilde{\pi}$ is denoted by $\tilde{\pi}^{\tilde{M}}$.

If $\tilde{\pi}$ belongs to $\Pi(\tilde{M}(F_S))$ and $\tilde{\lambda} \in \mathfrak{a}^*_{\tilde{M},\mathbf{C}}$ then the representation $\tilde{\pi}_{\tilde{\lambda}}$ given by

$$\tilde{\pi}_{\tilde{\lambda}}(\tilde{\gamma}) = \tilde{\pi}(\tilde{\gamma})e^{\tilde{\lambda}(H_{\tilde{M}}(\tilde{\gamma}))},\ \tilde{\gamma} \in \tilde{M}(F_S)$$

belongs to $\Pi(\tilde{M}(F_S))$ as well. The set of genuine standard representations $\Sigma(\tilde{M}(F_S))$ of $\tilde{M}(F_S)$ is defined to be the set of representations of the form $\tilde{\pi}^{\tilde{M}}_{\tilde{\lambda}}$, where $\tilde{\pi} \in \Pi_{\text{temp}}(\tilde{M}_1(F_S))$, $M_1 \in \mathcal{L}^M$ and $\tilde{\lambda} \in \mathfrak{a}^*_{\tilde{M}_1,\mathbf{C}}$. The set $\Sigma(M(F_S))$ is defined analogously.

We now discuss spaces of functions on our groups and on the set of tempered representations. A function $\tilde{f} : \tilde{M} \to \mathbf{C}$ is said to be antigenuine if

$$\tilde{f}(\gamma, \zeta) = \zeta^{-1}\tilde{f}(\gamma, 1),\ (\gamma, \zeta) \in \tilde{M}.$$

The Hecke space $\mathcal{H}(\tilde{M}(F_S))$ is the space of antigenuine smooth compactly supported functions on $\tilde{M}(F_S)$ which are $(\tilde{K}_S \cap \tilde{M}(F_S))$-finite under left and right multiplication.

We may compare functions in the Hecke algebras of $\tilde{M}(F_S)$ and $M(F_S)$ using the orbit map and the notion of orbital integrals. Let $\tilde{h}$ be a smooth compactly supported function on $\tilde{M}(F_S)$ and $\tilde{\gamma} \in \tilde{M}(F_S)$ such that $\mathbf{p}(\tilde{\gamma})$ is semisimple in $M(F_S)$. Let $\tilde{M}_{\tilde{\gamma}}(F_S)$ denote the centralizer of $\tilde{\gamma}$ in $\tilde{M}(F_S)$. We define

$$I^{\tilde{M}}_{\tilde{M}}(\tilde{\gamma}, \tilde{h}) = |D^M(\mathbf{p}(\tilde{\gamma}))|^{1/2} \int_{\tilde{M}_{\tilde{\gamma}}(F_S)\backslash \tilde{M}(F_S)} \tilde{h}\ (\tilde{x}^{-1}\tilde{\gamma}\tilde{x})d\tilde{x},$$

where

$$D^M(\gamma) = \prod_{v \in S} \det(1 - \mathrm{Ad}(\gamma_v))|_{\mathfrak{m}_v/\mathfrak{m}_{\gamma_v}},$$

and $\mathfrak{m}_v$ and $\mathfrak{m}_{\gamma_v}$ are the Lie algebras of $M(F_v)$ and $M_{\gamma_v}(F_v)$ respectively. This integral converges (§6 [15], §2 [8]) and is called an orbital integral.

Two functions, $\tilde{h} \in \mathcal{H}(\tilde{M}(F_S))$ and $h \in \mathcal{H}(M(F_S))$, are said to match if

$$I_M^M(\gamma, h) = I_{\tilde{M}}^{\tilde{M}}(\gamma^*, \tilde{h}),$$

for all semisimple elements $\gamma \in M(F_S)$ such that γ^n is G-regular.

The other important function space needed for the invariant trace formula is the Paley-Wiener space. Given $\tilde{f} \in \mathcal{H}(\tilde{M}(F_S))$ and $M_1 \in \mathcal{L}^M$, we define a map

$$\tilde{f}_{\tilde{M}_1} : \Pi_{\text{temp}}(\tilde{M}_1) \to \mathbf{C}$$

by $\tilde{f}_{\tilde{M}_1}(\tilde{\pi}) = \text{tr}(\tilde{\pi}^{\tilde{M}}(\tilde{f}))$. As assumed in [15], we assume that the trace Paley-Wiener theorem holds for $\tilde{M}$ ([17]). We may then take the Paley-Wiener space to be

$$\mathcal{I}(\tilde{M}(F_S)) = \{\tilde{h}_{\tilde{M}} : \tilde{h} \in \mathcal{H}(\tilde{M}(F_S))\}.$$

Let $\mathcal{V}$ be a topological vector space and suppose that a continuous map

$$\theta : \mathcal{H}(\tilde{M}(F_S)) \to \mathcal{V}$$

satisfies $\theta(\tilde{h}) = 0$ whenever $\tilde{h}_{\tilde{M}} = 0$. Then θ is said to be supported on characters. Moreover we can define a continuous linear map

$$\hat{\theta} : \mathcal{I}(\tilde{M}(F_S)) \to \mathcal{V},$$

such that $\hat{\theta}(\tilde{h}_{\tilde{M}}) = \theta(\tilde{h})$ for all $\tilde{h} \in \mathcal{H}(\tilde{M}(F_S))$. This construction will be used later on for the invariant maps occurring in the trace formulas.

The set of valuations S is said to have the closure property if

$$\mathfrak{a}_{M,S} = \{H_M(m) : m \in M(F_S)\}$$

is a closed subgroup of $\mathfrak{a}_{M(F_S)}$. If S contains an Archimedean valuation it has the closure property. If not, S has the closure property if and only if it is comprised entirely of valuations which divide a fixed rational prime. For the remainder of this paper S is assumed to have the closure property unless otherwise specified.

Put

$$i\mathfrak{a}^*_{\tilde{M},S} = i\mathfrak{a}^*_{M,S} = i\mathfrak{a}^*_{M(F_S)}/i\text{Hom}(\mathfrak{a}_{M,S}, \mathbf{Z}).$$

The group $i\mathfrak{a}^*_{M,S}$ inherits a measure from the Euclidean measure defined previously on $\mathfrak{a}_M$. Similarly $i\mathfrak{a}^*_{\tilde{M},S}$ inherits a measure from the measure which was designated for $\mathfrak{a}_{\tilde{M}}$. We identify any $\phi \in \mathcal{I}(M(F_S))$ with its Fourier transform,

$$\phi(\pi, X) = \int_{i\mathfrak{a}^*_{M,S}} \phi(\pi_\lambda) e^{-\lambda(X)} d\lambda, \;\; \pi \in \Pi_{\text{temp}}(M(F_S)), \;\; X \in \mathfrak{a}_{M,S}.$$

Likewise, we identify $\mathcal{I}(\tilde{M}(F_S)))$ with a space of functions on $\Pi_{\text{temp}}(\tilde{M}(F_S)) \times \mathfrak{a}_{\tilde{M},S}$.

The local metaplectic correspondence on tempered representations is an injection

$$\text{(3)} \qquad \Pi_{\text{temp}}(\tilde{M}(F_S)) \hookrightarrow \Pi_{\text{temp}}(M(F_S)),$$

$$\tilde{\pi} \mapsto \pi,$$

such that $\text{tr}\tilde{\pi}(\tilde{h}) = \text{tr}\pi(h)$ for any matching functions $\tilde{h} \in \mathcal{H}(\tilde{M}(F_S))$ and $h \in \mathcal{H}(M(F_S))$; or equivalently such that

$$\text{(4)} \qquad |D^M(\gamma^*)|^{1/2}\Theta_{\tilde{\pi}}(\gamma^*) = n/|n|^{r/2} \sum_{\{\delta \in M_\gamma(F_S)/A_M(F_S):\delta^*\zeta=\gamma^*\}} \zeta |D^M(\gamma)|^{1/2}\Theta_\pi(\delta),$$

where $\zeta \in \widetilde{A^n_M}(F_S)/A^*_M(F_S)$ (see Lemma 10.7), and $\Theta_{\tilde{\pi}}$ and Θ_π are the characters of $\tilde{\pi}$ and π respectively (Proposition 27.3 [15]). It follows from 4 and the equalities

$$\text{(5)} \qquad H_{\tilde{M}}(\gamma^*) = H_{\tilde{M}}((\gamma^n, 1)) = nH_M(\gamma) = (H_M(\gamma))^*, \;\; \gamma \in M(F_S),$$

that π_λ corresponds to $\tilde{\pi}_{\lambda^*}$ for any $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$.

The image of this correspondence is characterised in terms of matching functions. A representation $\pi \in \Pi_{\text{temp}}(M(F_S))$ is called metic if there exist matching functions $\tilde{h} \in \mathcal{H}(\tilde{M}(F_S))$ and $h \in \mathcal{H}(M(F_S))$ such that $\text{tr}\pi(h) \neq 0$. The image of (3) is the subset of representations in $\Pi_{\text{temp}}(M(F_S))$ which are metic. We may extend the definition of metic to apply to all admissible representations in the following way. A representation $\pi \in \Pi(M(F_S))$ is called metic if it is the Langlands quotient of $\pi_{1,\lambda}$, where $M_1 \in \mathcal{L}^M$, $\pi_1 \in \Pi_{\text{temp}}(\tilde{M}_1(F_S))$ is metic, and λ lies in a fixed Weyl chamber of $\mathfrak{a}^*_{M_1,\mathbf{C}}$. An admissible representation ρ of $M(F_S))$ is defined to be metic if all of its irreducible subquotients are metic. Our definition of metic coincides with the definition of metic in [15] for tempered representations. (*cf.* §27.3 [15]). Set $\Pi^{\rm met}_{\rm temp}(M(F_S))$ and $\Pi^{\rm met}(M(F_S))$ to be the subsets of metic representations in $\Pi_{\rm temp}(M(F_S))$ and $\Pi(M(F_S))$ respectively.

We now extend the local metaplectic correspondence to $\Pi(\tilde{M}(F_S))$. By using the Jacquet modules introduced in §14 [15], and following the proof of [29], it can be shown that the Langlands quotient theorem holds for $\tilde{M}(F_S)$. More precisely, every $\tilde{\pi} \in \Pi(\tilde{M}(F_S))$ may be written uniquely as the quotient of some induced representation $\tilde{\pi}^{M}_{1,\lambda}$, where $\tilde{\pi}_1 \in \Pi_{\rm temp}(M_1(F_S))$, λ is in a fixed positive Weyl chamber of $\mathfrak{a}_{\tilde{M}_1,\mathbf{C}}$ and $M_1 \in \mathcal{L}^M$. With the Langlands quotient theorem in place, we may extend injection (3) by assigning the Langlands quotient of a representation $\tilde{\pi}^{\tilde{M}}_{\lambda}$ to the Langlands quotient of π^{M}_{λ}, where π is the image of $\tilde{\pi} \in \Pi_{\rm temp}(\tilde{M}_1(F_S))$ under (3), $\lambda \in \mathfrak{a}^*_{M_1,\mathbf{C}}$ and $M_1 \in \mathcal{L}^M$. Broadly stated, this prescribes an injection

$$\Pi(\tilde{M}(F_S)) \hookrightarrow \Pi(M(F_S)). \tag{6}$$

This extension is compatible with (3) by Proposition 26.2 [15] [1].

It is natural to seek further criteria characterizing the image set of (3) or (6). In order to describe such criteria we define a family of important finite groups which are indexed by the elements of $\mathcal{L}$. Recall decomposition (2),

$$M \cong \prod_{i=1}^{\ell} \mathrm{GL}(r_i).$$

Under this isomorphism we find that

$$A_M \cong \prod_{i=1}^{\ell} Z_i,$$

where Z_i is the subgroup of scalar matrices in $\mathrm{GL}(r_i)$. Let μ^M_n be the finite subgroup of A_M corresponding to those matrices in each Z_i whose entries lie in μ_n. Notice that μ^M_n is the kernel of the map $a \mapsto a^n$ defined on A_M.

An implicit feature of the local metaplectic correspondence is that the central character of any $\pi \in \Pi^{\rm met}_{\rm temp}(M(F_S))$ is trivial on μ^M_n. The converse is in general not true. It is however true for essentially square integrable representations (Theorem 26.1 [15]) and

[1] There seems to be an error in the method of induction described in §26.2 [15]. Nonetheless, this method is valid under our assumption on n and r. See Appendix 19.

(5). In other words π_λ is metic for any square integrable representation $\pi \in \Pi(M(F_S)$ and $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$.

The set of genuine standard representations $\Sigma(\tilde{M}(F_S))$ of $\tilde{M}(F_S)$ is defined to be the set of representations of the form $\tilde{\pi}_\lambda^{\tilde{M}(F_S)}$, where $\tilde{\pi} \in \Pi_{\mathrm{temp}}(\tilde{M}_1(F_S))$, $M_1(F_S)$ is a Levi subgroup of $M(F_S)$ and $\lambda \in \mathfrak{a}^*_{\tilde{M}_1,\mathbf{C}}$. We wish to extend the local metaplectic correspondence to standard representations. Before we indicate how this is done we fix some notation from [22]. A segment Δ is an ordered m-tuple of representations,

$$[\sigma, \sigma(m)] = (\sigma, \sigma|\cdot|_v, \dots, \sigma|\cdot|_v^{m-1}),$$

where σ is an irreducible supercuspidal representation of $\mathrm{GL}(b, F_v)$, for some positive integers b and m, and a nonArchimedean valuation v of F. The unique irreducible quotient of the representation induced from Δ is denoted by $Q(\Delta)$. Given segments, $\Delta_1, \dots, \Delta_k$ (which do not precede each other), the unique irreducible quotient of the representation induced from $\otimes_{i=1}^k Q(\Delta_i)$ is denoted by $Q(\otimes_{i=1}^k Q(\Delta_i))$.

Lemma 3.1 *Suppose $\pi_1 \in \Pi_{\mathrm{temp}}(M_1(F_S))$, $M_1 \in \mathcal{L}^M$, $\lambda \in \mathfrak{a}^*_{M_1,\mathbf{C}}$ and $\rho = \pi^M_{1,\lambda}$. Then ρ is metic if and only if π_1 is metic.*

Proof. Since the local metaplectic correspondence is stable under twists by $\lambda \in \mathfrak{a}^*_{M_1,\mathbf{C}}$ and $\pi^{M_1}_{2,\lambda} = (\pi_2^{M_1})_\lambda$ for $M_2 \in \mathcal{L}^{M_1}$ and $\pi_2 \in \Pi(M_2(F_S))$, we may assume $\lambda = 0$ without any loss of generality. Moreover, it suffices to prove the lemma in two separate cases. In the first case we assume S to consist of a single complex valuation and in the second we assume S to consist of a single nonArchimedean valuation v.

We prove the complex case first. In this case we may take $M_1 = M_0$ as the irreducible tempered representations of $M_1(\mathbf{C})$ are principal series representations (Theorem 14.91 [20]). Thus $\pi_1 = \otimes_{i=1}^r \omega_i$, where $\omega_1, \dots, \omega_r$ are quasi-characters of $\mathbf{C}^\times$ satisfying the irreducibility criteria of Théorème 4.4 [12]. The irreducible representation ρ is metic if and only if there exists $\tilde{\rho} \in \Pi(\tilde{M}(\mathbf{C}))$ which corresponds to ρ. We may represent $\tilde{\rho}$ as $(\otimes_{i=1}^r \tilde{\omega}_i)^{\tilde{M}}$, where $\tilde{\omega}_1, \dots, \tilde{\omega}_r$ are again quasi-characters of $\mathbf{C}^\times$. Following the arguments of §2.1 [13], we find that $\tilde{\rho}$ corresponds to ρ if and only if $\otimes_{i=1}^r \tilde{\omega}_i$ corresponds to $\otimes_{i=1}^r \omega_i$ and $\omega_i = \tilde{\omega}_i^n$. This proves the complex case of the lemma.

Proposition 2.2.1 [22] specifies that $\pi_1 = \mathrm{Ind}_{P_2}^{M_1}(Q(\Delta_1)\otimes\cdots\otimes Q(\Delta_k))$ for some segments Δ_i, $1 \leq i \leq k$, $P_2 \in \mathcal{P}^{M_1}(M_2)$ and $M_2 \in \mathcal{L}^{M_1}$. By transitivity of induction we have $\rho = \mathrm{Ind}_{P_1}^{M}(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_k))$ for $P_1 \in \mathcal{P}^M(M_2)$ with $P_1 \cap M_1 = P_2$. Let π be an arbitrary subquotient of ρ. By Theorem 1.2.5 (c) [22], the irreducible subquotient π equals $Q(Q(\Delta_1')\otimes\cdots\otimes Q(\Delta_{k'}'))$ for some segments Δ_i', $1 \leq i \leq k'$. According to Theorem 7.1 of [33] (see also Théorème 5 of [26]), the segments $\Delta_1', \ldots, \Delta_{k'}$ are obtained from the segments $\Delta_1, \ldots, \Delta_k$ by "elementary operations". In other words, Δ_i' is either the union of two linked segments in $\{\Delta_i\}_{i=1}^k$, the intersection of two linked segments in $\{\Delta_i\}_{i=1}^k$ or equal to one of the segments in $\{\Delta_i\}_{i=1}^k$. In any event, the central characters of the supercuspidal representations occurring in the segments $\Delta_1', \ldots, \Delta_{k'}'$ are identical to those occurring in $\Delta_1, \ldots, \Delta_k$.

Suppose that π_1 is metic. This means that there exist matching functions $\tilde{h} \in \mathcal{H}\tilde{M}(F_v))$ and $h \in \mathcal{H}(M(F_v))$, such that $\mathrm{tr}\pi_1(h) \neq 0$. Using the Weyl integration formula, it can be shown (§24 [15]) that $\tilde{h}_{\tilde{M}_2}$ and h_{M_2} correspond to matching functions in $\mathcal{H}(\tilde{M}_2(F_v))$ and $\mathcal{H}(M_2(F_v))$ respectively. Furthermore

$$h_{M_2}(\otimes_{i=1}^k Q(\Delta_i)) = \mathrm{tr}\mathrm{Ind}_{P_2}^{M_1}(\otimes_{i=1}^k Q(\Delta_i))(h) = \mathrm{tr}\pi_1(h) \neq 0.$$

Hence $\otimes_{i=1}^k Q(\Delta_i)$ is metic. The identification of representations described in the appendix and the remarks immediately preceding Theorem 27.3 [5, FK]mply that each $Q(\Delta_i)$ is metic for $1 \leq i \leq k$. Suppose $\Delta_i = [\sigma_i, \sigma_i(s_i)]$, where the σ_i are supercuspidal and the s_i are non-negative integers for $1 \leq i \leq k$. Let ω_i be the restriction of the central character of σ_i to μ_n. The restrictions of the central character of $Q(\Delta_i)$ to μ_n is then seen to be $\omega_i^{s_i}$. As $Q(\Delta_i)$ is metic, $\omega_i^{s_i}$ is trivial. Since $1 \leq s_i \leq r$ we have $\gcd(s_i, n) = 1$ and so ω_i must be trivial. From the earlier description of Δ_i', $1 \leq i \leq k'$, it follows that the restrictions to μ_n of the central characters of the supercuspidal representations occuring in Δ_i' must be trivial as well. This implies that the central character of $Q(\Delta_i')$ is trivial on μ_n. According to Theorem 1.2.2 (ii) [22], the induced quotient $Q(\Delta_i')$ is essentially square integrable. Once again we appeal to Theorem 26.1 [15] and (5) to conclude that $Q(\Delta_i')$ is metic. We refer to the appendix again to conclude that $\otimes_{i=1}^{k'} Q(\Delta_i')$ is metic. Since π is the Langlands Quotient of this representation, it is metic.

Conversely suppose ρ is metic. This means that π is metic. By definition $\otimes_{i=1}^{k'} Q(\Delta_i')$ is metic. As earlier, we may conclude that Δ_i' and its supercuspidal factors are metic for $1 \leq i \leq k'$. As a result, the central characters of these supercuspidal representations are trivial on μ_n. As before this implies that Δ_i and in turn that $Q(\Delta_i)$ are metic for $1 \leq i \leq k$. This implies that π_i is metic by definition (*cf.* 26.2 [15] as well). □

The induction arguments of Proposition 26.2 [15] combined with Lemma 3.1 yield an injection

$$\Sigma(\tilde{M}(F_S)) \hookrightarrow \Sigma(M(F_S)), \tag{7}$$

whose image is the subset $\Sigma^{\text{met}}(M(F_S))$ of metic representations in $\Sigma(M(F_S))$.

Having specified the images of injections (6) and (7) we are free to define bijections

$$\Pi^{\text{met}}(M(F_S)) \xrightarrow{\sim} \Pi(\tilde{M}(F_S)) \tag{8}$$

and

$$\Sigma^{\text{met}}(M(F_S)) \xrightarrow{\sim} \Sigma(\tilde{M}(F_S)). \tag{9}$$

We define $\Pi^{\text{met}}_{\text{temp}}(M(F_S))$ and $\Pi^{\text{met}}_{\text{unit}}(M(F_S))$ to be the subsets of metic representations in $\Pi_{\text{temp}}(M(F_S))$ and $\Pi_{\text{unit}}(M(F_S))$ respectively.

We may now use (8) to define subspaces of $\mathcal{I}(M(F_S))$ and $\mathcal{H}(M(F_S))$ which pertain to $\Pi^{\text{met}}(M(F_S))$. Set

$$\mathcal{I}^{\text{met}}(M(F_S)) = \{\phi \in \mathcal{I}(M(F_S)) : \phi(\pi) = 0, \pi \in \Pi_{\text{temp}}(M(F_S)) \text{ but } \pi \notin \Pi^{\text{met}}_{\text{temp}}(M(F_S))\}$$

and

$$\mathcal{H}^{\text{met}}(M(F_S)) = \{h \in \mathcal{H}(M(F_S)) : h_M \in \mathcal{I}^{\text{met}}(M(F_S))\}.$$

We identify functions in $\mathcal{I}^{\text{met}}(M(F_S))$ with their restrictions to $\Pi^{\text{met}}_{\text{temp}}(M(F_S))$. These sets of functions may be compared to the corresponding sets of functions derived from the metaplectic coverings. Explicitly, If ϕ belongs to $\mathcal{I}^{\text{met}}(M(F_S))$, define

$$\phi^*(\pi^*, X^*) = n^{-\dim A_M} \phi(\pi, X), \ \pi \in \Pi^{\text{met}}_{\text{temp}}(M(F_S)), \ X \in \mathfrak{a}_{M,S}. \tag{10}$$

This definition produces a function in $\mathcal{I}(\tilde{M}(F_S))$ by virtue of the trace Paley-Wiener theorem ([17]). This is a transfer map of Paley-Wiener functions which is adjoint to bijection (8). As such, it is seen to be bijective as well.

The trace Paley-Wiener theorem and the Weyl integration formula suggest that a transfer map for Paley-Wiener spaces ought to yield a transfer map for Hecke spaces which is adjoint to map (1). Indeed, by using the trace Paley-Wiener theorem and the bijectivity of (10) we can define a map

$$\mathcal{H}^{\mathrm{met}}(M(F_S)) \xrightarrow{*} \mathcal{H}(\tilde{M}(F_S)),$$

$$h \mapsto h^*,$$

such that h and h^* match (*cf.* Corollary 27.3 [15]).

In order to show that this map is compatible with (10), let $h \in \mathcal{H}^{\mathrm{met}}(M(F_S))$ and $\pi \in \Pi^{\mathrm{met}}_{\mathrm{temp}}(M(F_S))$. Then

$$\begin{aligned}
(h_M)^*(\pi^*, X^*) &= n^{-(\dim A_M)} h_M(\pi, X) \\
&= n^{-(\dim A_M)} \int_{i\mathfrak{a}^*_{M,S}} \mathrm{tr}((\pi_\lambda(h)) e^{-\lambda(X)} d\lambda \\
&= n^{-(\dim A_M)} \int_{i\mathfrak{a}^*_{M,S}} h^*_{\tilde{M}}((\pi_\lambda)^*) e^{-\lambda^*(X^*)} d\lambda^* \\
&= \int_{i\mathfrak{a}^*_{\tilde{M},S}} h^*_{\tilde{M}}(\pi^*_{\lambda^*}) e^{-\lambda^*(X^*)} d\lambda^* \\
&= h^*_{\tilde{M}}(\pi^*, X^*),
\end{aligned}$$

as to be desired. The second from last equality follows from $d\lambda^* = n^{-(\dim A_M)} d\lambda$.

We remark that the functions $h \in \mathcal{H}^{\mathrm{met}}(M(F_S))$ are invariant under μ_n^M. This implies a certain invariance of the map $\theta : \mathcal{H}(M(F_S)) \to \mathcal{V}$ mentioned above. Indeed if we define ${}^\eta h$ and ${}^\eta \theta$ by

$${}^\eta h(\gamma) = h(\eta\gamma), \ \gamma \in M(F_S), \ \eta \in \mu_n^M,$$

$${}^\eta \theta(h) = \theta({}^\eta h),$$

then

$${}^\eta \theta(h) = \theta({}^\eta h) = \theta(h).$$

It follows that ${}^\eta \theta = \theta$ if θ vanishes outside of $\mathcal{H}^{\mathrm{met}}(M(F_S))$. Under this condition, $\widehat{{}^\eta \theta} = \theta$ as well.

The foregoing sets of representations and function spaces may easily be recast for groups over the adèles. We therefore write $\Pi(\tilde{G}(\mathbf{A}))$, $\mathcal{H}(\tilde{G}(\mathbf{A}))$, $\mathcal{I}(\tilde{G}(\mathbf{A}))$, $\Pi^{\mathrm{met}}(G(\mathbf{A}))$ etc. without explanation.

Henceforth, the functions f and $\tilde{f}$ will be taken to belong to either local or global versions of $\mathcal{H}^{\mathrm{met}}(G)$ and $\mathcal{H}(\tilde{G})$ respectively.

4 The Normalization of Intertwining Operators and the Plancherel Formula

Our goal here is to normalize the intertwining operators between induced representations. This is necessary for the definition of the invariant trace formula. In this section n and r are arbitrary positive integers and v is a nonArchimedean valuation.

What this normalization amounts to is the definition of functions

(11) $$r_{\tilde{Q}|\tilde{P}} : \Pi(\tilde{M}(F_S)) \times \mathfrak{a}^*_{\tilde{M},\mathbf{C}} \to \mathbf{C},\ Q, P \in \mathcal{P}(M),$$

which satisfy the conditions of Theorem 2.1 [9]. These functions are called normalizing factors. Such normalizing factors exist for general linear groups ([27], §4 [9]). We define candidates for normalizing factors of metaplectic coverings by setting

$$r_{\tilde{Q}|\tilde{P}}(\pi^*_{\lambda^*}) = r_{Q|P}(\pi_\lambda),$$

for all $\pi \in \Pi^{\mathrm{met}}(M(F_S))$ and $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$. In order to show that these proposed normalizing factors actually do satisfy Theorem 2.1 [9], we follow Lemma 2.1 [10]. This lemma relies on the Plancherel formula for reductive algebraic groups. As nontrivial metaplectic covering groups are not algebraic, we must justify the use of the Plancherel formula in the following lemma. This is done immediately afterwards.

Lemma 4.1 *The normalizing factors $r_{\tilde{Q}|\tilde{P}}(\pi^*_{\lambda^*})$ defined by (11) satisfy the properties of Theorem 2.1 [9].*

Proof. In §4 [9] it is explained that all of the properties of Theorem 2.1 [9] are satisfied if

(12) $$r_{\tilde{P}|\bar{\tilde{P}}}(\pi^*_{\lambda^*})r_{\bar{\tilde{P}}|\tilde{P}}(\pi^*_{\lambda^*}) = \mu_{\tilde{M}}(\pi^*_{\lambda^*})^{-1},$$

where $P \in \mathcal{P}(M)$, $\pi \in \Pi_{\text{temp}}^{\text{met}}(M(F_v))$, $\lambda \in \mathfrak{a}_{M,\mathbf{C}}^*$ and $\mu_{\tilde{M}}$ is the Harish-Chandra μ-function for the metaplectic group $\tilde{M}$. The normalizing factors for M, as defined in [27], already satisfy (12). That is

$$r_{P|\bar{P}}(\pi_\lambda) r_{\bar{P}|P}(\pi_\lambda) = \mu_M(\pi_\lambda)^{-1},$$

for all $P \in \mathcal{P}(M)$, $\pi \in \Pi_{\text{temp}}(M(F_v))$, and $\lambda \in \mathfrak{a}_{M,\mathbf{C}}^*$. Therefore it suffices to show that

$$\mu_{\tilde{M}}(\pi^*) = \mu_M(\pi), \ \pi \in \Pi_{\text{temp}}^{\text{met}}(M(F_v)). \tag{13}$$

Let $\Pi_{\text{disc}}(\tilde{M}(F_v))$ be the subset of representations in $\Pi_{\text{temp}}(\tilde{M}(F_v))$ which are square-integrable modulo $\widetilde{A_M^n}(F_v)$ and $\Pi_{\text{disc}}^{\text{met}}(M(F_v))$ be the subset of metic representations in $\Pi_{\text{disc}}(M(F_v))$. As mentioned in Proposition 27 [15], any $\tilde{\pi} \in \Pi_{\text{temp}}(\tilde{M}(F_v))$ is equal to $\tilde{\pi}_1^{\tilde{M}}$, where $\tilde{\pi}_1 \in \Pi_{\text{disc}}(\tilde{M}_1(F_v))$ and M_1 is a Levi subgroup of M. By definition

$$\mu_{\tilde{M}}(\tilde{\pi}) = \mu_{\tilde{M}_1}(\tilde{\pi}_1),$$

so it suffices to show (13) for $\pi \in \Pi_{\text{disc}}^{\text{met}}(M(F_v))$. This is achieved by means of the Plancherel formula. Let $f \in \mathcal{H}^{\text{met}}(G(F_v))$ be such that

$$\operatorname{tr} f_L(\pi) = 0, \ \pi \in \Pi_{\text{disc}}^{\text{met}}(L(F_v)), \ L \in \mathcal{L}, \ L \neq M.$$

It follows from the Plancherel formula that

$$f(1) = \gamma_M \int_{\Pi_{\text{disc}}^{\text{met}}(M(F_v))} d^M(\pi)\mu_M(\pi) f_M(\pi) d\pi,$$

where γ_M is a constant defined in terms of an integral over $\bar{U}$, $P = MU$ is a standard parabolic, $d^M(\pi)$ is the formal degree of $\pi \in \Pi_{\text{disc}}^{\text{met}}(M(F_v))$, and the measure on $\Pi_{\text{disc}}^{\text{met}}(M(F_v))$ is a product measure of infinitely many copies of the measure on $i\mathfrak{a}_{M,\{v\}}^*$. It follows easily from the definition of orbital integrals and Lemma 9.5 that

$$f(1) = I_G(1, f) = I_{\tilde{G}}(1, f^*) = f^*(1).$$

By the local metaplectic correspondence and the properties of f, we know that $f_{\tilde{G}}^*$ vanishes on any tempered representation which is not induced from some π^* where $\pi \in \Pi_{\text{disc}}^{\text{met}}(M(F_v))$ and that

$$f_{\tilde{M}}^*(\pi^*) = f_M(\pi).$$

This implies

$$f^*(1) = \gamma_{\tilde{M}} \int_{\Pi^{\rm met}_{\rm disc}(M(F_v))} d^{\tilde{M}}(\pi^*)\mu_{\tilde{M}}(\pi^*) f_M(\pi) d\pi.$$

The equality of γ_M with $\gamma_{\tilde{M}}$ will be argued in the discussion on the Plancherel formula immediately following this lemma. By varying f as specified above and using the trace Paley-Wiener theorem we find that

$$d^{\tilde{M}}(\pi^*)\mu_{\tilde{M}}(\pi^*) = d^M(\pi)\mu_M(\pi), \ \pi \in \Pi^{\rm met}_{\rm disc}(M(F_v)).$$

If $M = G$ then $\mu_G = \mu_{\tilde{G}} = 1$ and so $d^G(\pi) = d^{\tilde{G}}(\pi)$. It then follows that

$$d^M(\pi) = \sum_{i=1}^{\ell} d^{M(i)}(\pi_i) = \sum_{i=1}^{\ell} d^{\tilde{M}(i)}(\pi_i^*) = d^{\tilde{M}}(\pi^*),$$

in the notation of Appendix A, whence the lemma.□

We now list the properties of reductive algebraic groups which are used in Harish-Chandra's proof of the Plancherel formula ([16]) and show that they also hold for metaplectic coverings.

Let $P = MU$ be a standard parabolic subgroup of M. Then $\tilde{G}(F_v)$ splits over U (§2 [15]). The splitting homomorphism is s. In other words,

$$\mathrm{s}(U) = \{(u,1) : u \in U\}$$

forms a subgroup of $\tilde{G}(F_v)$. Every $Q \in \mathcal{P}(M)$ is of the form $P^w = w^{-1}Pw$ for some representative w of the Weyl group $W(\mathfrak{a}_M)$. It is easy to check that $\tilde{G}(F_v)$ splits over $U^w = w^{-1}Uw$ with splitting homomorphism s_w defined by

$$w^{-1}uw \mapsto \mathrm{s}(w)^{-1}\mathrm{s}(u)\mathrm{s}(w).$$

Clearly $\tilde{P^w} = \tilde{M}\mathrm{s}_w(U)$ as $\tilde{M}$ is stable under conjugation by $\mathrm{s}(w)$.

We define the Jacquet module of an admissible Hilbert space representation $(\tilde{\pi}, V)$ of finite length with respect to U^w in the following way. Let V_{U^w} be the linear span of

$$\{\tilde{\pi}(\mathrm{s}_w(w^{-1}uw))v - v : u \in U, v \in V\}.$$

It is a consequence of (2.2) [15] that $\tilde{M}$ normalizes $\mathrm{s}_w(U)$. Thus $\tilde{M}$ stabilizes V_{U^w}. We define the Jacquet module of $\tilde{\pi}$ with respect to U^w to be the representation obtained by twisting the quotient representation V/V_{U^w} with the modular function $\delta_{P^w}^{-1/2}$. We denote this representation by $\tilde{\pi}_{U^w}$. This is a mild generalization of §14 [15]. One may check that this definition yields the expected properties of Jacquet modules.

Another consequence of the splitting of $\tilde{G}(F_v)$ over U^w is the Iwasawa decomposition,

$$\tilde{G}(F_v) = \tilde{M}(F_v)\mathrm{s}_w(U^w)K_v = \tilde{M}(F_v)\mathrm{s}(w)^{-1}\mathrm{s}(U)\mathrm{s}(w)K_v.$$

The associated integration formula follows in the usual fashion.

Suppose for this paragraph that w is the representative of $W(\mathfrak{a}_M)$ such that $\bar{U} = U^w$. Then we obtain the Gelfand-Naimark decomposition. This is a decomposition of an open dense subset of $\tilde{G}(F_v)$ as

$$\mathrm{s}_w(U^w)\tilde{M}(F_v)\mathrm{s}(U).$$

Its associated integration formula is given by

$$\int_{\tilde{G}(F_v)} f(x)dx = \gamma_{\tilde{M}} \int_{U^w} \int_{\tilde{M}} \int_{U} f(\mathrm{s}_w(\bar{u})\tilde{m}\mathrm{s}(u))du d\tilde{m} d\bar{u},$$

where

$$\gamma_{\tilde{M}} = \gamma_M = \int_{\bar{U}} \delta_P(m_P(\bar{u}))d\bar{u},$$

$\bar{u} = u_P(\bar{u})m_P(\bar{u})k_P(\bar{u})$, $u_P(\bar{u}) \in U$, $m_P(\bar{u}) \in M(F_v)$ and $k_P(\bar{u}) \in K_v$. The results of this paragraph do not rely on the assumption that P is standard.

Observe that since $\tilde{M}(F_v)$ normalizes $\mathrm{s}_w(U^w)$ and $\widetilde{A_M^n}(F_v)$ is an abelian subgroup of $\tilde{M}(F_v)$, we may obtain a root space decomposition of $\mathrm{s}_w(U^w)$. Using this root space decomposition we may define a subset $\widetilde{A_M^n}^-$ of $\widetilde{A_M^n}$ as in §4 [11]. In fact all one needs to prove the remaining results concerning the asymptotic behaviour of matrix coefficients in §4 [11] is the Iwahori decomposition for arbitrarily small compact open subgroups. These Iwahori decompositions exist in $\tilde{G}(F_v)$ because there exists a compact open subgroup K_v' of $G(F_v)$ over which $\tilde{G}(F_v)$ splits (§[15]), and the Iwahori decomposition holds for $G(F_v)$.

The only decomposition which still needs to be addressed for $\tilde{G}(F_v)$ is the Cartan decomposition. This may be recast as

$$\tilde{G}(F_V) = \bigcup_{\gamma} K_v \gamma \widetilde{A_{M_0}^n}(F_v) K_v,$$

where γ runs over a set of representatives of $\widetilde{A_{M_0}}(F_v)/\widetilde{A^n_{M_0}}(F_v)$. This union is finite and disjoint. It is the finiteness which allows us to restrict our attention to $K_v\widetilde{A^n_{M_0}}(F_v)K_v$ when proving the convergence of integrals or bounds of certain functions.

We make one further remark concerning bounding functions on $\tilde{G}(F_v)$. If h is a genuine or antigenuine function on $\tilde{G}(F_v)$ then clearly

$$\sup_{\tilde{\gamma}\tilde{G}(F_v)} |h(\tilde{\gamma})| = \sup_{\tilde{\gamma}\in\tilde{G}(F_v)} |h(\mathrm{p}(\tilde{\gamma}),1)| = \sup_{\gamma\in G(F_v)} |h(\mathrm{s}(\gamma))|.$$

Therefore, in cases where one is interested in finding uniform bounds of such functions, the techniques of the non-metaplectic groups may be used.

This concludes the discussion of the properties necessary for the proof of the Plancherel formula. The proof may now be imitated after making some apparent definitions.

5 The Invariant Trace Formula

The purpose of this section is to present the metaplectic version of the invariant trace formula of Arthur ([6], [7]), and to serve as a more detailed introduction to the following sections.

Without further delay, we set forth the invariant trace formula of a function $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{A}))$ as the equality of

$$(14) \qquad n\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\tilde{\gamma}\in(\mathrm{s}_0(M(F)))_{\tilde{M},S}} a^{\tilde{M}}(S,\tilde{\gamma})I_{\tilde{M}}(\tilde{\gamma},\tilde{f}),$$

with

$$(15) \qquad \sum_t \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(\tilde{M},t)} a^{\tilde{M}}(\tilde{\pi})I_{\tilde{M}}(\tilde{\pi},\tilde{f})d\tilde{\pi}.$$

It will be convenient to denote (14) as $I(\tilde{f})$, and $I(f^*)$ as $I^{\mathcal{M}}(f)$. This trace formula is extrapolated from the trace formula given in [7] for algebraic groups. Since nontrivial metaplectic coverings of algebraic groups are not algebraic, one ought to verify the results of Arthur for metaplectic groups in order to rigorously assert the existence of a trace formula as we have done above. There are unfortunately too many results that need to be checked to be included in this thesis. Those results which have been checked (§4 for example) follow in a straightforward manner. There is no reason to doubt that the other results do not follow in the same way. We will therefore be assuming that the invariant trace formula is correct as stated.

Expansion (14) is known as the geometric side of the trace formula, as its terms depend on (in our case) conjugacy classes in $\tilde{M}(F)$. It will be further elaborated on in §9. The other sum, (15), is known as the spectral side of the trace formula, as its terms depend on representations of $\tilde{M}(\mathbf{A})$. It will be further elaborated on in §12. Both the geometric and spectral sides of the trace formula contain terms which are local, *i.e.* determined by $\tilde{M}(F_S)$, and global, *i.e.* determined by the subgroup $\mathrm{s}_0(M(F))$ of $\tilde{M}(\mathbf{A})$.

In sections 6 and 9 we examine and compare the local geometric terms of the trace formulas of $\tilde{G}$ and G. In Theorem 9.1, we state the precise fashion in which the geometric terms of these trace formulas are equal. The proof of Theorem 9.1 is completed in the final section. Its proof is inductive and involves the spectral side of the trace formula as well as the geometric side. Theorem 12.1 is the spectral analogue of Theorem 9.1 and is also completed in the final section. It is shown to partially follow from Theorem 9.1. The remaining sections are all steps in the inductive proof of Theorem 9.1 (i).

6 The Local Geometric Terms

The local geometric terms of (14) have the form $I_{\tilde{M}}(\tilde{\gamma}, \tilde{f})$, where $\tilde{\gamma}$ is a conjugacy class in $\tilde{M}(F_S)$ and $\tilde{f} \in \mathcal{H}(\tilde{M}(F_S))$. For a fixed function $\tilde{f}$, these terms are derived from weighted orbital integrals (*cf.* §2 [6]). On the other hand, if $\tilde{\gamma}$ is fixed then $I_{\tilde{M}}(\tilde{\gamma})$ may be viewed as an invariant distribution on $\mathcal{H}(\tilde{G}(F_S))$. Both points of view are valuable. We will abuse notation from time to time by identifying an element of $\tilde{M}(F_S))$ with its conjugacy class.

These distributions satisfy two useful properties. The first of these is the descent property (Theorem 8.1 [6]). This implies that if $M_1 \in \mathcal{L}$ is a subgroup of M and $\tilde{\gamma} \in \tilde{M}_1(F_S)$ such that $\tilde{M}_{1,\tilde{\gamma}} = M_{\tilde{\gamma}}$ then

$$(16) \qquad I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) = \sum_{L \in \mathcal{L}(M_1)} d^G_{M_1}(M, L) \hat{I}^{\tilde{L}}_{\tilde{M}_1}(\tilde{\gamma}, \tilde{f}_{\tilde{L}}).$$

See [6] for the definition of $d^G_{M_1}(M, L)$.

The second property satisfied by these distributions is one of splitting. That is, if S is a disjoint union of nonempty sets S_1 and S_2, and $\tilde{f} = \tilde{f}_1\tilde{f}_2$, $\tilde{\gamma} = \tilde{\gamma}_1\tilde{\gamma}_2$ are the corresponding decompositions, then

$$I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) = \sum_{\tilde{L}_1, \tilde{L}_2 \in \mathcal{L}(M)} d^G_{M_1}(L_1, L_2) \hat{I}^{\tilde{L}_1}_{\tilde{M}}(\tilde{\gamma}_1, \tilde{f}_{1,\tilde{L}_1}) \hat{I}^{\tilde{L}_2}_{\tilde{M}}(\tilde{\gamma}_2, \tilde{f}_{2,\tilde{L}_2}). \tag{17}$$

Let us speculate on how the local geometric terms of the trace formula for G might match up with those of the trace formula of $\tilde{G}$. Our means of relating conjugacy classes of these groups is the transfer map (1). The restriction of this map to $M(F_S)$ is invariant under μ_n^M. We prove this as a lemma.

Lemma 6.1 *Suppose $\gamma \in M(F_S)$ and $\eta \in \mu_n^M$. Then*

$$(\eta\gamma)^* = \gamma^*.$$

Proof. It is sufficient to prove the lemma in the case that S consists of a single valuation v. Let

$$(\cdot, \cdot)_{F_v} : G \times G \to \mu_n$$

be the nth Hilbert symbol of F_v. The cocycle τ_{0v} is given by

$$\tau_{0v}(\delta', \delta) = \prod_{1 \le i < j \le r} (\delta'_i, \delta_j)_{F_v} \kappa_v(\gamma)\kappa_v(\delta)/\kappa_v(\delta'\delta),$$

where $\delta', \delta \in A_{M_0}(F_v)$, δ'_i and δ_j are the respective diagonal entries of δ' and δ, and κ_v is a map taking values in μ_n (§2 [15]). It may be shown by induction on n that

$$\begin{aligned}
(\eta\gamma)^* &= (\eta\gamma, 1)^n \\
&= (1, \tau_S(\eta, \gamma)^{-n(n+1)/2} \tau_{0v}(\gamma, \eta)^{n(n-1)/2})(\eta^n, \prod_{k=1}^{n-1} \tau_{0v}(\eta^k, \eta))(\gamma, 1)^n \\
&= (1, \prod_{k=1}^{n-1} \prod_{i<j} (\eta_i, \eta_j)^k_{F_v} \kappa_v(\eta^k)\kappa_v(\eta)/\kappa_v(\eta^{k+1}))\gamma^* \\
&= (1, \prod_{i<j} (\eta_i, \eta_j)^{n(n-1)/2)}_{F_v} \kappa_v(\eta)^n/\kappa_v(\eta^n))\gamma^* \\
&= \gamma^*. \Box
\end{aligned}$$

Thus, if we wish to compare $I_{\tilde{M}}(\gamma^*)$ with the invariant distributions from $M(F_S)$, we had best group the latter into μ_n^M invariant sums. An obvious such grouping would be

$$\sum_{\eta \in \mu_n^M} I_M(\eta\gamma, f), \ f \in \mathcal{H}^{\mathrm{met}}(M(F_S)).$$

This grouping has only one shortcoming. If $M = G$ then

$$\sum_{\eta \in \mu_n^G} I_G(\eta\gamma, f) \neq I_G(\gamma, f) = I_{\tilde{G}}(\gamma^*, f^*).$$

We can correct this shortcoming if we take into account that

$$I_M(\eta\gamma, f) = I_M(\gamma, f), \ \eta \in \mu_n^G. \tag{18}$$

Indeed this invariance follows from the μ_n^G-invariance of the functions in $\mathcal{H}^{\mathrm{met}}(G(F_S))$. It then makes sense to define

$$I_M^{\mathcal{E}}(\gamma, f) = \sum_{\eta \in \mu_n^M/\mu_n^G} I_M(\eta\gamma, f), \ \gamma \in M(F_S).$$

As we shall see in Theorem 9.1, this is the sought-after grouping of invariant distributions. Notice that if $L \in \mathcal{L}(M)$ then

$$\hat{I}_M^{L,\mathcal{E}}(\gamma, f_L) = \sum_{\eta \in \mu_n^M/\mu_n^L} \hat{I}_M^L(\eta\gamma, f_L),$$

for all $\gamma \in M(F_S)$.

Before examining the properties of $I_M^{\mathcal{E}}(\gamma)$, let us reconsider (18), that is the μ_n^G-invariance of $I_M(\gamma)$. It is a direct consequence of the inductive definition of I_M that

$$I_M(\eta\gamma, f) = I_M(\gamma, {}^{\eta}f) = I_M(\gamma, f), \ \gamma \in M(F_S), \ \eta in \mu_n^G.$$

In the notation of §3 this becomes

$$I_M(\eta\gamma) = {}^{\eta}I_M(\gamma) = I_M(\gamma).$$

It is useful to note the reformulation in terms of Paley-Wiener functions, namely

$$\hat{I}_M(\eta\gamma, f_M) = {}^{\eta}\hat{I}_M(\gamma, f_M) = \hat{I}_M(\gamma, f_M).$$

We now show that the distributions $I_M^{\mathcal{E}}(\gamma)$ satisfy descent and splitting properties as well.

Lemma 6.2 *Suppose M and M_1 belong to $\mathcal{L}$ and $M_1 \subset M$. Suppose further that $L \in \mathcal{L}(M_1)$ such that $d^G_{M_1}(M, L) \neq 0$. Then the map*

$$\mu_n^M/\mu_n^G \times \mu_n^L/\mu_n^G \to \mu_n^{M_1}/\mu_n^G,$$

given by $(\eta_1\mu_n^G, \eta_2\mu_n^G) \mapsto \eta_1\eta_2\mu_n^G$, is an isomorphism.

Proof. If $d^G_{M_1}(M, L) \neq 0$ as above, then by definition, $\mathfrak{a}^M_{M_1} \oplus \mathfrak{a}^L_{M_1} \cong \mathfrak{a}^G_{M_1}$ (§7 [6]). The vector spaces $\mathfrak{a}^G_M$ and $\mathfrak{a}^G_L$ may be regarded as the respective orthogonal complements of $\mathfrak{a}^M_{M_1}$ and $\mathfrak{a}^L_{M_1}$ in $\mathfrak{a}^G_{M_1}$. As a consequence we also have $\mathfrak{a}^G_M \oplus \mathfrak{a}^G_L \cong \mathfrak{a}^G_{M_1}$. Consider the homomorphism

$$H_{M_1} : M_1 \to \mathfrak{a}_{M_1}.$$

It is readily verified that it passes to a homomorphism $H'_{M_1} : A_{M_1}/A_G \to \mathfrak{a}^G_{M_1}$ such that $H'_{M_1}(A_M/A_G) \subset \mathfrak{a}^G_M$ and $H'_{M_1}(A_L/A_G) \subset \mathfrak{a}^G_L$. Accordingly

$$H'_{M_1}((A_M \cap A_L)/A_G) \subset \mathfrak{a}^G_M \cap \mathfrak{a}^G_L = \{0\}.$$

In other words, $|\xi(\gamma)| = 1$ for all γ belonging to the split torus $A_M \cap A_L$, and all characters $\xi \in X(M_1)_F$ which are trivial when restricted to G. This implies that $A_L \cap A_M \subset A_G$. As a result, the multiplication map

$$A_M/A_G \times A_L/A_G \to A_{M_1}/A_G$$

is injective. It now follows from the commutative diagram,

$$\begin{array}{ccc} \mu_n^M/\mu_n^G \times \mu_n^L/\mu_n^G & \hookrightarrow & A_M/A_G \times A_L/A_G \\ \downarrow & & \downarrow \\ \mu_n^{M_1}/\mu_n^G & \hookrightarrow & A_{M_1}/A_G \end{array},$$

that the map of the lemma is injective. The surjectivity of the map can be seen from the following equalities.

$$|\mu_n^M/\mu_n^G \times \mu_n^L/\mu_n^G| = n^{\dim \mathfrak{a}^G_M} n^{\dim \mathfrak{a}^G_L} = n^{\dim \mathfrak{a}^G_{M_1}} = |\mu_n^{M_1}/\mu_n^G|. \square$$

Lemma 6.3 *Let M, M_1 and L be as in Lemma 6.2. The map*

$$\mu_n^M/\mu_n^G \to \mu_n^{M_1}/\mu_n^L,$$

given by $\eta\mu_n^G \mapsto \eta\mu_n^L$ for $\eta \in \mu_n^M$, is an isomorphism.

Proof. The proof of this lemma follows from arguments similar to those of Lemma 6.2.□

The following proposition proves a descent property for $I_M^{\Sigma}(\gamma)$. For this we need the notion of an induced conjugacy class. Given $\gamma \in M(F_S)$ define the induced conjugacy class of γ, γ^G, to be the union of the $G(F_S)$-conjugacy classes which intersect γU in an open set. Here $P = MU$ and $P \in \mathcal{P}(M)$ is arbitrary. This definition is well-defined (§6 [8]) and in our case γ^G is always a single $G(F_S)$-conjugacy class.

Proposition 6.1 *Suppose M and M_1 belong to $\mathcal{L}$ with $M_1 \subset M$. Moreover suppose $\gamma \in M_1(F_S)$. Then*

$$I_M^{\Sigma}(\gamma^M, f) = \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L)\hat{I}_{M_1}^{L,\Sigma}(\gamma, f_L).$$

Proof. Let $\eta \in \mu_n^M$. Since η lies in the centre of $M(F_S)$, $\eta\gamma^M = \{\eta\delta : \delta \in \gamma^M\}$ is a conjugacy class in $M(F_S)$. Let U be the unipotent radical of some $P \in \mathcal{P}^M(M_1)$. Clearly, left multiplication by η is a homeomorphism between γU and $\eta\gamma U$. It follows that $\eta\gamma^M$ is a conjugacy class of $M(F_S)$ which intersects $\eta\gamma U$ in an open set. In other words $\eta\gamma^M = (\eta\gamma)^M$.

The descent property for $I_M(\gamma)$ (Theorem 8.1 [6]) together with Lemma 6.3 yield

$$\begin{aligned}
I_M^{\Sigma}(\gamma^M, f) &= \sum_{\eta \in \mu_n^M/\mu_n^G} I_M(\eta\gamma^M, f) \\
&= \sum_{\eta \in \mu_n^M/\mu_n^G} I_M((\eta\gamma)^M, f) \\
&= \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L) \sum_{\eta \in \mu_n^M/\mu_n^G} \hat{I}_{M_1}^L(\eta\gamma, f_L) \\
&= \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L) \sum_{\eta \in \mu_n^{M_1}/\mu_n^L} \hat{I}_{M_1}^L(\eta\gamma, f_L) \\
&= \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L)\hat{I}_{M_1}^{L,\Sigma}(\gamma, f_L).\square
\end{aligned}$$

Corollary 6.1 *Suppose M and M_1 belong to $\mathcal{L}$ with $M_1 \subset M$. Moreover suppose $\gamma \in M_1(F_S)$. Then*

$$I_M^{\Sigma}(\gamma, f) = \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L)\hat{I}_{M_1}^{L,\Sigma}(\gamma, f_L).$$

Proof. It follows from rational canonical form that γ^M is equal to the conjugacy class of γ in $M(F_S)$.□

Proposition 6.2 *Suppose S is the disjoint union of nonempty sets S_1 and S_2, and that $f = f_1 f_2$, $\gamma = \gamma_1\gamma_2 \in M(F_S)$ are corresponding decompositions. Then*

$$I_M^{\mathcal{E}}(\gamma, f) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2)\hat{I}_M^{L_1,\mathcal{E}}(\gamma_1, f_{1,L_1})\hat{I}_M^{L_2,\mathcal{E}}(\gamma_2, f_{2,L_2}).$$

Proof. We begin by applying the splitting property to the summands of $I_M^{\mathcal{E}}(\gamma)$.

$$\begin{aligned} I_M^{\mathcal{E}}(\gamma, f) &= \sum_{\eta \in \mu_n^M/\mu_n^G} I_M(\eta\gamma, f) \\ &= \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \sum_{\eta \in \mu_n^M/\mu_n^G} \hat{I}_M^{L_1}(\eta\gamma_1, f_{1,L_1})\hat{I}_M^{L_2}(\eta\gamma_2, f_{2,L_2}). \end{aligned}$$

Suppose the expression $d_M^G(L_1, L_2)$ of these sums is not zero. By using arguments similar to those of the proof of Lemma 6.2, it may be established that

$$\mu_n^M/\mu_n^G \cong \mu_n^M/\mu_n^{L_1} \times \mu_n^M/\mu_n^{L_2},$$

$$\mu_n^M/\mu_n^G \cong \mu_n^{L_1}/\mu_n^G \times \mu_n^{L_2}/\mu_n^G,$$

and $\mu_n^{L_1} \cap \mu_n^{L_2} = \mu_n^G$. From $\mu_n^{L_1} \cap \mu_n^{L_2} = \mu_n^G$ it follows that the homomorphism

$$\mu_n^{L_1}/\mu_n^G \to \mu_n^M/\mu_n^{L_2},$$

given by $\eta\mu_n^G \mapsto \eta\mu_n^{L_2}$, is injective. This homomorphism is also surjective as

$$|\mu_n^{L_1}/\mu_n^G| = n^{\dim \mathfrak{a}_M^{L_1} - \dim \mathfrak{a}_M^G} = n^{\dim \mathfrak{a}_M^{L_2}} = |\mu_n^M/\mu_n^{L_2}|.$$

It may be deduced in the same manner that $\mu_n^{L_2}/\mu_n^G \cong \mu_n^M/\mu_n^{L_1}$.

Thus the previous sum is equal to

$$\begin{aligned} &\sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \sum_{\eta_1 \in \mu_n^M/\mu_n^{L_1}} \sum_{\eta_2 \in \mu_n^M/\mu_n^{L_2}} \hat{I}_M^{L_1}(\eta_1\eta_2\gamma_1, f_{1,L_1})\hat{I}_M^{L_2}(\eta_1\eta_2\gamma_2, f_{2,L_2}) \\ &= \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \sum_{\eta_1 \in \mu_n^{L_2}/\mu_n^G} \sum_{\eta_2 \in \mu_n^{L_1}/\mu_n^G} \hat{I}_M^{L_1}(\eta_1\eta_2\gamma_1, f_{1,L_1})\hat{I}_M^{L_2}(\eta_1\eta_2\gamma_2, f_{2,L_2}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{L_1,L_2\in\mathcal{L}(M)} d_M^G(L_1,L_2) \sum_{\eta_1\in\mu_n^{L_2}/\mu_n^G} \hat{I}_M^{L_1}(\eta_1\eta_2\gamma_1, f_{1,L_1}) \sum_{\eta_2\in\mu_n^{L_1}/\mu_n^G} \hat{I}_M^{L_2}(\eta_1\eta_2\gamma_2, f_{2,L_2}) \\
&= \sum_{L_1,L_2\in\mathcal{L}(M)} d_M^G(L_1,L_2) \sum_{\eta_1\in\mu_n^{M}/\mu_n^{L_1}} \hat{I}_M^{L_1}(\eta_1\gamma_1, f_{1,L_1}) \sum_{\eta_2\in\mu_n^{M}/\mu_n^{L_2}} \hat{I}_M^{L_2}(\eta_2\gamma_2, f_{2,L_2}) \\
&= \sum_{L_1,L_2\in\mathcal{L}(M)} d_M^G(L_1,L_2) \hat{I}_M^{L_1,\Sigma}(\gamma_1, f_{1,L_1}) \hat{I}_M^{L_2,\Sigma}(\gamma_2, f_{2,L_2}). \square
\end{aligned}
$$

7 A Fundamental Lemma

We may already prove a specific, yet crucial identity between the local geometric terms of $\tilde{G}$ and G. For the duration of this section we fix v to be a valuation of F such that $|n|_v = 1$. Define f_v^0 to be the product of the characteristic function of K_v with the scalar $\text{vol}(K_v)^{-1}$. Define $\tilde{f}_v^0$ to be the unique antigenuine function such that $\tilde{f}_v^0 \circ \mathbf{p} = f_v^0$. The following proposition is a type of weighted fundamental lemma, as it is an identity of weighted orbital integrals defined in [8].

We first prove a technical lemma which concerns the weight $v_{\tilde{M}}$ of the weighted orbital integrals (*cf.* p. 36 [1], §§1-2 [8]).

Lemma 7.1 *Suppose $\gamma \in M(F_v)$. Then*

$$v_{\tilde{M}}(\mathrm{s}(\gamma)) = n^{\dim(A_M/A_G)} v_M(\gamma).$$

Proof. The $(\tilde{G}, \tilde{M})$ family from which $v_{\tilde{M}}$ is derived is defined as

$$
\begin{aligned}
v_{\tilde{P}}(\lambda^*, H_{\tilde{M}}(\mathrm{s}(\gamma))) &= e^{\lambda^*(H_{\tilde{M}}(\mathrm{s}(\gamma)^*)} \\
&= e^{\lambda^*(nH_{\tilde{M}}(\mathrm{s}(\gamma)))} \\
&= e^{\lambda(H_M(\gamma))},
\end{aligned}
$$

where $P \in \mathcal{P}(M)$ and $\lambda \in i\mathfrak{a}_M^*$. In particular

$$v_{\tilde{P}}(\lambda^*, \mathrm{s}(\gamma)) = v_P(\lambda, \gamma).$$

By definition then

$$v_{\tilde{M}}(\mathrm{s}(\gamma)) = \lim_{\lambda\to 0} \sum_{P\in\mathcal{P}(M)} v_{\tilde{P}}(\lambda^*, \mathrm{s}(\gamma)) \text{vol}\left(\mathfrak{a}_{\tilde{M}}/\mathrm{Z}(\Delta_M^\vee)^*\right) \prod_{\alpha\in\Delta_M} (\lambda^*(\alpha^{\vee *}))^{-1},$$

where Δ_M is the set of simple roots of (P, A_M) (§6 [1]). Since the measures on $\mathfrak{a}_M$ and $\mathfrak{a}_{\tilde{M}}$ are identical, we have

$$\mathrm{vol}\,(\mathfrak{a}_{\tilde{M}}/\mathbf{Z}(\Delta_M^\vee)^*) = n^{\dim(A_M/A_G)}\mathrm{vol}\,(\mathfrak{a}_M/\mathbf{Z}(\Delta_M^\vee)).$$

Moreover $\lambda^*(\alpha^{\vee *}) = \lambda(\alpha^\vee)$. Hence

$$\begin{aligned} v_{\tilde{M}}(\mathrm{s}(\gamma)) &= \lim_{\lambda\to 0} \sum_{P\in\mathcal{P}(M)} v_P(\lambda,\gamma)(n^{\dim(A_M/A_G)}\mathrm{vol}\,(\mathfrak{a}_M/\mathbf{Z}(\Delta_M^\vee)) \prod_{\alpha\in\Delta_M} (\lambda(\alpha^\vee)^{-1})) \\ &= n^{\dim(A_M/A_G)} v_M(\gamma)\Box \end{aligned}$$

In what follows, an element $\tilde{\gamma} \in \tilde{M}(F_S))$ is said to be F_v-elliptic in $\tilde{M}$ if $\tilde{\gamma} \in \tilde{M}_{\tilde{\delta}}(F_v)$ for some regular element $\tilde{\delta} \in \tilde{M}(F_v)$ and $\tilde{M}_{\tilde{\delta}}(F_v)/\widetilde{A_M^n}(F_v)$ is compact.

Proposition 7.1 *Suppose $\gamma \in M(F_v)$ such that γ^n is semisimple and G-regular. Then*

$$J_{\tilde{M}}(\gamma^*, \tilde{f}_v^0) = n^{\dim(A_M/A_G)} J_M(\gamma, f_v^0).$$

Proof. We first focus our attention on $J_{\tilde{M}}(\gamma^*, \tilde{f}_v^0)$. Without loss of generality $\gamma \in K_v$. By Lemma 1.1.2 [13], we have $\tilde{G}_{\gamma^n} = p^{-1}(G_\gamma)$. In consequence of this and the normalization of measures in §3, $\tilde{G}_{\gamma^n}(F_v)\backslash\tilde{G}(F_v)$ and $G_\gamma(F_v)\backslash G(F_v)$ are identified as measure spaces. We retain the notation of the latter space. It now follows that

$$J_{\tilde{M}}(\gamma^*, \tilde{f}_v^0) = n^{\dim(A_M/A_G)}|D(\gamma^n)|_v^{1/2} \int_{G_\gamma(F_v)\backslash G(F_v)} \tilde{f}_v^0(\mathrm{s}(x^{-1})\gamma^*\mathrm{s}(x))v_M(x)dx.$$

Let $A_{G_\gamma}(F_v)$ be the maximal split component of the centre of G_γ. The centralizer of $A_{G_\gamma}(F_v)$ in $G(F_v)$ is a Levi subgroup. Let M_1 denote this Levi subgroup and let N_1 be its unipotent radical. The inclusion $M_1 \subset M$ follows from the fact that the centre of M, which is a split torus, is contained in A_{G_γ}, the centre of M_1. Observe also that $M_1 \supset G_\gamma$ and that γ is F_v-elliptic in $M_1(F_v)$. If we apply the Iwasawa decomposition, $G = M_1N_1K$, to $J_{\tilde{M}}(\gamma^*, \tilde{f}^0)$, we obtain the product of $n^{\dim(A_M/A_G)}$ with

$$|D(\gamma^n)|_v^{1/2} \int_{M_{1,\gamma}(F_v)\backslash M_1(F_v)} \int_{N_1(F_v)} \int_{K_v} \tilde{f}_v^0(\mathrm{s}(k^{-1}n^{-1}m^{-1})\gamma^*\mathrm{s}(mnk))v_M(mnk)dk\,dn\,dm$$

(with apologies for the double usage of n). By taking into account that $\bar{f}_v^0$ is bi-invariant under $\mathrm{s}(K_v)$ and that v_M is right K_v-invariant (§2 [8]), we may ignore the above integral over K_v to arrive at

$$n^{\dim(A_M/A_G)}|D(\gamma^n)|_v^{1/2}\int_{M_{1,\gamma}(F_v)\backslash M_1(F_v)}\int_{N_1(F_v)}\bar{f}_v^0(\mathrm{s}(n^{-1}m^{-1})\gamma^*\mathrm{s}(mn))v_M(mn)dn\,dm.$$

Define a map $\varphi_m : N_1 \to N_1$ by $\varphi_m(n) = m_0^{-1}n^{-1}m_0 n$, where $m_0 = m^{-1}\gamma^n m$, $m \in M_1$ and $n \in N_1$. It is not hard to show that φ_m is injective from the fact that γ^n is G-regular. By Proposition 7 [15], the last integral may be expressed as the product of $n^{\dim(A_M/A_G)}|D^{M_1}(\gamma^n)|_v^{1/2}$ with

$$\int_{M_{1,\gamma}(F_v)\backslash M_1(F_v)}\delta_{P_1}^{1/2}(m^{-1}\gamma^n m)\int_{N_2(F_v)}\bar{f}_v^0(\mathrm{s}(m^{-1})\gamma^*\mathrm{s}(mn))v_M(m\varphi_m^{-1}(n))dn\,dm,$$

where N_2 is the image of φ_m, δ_{P_1} is the modular function of the parabolic subgroup $P_1 = M_1N_1$, and D^{M_1} is the Weyl discriminant for the group M_1.

It is obvious that $\bar{f}_v^0(\mathrm{s}(m^{-1})\gamma^*\mathrm{s}(mn))v_M(m\varphi_m^{-1}(n))$ vanishes unless

$$m_0 n = m^{-1}\gamma^n mn$$

lies in K_v. Now observe that if $m_0 n$ lies in K_v, then so do m_0 and n. Indeed, since M_1N_1 may be taken to be a standard parabolic subgroup, the entries of $m_0 n$ lie in R_v if and only if the entries of m_0 and n lie in R. If $m_0 \in K$ then $\delta_{P_1}(m_0) = 1$ and our integral becomes

$$n^{\dim(A_M/A_G)}|D^{M_1}(\gamma^n)|_v^{1/2}\int_{M_{1,\gamma}(F_v)\backslash M_1(F_v)}\bar{f}_v^0(\mathrm{s}(m^{-1})\gamma^*\mathrm{s}(m))\int_{N_2(F_v)\cap K_v}v_M(m\varphi_m^{-1}(n))dn\,dm$$
$$= n^{\dim(A_M/A_G)}|D^{M_1}(\gamma^n)|_v^{1/2}\int_{M_{1,\gamma}(F_v)\backslash M_1(F_v)}\bar{f}_v^0(\mathrm{s}(m^{-1})\gamma^*\mathrm{s}(m))\int_{N_2(F_v)\cap K_v}v_M(\varphi_m^{-1}(n))dn\,dm.$$

The latter equality arises from the left M-invariance of v_M (§2 [8]). The preceding calculation can easily be adapted to $J_M(\gamma, f_v^0)$ by setting $n = 1$. Consequently,

$$J_M(\gamma, f_v^0) = |D^{M_1}(\gamma)|_v^{1/2}\int_{M_{1,\gamma}(F_v)\backslash M_1(F_v)}f_v^0(m^{-1}\gamma m)\int_{N_2(F_v)\cap K_v}v_M(\varphi_m^{-1}(n))dn\,dm.$$

We now follow the arguments of §12 [15]. We may decompose M_1 as $\prod_{i=1}^{\ell_1}M_1(i)$, where $M_1(i) \cong \mathrm{GL}(b_i, F_v)$ and $\sum_{i=1}^{\ell_1}b_i = r$. Let su be the topological Jordan decomposition of

γ (§3 Lemma 2 [18]). Then $s^n u^n$ is the topological Jordan decomposition of γ^n. We will show that $M_{1,s^n}(F_v) = M_{1,s}$. Since γ is semisimple, F_v-elliptic in M_1, and commutes with s, s itself is semisimple and F_v-elliptic in $M_1(F_v)$. Thus $M_{1,s^n}(F_v)$ is a reductive group isomorphic to $\prod_{i=1}^{\ell_1'} \mathrm{GL}(b_i', F_i')$, where F_i' is a field extension of F_v and $\sum_{i=1}^{\ell_1'}[F_i' : F_v]b_i' = r$. Since it contains the elliptic torus $M_{1,\gamma}(F_v)$, we must have $\ell_1' = \ell_1$ and $[F_i' : F_v]b_i' = b_i$. In particular s^n may be regarded as an element of $\prod_{i=1}^{\ell_1} F_i'$. Let F_i'' be the field obtained from F_i' by adjoining an nth root of s^n. Then F_i'' is a field extension of F_i' whose degree divides n (*cf.* proof of Lemma 9.1). However $[F_i'' : F_i'] \leq r$, so we must have $[F_i'' : F_i'] = 1$, that is $F_i'' = F_i'$. Since $M_{1,s}(F_v)$ is isomorphic to $\prod_{i=1}^{\ell_1} \mathrm{GL}(b_i'', F_i'')$, where $b_i''[F_i'' : F_i'] = b_i'$, we may conclude that $M_{1,s^n}(F_v) = M_{1,s}(F_v)$.

We now reduce the earlier weighted orbital integrals over $M_1(F_v)$ to weighted orbital integrals over $M_{1,s}(F_v)$. Indeed according to §3 Lemma 3 [18], if $m^{-1}\gamma m \in K_v$ for $m \in M_1(F_v)$, then $m \in M_{1,s}(F_v)K_v$. We set

$$\psi(m) = \int_{N_2(F_v)\cap K_v} v_M(\varphi_m^{-1}(n))dn, \ m \in M_1(F_v)$$

in order to render the ensuing computations more readable. With this notation we find

$$|D^{M_1}(\gamma^n)|_v^{1/2} \int_{M_{1,\gamma}(F_v)\backslash M_1(F_v)} \bar{f}_v^0(\mathrm{s}(m^{-1})\gamma^* \mathrm{s}(m))\psi(m)dm$$

$$(19) = \mathrm{vol}\left(\frac{K_v \cap M_1(F_v)}{(K_v \cap M_{1,s}(F_v))}\right) |D^{M_1}(s^n u^n)|_v^{1/2} \int_{M_{1,\gamma}(F_v)\backslash M_{1,s}(F_v)} \bar{f}_v^0(\mathrm{s}(m^{-1})\gamma^* \mathrm{s}(m))\psi(m)dm.$$

In the following we identify $M_{1,s}(F_v)$ with $\prod_{i=1}^{\ell_1} \mathrm{GL}(b_i', F_i')$. We may replace s^* and u^* with $(s^n, 1)$ and $(u^n, 1)$ respectively since K_v splits over $\tilde{G}(F_v)$ (§2 [15]). We will abuse notation slightly by suppressing the second coordinate. By a variation Proposition 0.1.5 [19],

$$m^{-1}s^n m = (1, (\det(s^n), \det(m))_{F_v} / \prod_{i=1}^{\ell_1} (s_i^n, \det(m_i))_{F_i'}),$$

where $m \in \prod_{i=1}^{\ell_1} \mathrm{GL}(b_i', F_i')$, m_i is the image of m in $\mathrm{GL}(b_i', F_i')$ and $(\cdot,\cdot)$ is the nth Hilbert symbol of the field following it in subscript. Since the Hilbert symbol is trivial on nth powers, the above expression is easily seen to be the identity. Hence (19) becomes

$$\mathrm{vol}\left(\frac{K_v \cap M_1(F_v)}{K_v \cap M_{1,s}(F_v)}\right) |D^{M_{1,s}}(u^n)|_v^{1/2} \int_{\prod_{i=1}^{\ell_1} \mathrm{PGL}(b_i', F_i')} \bar{f}_v^0(\mathrm{s}(m^{-1})u^n \mathrm{s}(m))\psi(m)dm.$$

By following the arguments of the Proposition of §12 [15] verbatim, we can show this to be equal to

$$\mathrm{vol}\left(\frac{K_v \cap M_1(F_v)}{(K_v \cap M_{1,s}(F_v)}\right) |D^{M_{1,s}}(u)|_v^{1/2} \int_{\prod_{i=1}^{\ell_1} \mathrm{PGL}(b'_i, F'_i)} f_v^0(m^{-1}um)\psi(m)dm.$$

Working backwards from this integral, we obtain the proposition. ◻

Proposition 7.2 *Suppose $\gamma \in M(F_v)$. Then*

$$I_{\tilde{M}}(\gamma^*, \tilde{f}_v^0) = I_M^{\Sigma}(\gamma, f_v^0). \tag{20}$$

Proof. Once again, without loss of generality, we may assume that γ is in K_v. By Proposition 7.1 and Lemma 6.1, it is apparent that

$$n^{-\dim(A_M/A_G)} J_{\tilde{M}}(\gamma^*, \tilde{f}_v^0) = J_M(\eta\gamma, f_v^0),\ \eta \in \mu_n^M.$$

According to Lemma 2.1 [6]

$$I_{\tilde{M}}(\gamma^*, \tilde{f}_v^0) = J_{\tilde{M}}(\eta\gamma, \tilde{f}_v^0).$$

Thus

$$\begin{aligned}
I_{\tilde{M}}(\gamma^*, \tilde{f}_v^0) &= J_{\tilde{M}}(\gamma^*, \tilde{f}_v^0) \\
&= \sum_{\eta \in \mu_n^M/\mu_n^G} n^{-\dim(A_M/A_G)} J_{\tilde{M}}(\gamma^*, \tilde{f}_v^0) \\
&= \sum_{\eta \in \mu_n^M/\mu_n^G} J_M(\eta\gamma, f^0) \\
&= \sum_{\eta \in \mu_n^M/\mu_n^G} I_M(\eta\gamma, f^0) \\
&= I_M^{\Sigma}(\gamma, f^0),
\end{aligned}$$

if γ^n is G-regular.

If γ^n is not assumed to be G-regular, let $\gamma = \sigma u$ be the Jordan decomposition of $\gamma \in M$. By the definition of $r_{\tilde{P}}(\nu^*, \gamma^*, a^*)$ (3.4 [8]), we have

$$r_{\tilde{P}}(\nu^*, \gamma^*, a^*) \;=\; \prod_{\beta} |(a^n)^{\beta^*} - (a^{-n})^{\beta^*}|^{\rho(\beta^*, u^n)(\frac{\nu^*}{2})(\beta^\vee)^*}$$

$$
\begin{aligned}
&= \prod_{\beta} |(a^n)^{\beta/n} - (a^{-n})^{\beta/n}|^{\rho(\beta/n, u^n)(\frac{\nu}{2n})(n\beta^\vee)} \\
&= \prod_{\beta} |a^{\beta} - (a^{-1})^{\beta}|^{\rho(\beta, u)(\frac{\nu}{2})(\beta^\vee)} \\
&= r_P(\nu, \gamma, a),
\end{aligned}
$$

where $a \in A_{M,\mathrm{reg}}(F_v)$, $P \in \mathcal{P}(M)$, and the products are taken over the roots of (P_σ, A_{M_σ}). From the definition of (G, M) families, we have

$$
\begin{aligned}
r_M^L(\gamma, a) &= \lim_{\nu \to 0} \sum_{P \in \mathcal{P}^L(M)} r_P^L(\nu, \gamma, a)(\mathrm{vol}\left(\mathfrak{a}_M^L/\mathbf{Z}((\Delta_M^L)^\vee)\right) \prod_{\alpha \in \Delta_M^L} \nu(\alpha^\vee)^{-1}) \\
&= \lim_{\nu \to 0} \sum_{P \in \mathcal{P}^L(M)} r_{\tilde{P}}^{\tilde{L}}(\nu^*, \gamma^*, a^*)(\mathrm{vol}\left(\mathfrak{a}_M^L/\mathbf{Z}((\Delta_M^L)^\vee)\right) \prod_{\alpha \in \Delta_M^L} \nu(\alpha^\vee)^{-1}) \\
&= \lim_{\nu^* \to 0} \sum_{P \in \mathcal{P}^L(M)} r_{\tilde{P}}^{\tilde{L}}(\nu^*, \gamma^*, a^*)(n^{-\dim(A_M/A_L)}(\mathrm{vol}\left(\mathfrak{a}_{\tilde{M}}^{\tilde{L}}/\mathbf{Z}((\Delta_M^L)^{\vee *})\right) \prod_{\alpha \in \Delta_M^L} \nu^*(\alpha^{\vee *})^{-1}) \\
&= n^{-\dim(A_M/A_L)} r_{\tilde{M}}^{\tilde{L}}(\gamma^*, a^*)
\end{aligned}
$$

In consequence,

$$
\begin{aligned}
&n^{\dim(A_M/A_G)} I_M(\gamma, f_v^0) \\
&\quad = \lim_{a \to 1} n^{\dim(A_M/A_G)} \sum_{L \in \mathcal{L}(M)} r_M^L(\gamma, a) I_L(a\gamma, f_v^0) \\
&\quad = \lim_{a \to 1} \sum_{L \in \mathcal{L}(M)} n^{\dim(A_M/A_G) - \dim(A_M/A_L)} r_{\tilde{M}}^{\tilde{L}}(\gamma^*, a^*) I_L(a\gamma, f_v^0) \\
&\quad = \lim_{a \to 1} \sum_{L \in \mathcal{L}(M)} r_{\tilde{M}}^{\tilde{L}}(\gamma^*, a^*)(n^{\dim(A_L/A_G)} I_L(a\gamma, f_v^0)) \\
&\quad = \lim_{a \to 1} \sum_{L \in \mathcal{L}(M)} r_{\tilde{M}}^{\tilde{L}}(\gamma^*, a^*) I_L^{\Sigma}(a\gamma, f_v^0) \\
&\quad = \lim_{a \to 1} \sum_{L \in \mathcal{L}(M)} r_{\tilde{M}}^{\tilde{L}}(\gamma^*, a^*) I_{\tilde{L}}((a\gamma)^*, \tilde{f}_v^0) \\
&\quad = I_{\tilde{M}}(\gamma^*, \tilde{f}_v^0).
\end{aligned}
$$

The lemma now follows from an application of Lemma 6.1.□

We close this section with two partial results that are true for arbitrary positive integers n and $r \geq 2$. We stress that n need not be relatively prime to any number in Lemma 7.2 and Lemma 7.3.

Lemma 7.2 *Suppose $\gamma \in M(F_v)$ such that γ^n is diagonal and G-regular. Then*

$$J_{\tilde{M}}(\gamma^*, \tilde{f}_v^0) = n^{\dim(A_M/A_G)} J_M(\gamma, f_v^0).$$

Proof. The first portion of the proof of Proposition 7.1 does not rely on any assumptions between n and r, and so we may write

$$\begin{aligned} & J_{\tilde{M}}(\gamma^*, \tilde{f}_v^0) \\ = \; & n^{\dim(A_M/A_G)} |D^{M_1}(\gamma^n)|_v^{1/2} \int_{M_{1,\gamma}(F_v)\backslash M_1(F_v)} \tilde{f}_v^0(\mathrm{s}(m^{-1})\gamma^n \mathrm{s}(m)) \int_{N_2 \cap K_v} v_M(\varphi_m^{-1}(n)) dn dm \end{aligned}$$

and

$$J_M(\gamma, f_v^0) = |D^{M_1}(\gamma)|_v^{1/2} \int_{M_{1,\gamma}(F_v)\backslash M_1(F_v)} f_v^0(m^{-1}\gamma m) \int_{N_2 \cap K_v} v_M(\varphi_m^{-1}(n)) dn dm$$

as before. Since γ is diagonal, M_1 is the diagonal Levi subgroup M_0 and so φ_m is, despite appearances, independent of $m \in M_0$. Moreover $M_{0,\gamma}(F_v) = M_0(F_v)$. Hence

$$J_{\tilde{M}}(\gamma^*, \tilde{f}_v^0) = n^{\dim(A_M/A_G)} |D^{M_0}(\gamma^n)|_v^{1/2} \int_{N_2 \cap K_v} v_M(\varphi_m^{-1}(n)) dn,$$

and

$$J_M(\gamma, f_v^0) = |D^{M_0}(\gamma)|_v^{1/2} \int_{N_2 \cap K_v} v_M(\varphi_m^{-1}(n)) dn.$$

The desired equality follows from

$$|D^{M_0}(\gamma^n)|_v = |D^{M_0}(\gamma)|_v = 1. \square$$

Lemma 7.3 (Flicker, Kazhdan, Waldspurger) *Suppose $\gamma \in G(F_v)$ such that γ^n is G-regular. Then*

$$J_{\tilde{G}}(\gamma^*, \tilde{f}_v^0) = J_G(\gamma, f_v^0).$$

Proof. This is proven in Proposition 12 of [15] modulo an assertion which is proven in [32]. See the first appendix of [14] for details.$\square$

8 A Vanishing Property

Given that the orbit map, (1), is our means of relating conjugacy classes of G to conjugacy classes $\tilde{G}$, we should hope that the distributions $I_{\tilde{M}}(\tilde{\gamma}, \tilde{f})$ of (14) vanish unless $\tilde{\gamma}$ lies in the image of (1). This is referred to as a vanishing property. Note that by §3 [15], we have that $I_{\tilde{G}}(\tilde{\gamma}, \tilde{f})$ vanishes unless $\mathbf{p}(\tilde{\gamma}) = \gamma^n$ for some $\gamma \in G(F_S)$. Thus we already have a local vanishing property for the case $M = G$. The formulation and proof of the vanishing property for arbitrary M are somewhat roundabout and require a several additional definitions and lemmas. This section follows §10 [6] closely.

Let v be a nonArchimedean valuation and let $M(F_v)_{\text{ell}}$ be the set of F_v-elliptic elements in $M(F_v)$. By rational canonical form, any $\delta \in G(F_v)$ is in some induced conjugacy class τ^G, where $\tau \in L(F_v)_{\text{ell}}$ and $L \in \mathcal{L}$. The pair (L, τ) is uniquely determined by δ up to $G(F_v)$-conjugacy. Let $\tilde{\delta} \in \tilde{G}(F_v)$ such that $\mathbf{p}(\tilde{\delta}) = \delta$ as above. We define $\tilde{G}(F_v)_G \subset \tilde{G}(F_v)$ by specifying that $\tilde{\delta} \in \tilde{G}(F_v)_G$ if and only if $\xi(\tau) \in F_v^{\times n}$ for all $\xi \in X(L(F_v))_F$. We also define $\tilde{G}(F_v)^G$ by specifying that $\tilde{\delta} \in \tilde{G}(F_v)^G$ if and only if $\xi(\tau) \in F_v^{\times n}$ for all $\xi \in X(G(F_v))_F$. It follows from $X(G)_F \subset X(L)_F$ that $\tilde{G}(F_v)^G \subset \tilde{G}(F_v)_G$. Clearly, we may define $\tilde{M}(F_v)_M$ and $\tilde{M}(F_v)^M$ as above by replacing G with M.

Lemma 8.1 *Let E/F_v be an extension of degree t such that $t \leq n$, and $\gcd(n,t) = 1$. If $x \in E^\times$ such that $N_{E/F_v}(x) \in F_v^{\times n}$ then $x \in E^{\times n}$.*

Proof. We first show that $E^\times/E^{\times n} \cong F_v^\times/F_v^{\times n}$ and that we may take coset representatives of $E^\times/E^{\times n}$ to be in $F_v^\times$. The homomorphism

$$F_v^\times/F_v^{\times n} \to E^\times/E^{\times n}$$

given by $zF_v^{\times n} \mapsto zE^{\times n}$, for $z \in F_v^\times$, is injective. Indeed, suppose z does not belong to $F^{\times n}$, but does belong to $E^{\times n}$. Then $[F(z^{1/n}) : F_v]$ divides n by Theorem 10 (b) VIII §6 [23]. Moreover $z^{1/n} \in E$, so

$$t = [E : F_v] = [E : F_v(z^{1/n})][F_v(z^{1/n}) : F_v],$$

which contradicts $\gcd(n,t) = 1$. The surjectivity of this map follows at once from the fact that (Corollary II §3 [24])

$$|E^\times/E^{\times n}| = |F_v^\times/F_v^{\times n}| = n^2/|n|_v. \tag{21}$$

The enunciation of the lemma amounts to showing the injectivity of the homomorphism

$$N_{E/F_v} : E^\times/E^{\times n} \to F_v^\times/F_v^{\times n}$$

given by

$$xE^{\times n} \mapsto (N_{E/F_v}(x))F_v^{\times n} = x^t F_v^{\times n},$$

where $x \in F_v^\times$. If $x^t F_v^{\times n} = F_v^{\times n}$ and $x \notin F_v^{\times n}$, then t must divide $n^2/|n|_v$ by (21), thereby contradicting $\gcd(t,n) = 1$. Thus this map is injective. ◫

Proposition 8.1 *Let $\tilde{\gamma} \in \tilde{M}(F_v)$. Then $\tilde{\gamma} \in \tilde{M}(F_v)_M$ if and only if $\mathbf{p}(\tilde{\gamma}) = \delta^n$ for some $\delta \in M(F_v)$.*

Proof. The proposition may plainly be deduced from the case that $M = G$, if decomposition (2) is kept in mind. Let $\tilde{\gamma} \in \tilde{G}(F_v)$ such that $\mathbf{p}(\tilde{\gamma}) = \delta^n$ for some $\delta \in G(F_v)$. Suppose $\delta \in \tau^G$, where $\tau \in L(F_v)_{\text{ell}}$ and $L \in \mathcal{L}$. Then $\delta^n \in (\tau^n)^G$ and $\tau^n \in L(F_v)_{\text{ell}}$. Clearly, $\xi(\tau^n) = (\xi(\tau))^n \in F_v^{\times n}$ for all $\xi \in X(L)_F$. Consequently $\tilde{\gamma} \in \tilde{G}(F_v)_G$.

Conversely, suppose $\delta \in \mathbf{p}(\tilde{G}(F_v)_G)$ and $\delta \in \tau^G$ for τ and L as above. We must show that $\delta = \sigma^n$ for some $\sigma \in G(F_v)$. Suppose that L has a decomposition $L = \prod_{i=1}^b L(i)$, where $L(i)(F_v) \cong \mathrm{GL}(k_i, F_v)$. Let $\{\xi_i\}_{i=1}^b$ be a base for $X(L)_F$. We identify $L(i)$ with $\mathrm{GL}(k_i)$ by the above isomorphism, take $\xi_i = \det_{|L_i}$ and set $\tau = (\tau_i)_{i=1}^b$, where $\tau_i \in \mathrm{GL}(k_i, F_v)$. Since τ is F_v-elliptic in $L(F_v)$, there exist elliptic tori $T_i \subset L(i)$ and extension fields E_i/F_v, such that $[E_i : F_v] = k_i$ and $T_i \cong E_i^\times$. If we consider $\tau_i \in E_i$ to be the image of τ under these isomorphisms then

$$N_{E_i/F_v}(\tau_i) = \det(\tau_i) = \xi_i(\tau_i) = \xi_i(\tau) \in F_v^{\times n}.$$

Thus, by Lemma 8.1, $\tau_i \in E_i^{\times n}$, i.e. $\tau_i = \beta_i^n$, where $\beta_i \in E_i^\times \cong T_i \subset L_i(F_v)_{\text{ell}}$. This implies that $\tau = \beta^n$, where $\beta = (\beta_i)_{i=1}^b \in L(F_v)_{\text{ell}}$. As a result $\delta \in (\beta^n)^G$, and so there exists $\sigma \in \beta^G \subset G(F_v)$ such that $\delta = \sigma^n$.□

This next proposition establishes a local vanishing property (*cf.* §10 [7]).

Proposition 8.2 *Suppose $\tilde{\delta} \in \tilde{M}(F_v)^M$. Then $I_{\tilde{M}}(\tilde{\delta}, \tilde{f}) = 0$ for all $\tilde{f} \in \mathcal{H}^{\mathrm{met}}(G(F_v))$ unless $\tilde{\delta} \in \tilde{M}(F_v)_M$.*

Proof. Assume $I_{\tilde{M}}(\tilde{\delta}, \tilde{f}) \neq 0$ for some $\tilde{\delta} \in \tilde{M}(F_v)^M$ and $\tilde{f} \in \mathcal{H}(\tilde{G}(F_S))$. Let $L_1 \in \mathcal{L}^M$ and $\tau \in L_1(F_v)_{\mathrm{ell}}$ such that $\mathrm{p}(\tilde{\delta}) \in \tau^M$. Fix $\xi_1 \in X(L_1)_F$. We must show that $\xi_1(\tau) \in F_v^{\times n}$. By the descent descent property (§6)

$$I_{\tilde{M}}(\mathrm{p}(\tilde{\delta}), \tilde{f}) = \sum_{L_2 \in \mathcal{L}(L_1)} d^G_{L_1}(M, L_2) \hat{I}^{\tilde{L}_2}_{\tilde{L}_1}(\tau, \tilde{f}_{\tilde{L}_2}) \neq 0.$$

Hence there is some $L_2 \in \mathcal{L}(L_1)$ such that $d^G_{L_1}(M, L_2) \hat{I}^{\tilde{L}_2}_{\tilde{L}_1}(\tau, \tilde{f}_{\tilde{L}_2}) \neq 0$. Since $d^G_{L_1}(M, L_2) \neq 0$, we may decompose ξ_1 as $\xi + \xi_2$, where $\xi \in X(M)_F$ and $\xi_2 \in X(L_2)_F$. The distribution $\hat{I}^{\tilde{L}_2}_{\tilde{L}_1}(\tau)$ is in the closed linear span of $\{\hat{I}^{\tilde{L}_2}_{\tilde{L}_2}(\eta)\}$, where η ranges over the G-regular points in $L_2(F_v)$ such that $\xi_2(\tau) = \xi_2(\eta)$ (*cf.* Proposition 10.2 [6]). Therefore there exists such an η with

$$\hat{I}^{\tilde{L}_2}_{\tilde{L}_2}(\eta, \tilde{f}_{\tilde{L}_2}) = I_{\tilde{G}}(\eta, \tilde{f}) \neq 0.$$

From the remark at the beginning of this section, this implies that $\xi_2(\tau) = \xi_2(\eta) \in F_v^{\times n}$. By assumption $\xi(\tau) \in F_v^{\times n}$. Consequently $\xi_1(\tau) = \xi(\tau)\xi_2(\tau) \in F_v^{\times n}$, and the proposition follows.□

We transfer this local vanishing property to a global vanishing property by using splitting and some local-global results on n-th roots in F and $G(F)$.

Lemma 8.2 *Let $x \in F^\times$ such that $x \in F_v^{\times n}$ for almost all valuations v. Then $x \in F^{\times n}$.*

Proof. We will prove this lemma by contradiction. To this end, let σ be a non-trivial element in the Galois group of the abelian extension $F(x^{1/n})$. Observe that $F(x^{1/n})$ is well-defined since $\mu_n \subset F$. It is immediate from the hypothesis of the lemma that the density of the valuations for which σ is the Frobenius automorphism is zero. However the Tchebotarev density theorem (VIII, §4 Theorem 10 [24]) tells us that this density is $1/[F(x^{1/n}) : F]$.□

Lemma 8.3 *Let E be any field containing μ_n. Let $\gamma \in \mathrm{GL}(r, E)$ be such that γ is E-elliptic in some Levi subgroup L of $\mathrm{GL}(r, E)$. If γ^n is in the centre of $\mathrm{GL}(r, E)$ then $\gamma \in A_L(E)$.*

Proof. Suppose first that γ is E-elliptic in $\mathrm{GL}(r,E)$. Then we may view γ as an element of $E_1^\times$, where E_1 is a field extension of E such that $[E_1 : E]$ divides r. The minimal polynomial of γ over E divides the polynomial $g(X) = X^n - \gamma^n$. From this it is clear that the norm of γ is $\gamma^{[E_1:E]}\zeta \in E$, for some $\zeta \in \mu_n$. Since $\zeta \in \mu_n \subset E$ we have $\gamma^{[E_1:E]} \in E$. Writing $\gamma^n = \gamma^{k[E_1:E]}\gamma^b$ for some integers k and $0 \leq b < [E_1 : E]$, we find $\gamma^b = \gamma^n\gamma^{-k[E_1:E]} \in E$. This implies that $b = 0$ and $n = k[E_1 : E]$. Since we are assuming $\gcd(n,r) = 1$, we must have $[E_1 : E] = 1$. In particular, γ belongs to $E^\times$. In the context of the group $\mathrm{GL}(r,E)$, this means that γ lies in the centre.

The proof of the lemma for arbitrary γ may be obtained by combining the above argument with the fact that γ is E-elliptic in some Levi subgroup $L = \prod_{i=1}^{j} \mathrm{GL}(k_i)$ of $\mathrm{GL}(r)$.□

Lemma 8.4 *Let δ be an element of $M(F)$ such that $\delta = \gamma_v^n$ for some $\gamma_v \in M(F_v)$ and almost all valuations v of F. Then $\delta = \gamma^n$ for some $\gamma \in M(F)$.*

Proof. This lemma follows easily from the case $M = G$. Suppose first that $\delta \in G(F)$ is semisimple. We may then, by rational canonical form, take δ to be a diagonal block matrix of the form

$$\begin{pmatrix} \delta_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \delta_1 & & & & \\ & & & \ddots & & & \\ & & & & \delta_k & & \\ & & & & & \ddots & \\ 0 & & & & & & \delta_k \end{pmatrix},$$

where $\delta_i \in \mathrm{GL}(m_i, F)$ generates a field extension F_i/F of degree m_i, and appears b_i times. The centralizer $G_\delta(F)$ of δ in $G(F)$ is isomorphic to $\prod_{i=1}^{k} \mathrm{GL}(b_i, F_i)$. It is not difficult to see that this lemma can be solved for the semisimple case if it is solved for the case $k = 1$. Let us then restrict our proof to this case. If $F_1 = F$ we may identify δ with $\delta_1 \in F^\times$. By Lemma 8.3, γ_v belongs to $A_G(F_v)$, so we may identify it with a scalar in $F_v^\times$. Thus we are in the same circumstance as Lemma 8.2 with $x = \delta_1$. Next let F_1/F be an arbitrary finite field extension. Let $w_1, \ldots, w_d$ be the valuations of F_1 which divide v. Then δ_1 is conjugate to

$$\begin{pmatrix} \delta_{11} & & 0 \\ & \ddots & \\ 0 & & \delta_{1d} \end{pmatrix}$$

in $\mathrm{GL}(m_1, F_v)$, where δ_{1i} generates F_{1,w_i} over F_v. This implies that

$$G_\delta(F_v) \cong \prod_{i=1}^{d} \mathrm{GL}(r_1, F_{1,w_i})$$

Clearly, $\gamma_v \in G_\delta(F_v)$. Via this last isomorphism, we may decompose $\gamma_v = (\gamma_{v,i})_{i=1}^d$, where $\gamma_{v,i} \in \mathrm{GL}(r_1, F_{1,w_i})$, $1 \leq i \leq d$. Observe also that the map $\delta_1 \mapsto \delta_{1i}$ corresponds to an embedding $F_1 \hookrightarrow F_{1,w_i}$, which in turn yields an embedding $\mathrm{GL}(r_1, F_1) \hookrightarrow \mathrm{GL}(r_1, F_{1,w_i})$, $1 \leq i \leq d$. With respect to these embeddings, we have $\gamma_{v,i}^n = \delta$, $1 \leq i \leq d$. This places us once more in the same circumstance as $F_1 = F$, which has been taken care of.

For general $\delta \in G(F)$ let $\delta = \sigma u$ be the Jordan decomposition where σ is semisimple and u is unipotent. Similarly let $\sigma_v u_v$ be the Jordan decomposition of $\gamma_v \in G(F_v)$. It follows clearly from $\gamma_v^n = \delta$ that $\sigma_v^n = \sigma$ and $u_v^n = u$. We first assume that σ is a scalar matrix in $G(F)$. Then, once again by Lemma 8.2, there exists a scalar matrix $\sigma_1 \in A_G(F)$ such that $\sigma_1^n = \sigma$. Let $u_1 = \exp(\frac{1}{n}\log(u))$. Then $u_1^n = u$ and $\sigma_1 u_1 = u_1 \sigma_1$ together imply that $(\sigma_1 u_1)^n = \sigma u = \delta$. For arbitrary semisimple $\sigma \in G(F)$ we may follow the decomposition of $G_\sigma(F)$ as before since $u \in G_\sigma(F)$. This decomposition allows us once more to restrict our proof to the case that σ is a scalar matrix and we may argue as above to complete the proof.□

Proposition 8.3 *Let S be a large set of valuations containing the Archimedean valuations and $\delta \in \mathrm{s}_0(M(F)) \cap \tilde{M}(F_S)$. Then $I_{\tilde{M}}(\delta, \tilde{f}) = 0$ for all $\tilde{f} \in \mathcal{H}(\tilde{G}(F_S))$ unless $\mathrm{p}(\delta) = \gamma^n$ for some $\gamma \in M(F)$.*

Proof. We may assume by §3 [15] that the proposition holds for $L \in \mathcal{L}(M)$ such that $L \neq G$. Suppose $I_{\tilde{M}}(\delta, \tilde{f}) \neq 0$. We first show that $\delta \in \tilde{M}(F_S)^M = \prod_{v \in S} \tilde{M}(F_v)^M$. Suppose the contrary, i.e. suppose that $\tilde{\xi}(\delta)$ does not belong to $F_{v_1}^{\times n}$ for some $\tilde{\xi} \in X(\tilde{M}(F_S))_F$ and $v_1 \in S$. Then $\tilde{\xi}(\delta)$ does not belong to $F^{\times n}$. By Lemma 8.2 there is another place v_2, which we may assume to be in S, such that $\tilde{\xi}(\delta)$ does not belong to $F_{v_2}^{\times n}$. The sets $S_1 =$

$S-\{v_2\}$ and $S_2=\{v_2\}$ both have the closure property, as they contain Archimedean valuations. Decompose $\tilde{f}\in\mathcal{H}(\tilde{G}(F_S))$ into $\tilde{f}_1\tilde{f}_2$ such that $\tilde{f}_1\in\mathcal{H}(\tilde{G}(F_{S_1}))$, $\tilde{f}_2\in\mathcal{H}(\tilde{G}(F_{S_2}))$. Applying the splitting property (6), we obtain

$$I_{\tilde{M}}(\delta,\tilde{f})=\sum_{L_1,L_2\in\mathcal{L}(M)} d_M^G(L_1,L_2)\hat{I}_{\tilde{M}}^{\tilde{L}_1}(\delta,\tilde{f}_{1,\tilde{L}_1})\hat{I}_{\tilde{M}}^{\tilde{L}_2}(\delta,\tilde{f}_{2,\tilde{L}_2})\neq 0.$$

Thus there is a pair $L_1,L_2\in\mathcal{L}(L)$ such that $d_{\tilde{L}}^{\tilde{G}}(\tilde{L}_1,\tilde{L}_2)\neq 0$ and $\hat{I}_{\tilde{L}}^{\tilde{L}_i}(\delta,\tilde{f}_{i,\tilde{L}_i})\neq 0,\ i=1,2$. As in Proposition 8.2

$$\tilde{\xi}(\delta)=\tilde{\xi}_1(\delta)\tilde{\xi}_2(\delta),\ \tilde{\xi}_i\in X(\tilde{L}_i(F_S))_F,\ i=1,2.$$

Suppose that $\tilde{\xi}_1(\delta)\in F^{\times n}$. Then it belongs to $F_v^{\times n}$ for all valuations v of F so we must have $\tilde{\xi}_2(\delta)\notin F_{v_2}^{\times n}$. It follows by induction that $\hat{I}_{\tilde{M}}^{\tilde{L}_2}(\delta,\tilde{f}_{2,\tilde{L}_2})=0$ and this is a contradiction. On the other hand if $\tilde{\xi}_1(\delta)\notin F^{\times n}$, then we may assume as we did before for S that $\tilde{\xi}_1(\delta)\notin F_{v_3}^{\times n}$ for some valuation $v_3\in S_1$. This implies $\hat{I}_{\tilde{M}}^{\tilde{L}_1}(\delta,\tilde{f}_{1,\tilde{L}_1})=0$. This is also a contradiction. Consequently $\delta\in\tilde{M}(F_S)^M$.

Now we may apply Proposition 8.2 to conclude that

$$\delta\in\tilde{M}(F_S)_M=\prod_{v\in S}\tilde{M}(F_v)_M.$$

According to Proposition 8.1, there exist $\gamma_v\in G(F_v)$ such that $\gamma_v^n=\mathbf{p}(\delta)$ for all $v\in S$. Since S can be made arbitrarily large, we may assume that $\mathbf{p}(\delta)$ is an nth power at all of the valuations of F. Therefore we may apply Lemma 8.4 to $\mathbf{p}(\delta)$ and conclude that there exists an element γ of $G(F)$ such that $\gamma^n=\mathbf{p}(\delta)$.□

9 The Geometric Side of the Trace Formula

Before giving the details of the geometric sides of the trace formulas, we give some motivation. According to §2 [19], the map s_0 of §2 is a homomorphism of $G(F)$ into $\tilde{G}(\mathbf{A})$. In particular $G(F)$ splits over $\tilde{G}(\mathbf{A})$. Let $L^2(\mathbf{s}_0(G(F))\backslash\tilde{G}(\mathbf{A}))$ be the space of square-integrable functions on $\tilde{G}(\mathbf{A})$, which are genuine and left-invariant under $\mathbf{s}_0(G(F))$. We can now form a theory of automorphic representations on $\tilde{G}(\mathbf{A})$ by examining the (right) regular representation R on $L^2(\mathrm{s}_0(G(F))\backslash\tilde{G}(\mathbf{A}))$. The geometric side of the trace formula originates from the following calculation. Let $\varphi \in L^2(\mathrm{s}_0(G(F))\backslash\tilde{G}(\mathbf{A}))$. Then

$$\begin{aligned}
(R(\tilde{f})\varphi)(y) &= \int_{\tilde{G}(\mathbf{A})} \tilde{f}(x)(R(x)\varphi)(y)dx \\
&= \int_{\tilde{G}(\mathbf{A})} \tilde{f}(x)\varphi(yx)dx \\
&= n\int_{\mathrm{i}(\mu_n)\backslash\tilde{G}(\mathbf{A})} \tilde{f}(x)\varphi(yx)dx \\
&= n\int_{\mathrm{i}(\mu_n)\backslash\tilde{G}(\mathbf{A})} \tilde{f}(y^{-1}x)\varphi(x)dx \\
&= n\int_{\mathrm{i}(\mu_n)\mathrm{s}_0(G(F))\backslash\tilde{G}(\mathbf{A})} \sum_{\gamma\in \mathrm{s}_0(G(F))} \tilde{f}(y^{-1}\gamma x)\varphi(\gamma x)dx \\
&= n\int_{\mathrm{i}(\mu_n)\mathrm{s}_0(G(F))\backslash\tilde{G}(\mathbf{A})} \left(\sum_{\gamma\in \mathrm{s}_0(G(F))} \tilde{f}(y^{-1}\gamma x)\right)\varphi(x)dx
\end{aligned}$$

Roughly speaking, the trace of the operator $R(\tilde{f})$ is obtained by integrating the integral kernel

$$(x,y)\mapsto \sum_{\gamma\in \mathrm{s}_0(G(F))} \tilde{f}(y^{-1}\gamma x)$$

over the diagonal. The only novelty in this calculation is the coefficient n in front of the integral. This justifies its appearance in (14).

If n is prime let $S_{(n)}$ be the set of nonArchimedean valuations v of F such that $|n|_v \neq 1$. If n is not prime let $S_{(n)}$ be the set of nonArchimedean valuations such that $|n|_v \neq 1$ together with a single Archimedean valuation. In either case $S_{(n)}$ has the closure property. Suppose S contains $S_{(n)}$ and the Archimedean valuations of F, and $\tilde{f} \in \mathcal{H}(\tilde{G}(F_S))$. We may embed $\tilde{f}$ into $\mathcal{H}(\tilde{G}(\mathbf{A}))$ by taking its product with $\prod_{v\notin S}\tilde{f}_v^0$, where f_v^0 is as in §7. If S satisfies some additional properties, which are given in §3 [7], then $I(\tilde{f})$ equals

$$n\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\tilde{\gamma}\in(\mathrm{s}_0(M(F)))_{\tilde{M},S}} a^{\tilde{M}}(S,\tilde{\gamma})I_{\tilde{M}}(\tilde{\gamma},\tilde{f}).$$

The set $(\mathrm{s}_0(M(F)))_{\tilde{M},S}$ denotes the set of $(\tilde{M},S)$-equivalence classes in $\mathrm{s}_0(M(F))$ (§8 [4]), which in the present case are just the conjugacy classes of $\mathrm{s}_0(M(F))$. The coefficient

$a^{\tilde{M}}(S,\tilde{\gamma})$ requires more explanation. Let σu be the Jordan decomposition of $\mathbf{p}(\tilde{\gamma}) \in M(F)$. Set $i^M(S,\sigma) = 1$ if σ is F-elliptic in $M(F)$, and the $M(F_v)$-orbit of σ meets $K_v \cap M(F_v)$ for every valuation $v \notin S$. Otherwise set $i^M(S,\sigma) = 0$. It follows from the nature of the conjugacy classes of $M(F)$ and 3.2 [7] that

$$a^{\tilde{M}}(S,\gamma) = i^M(S,\sigma) a^{\tilde{M}_{s_0(\sigma)}}(S,u). \tag{22}$$

For a description of $a^{\tilde{M}_{s_0(\sigma)}}(S,u)$ see §7 [4].

Consider the summand of (14) indexed by $M = G$, namely

$$\sum_{\tilde{\gamma} \in (s_0(G(F)))_{\tilde{G},S}} a^{\tilde{G}}(S,\tilde{\gamma}) I_{\tilde{G}}(\tilde{\gamma},\tilde{f}).$$

If we are to have any hope in comparing this term with its counterpart,

$$\sum_{\gamma \in (G(F))_{G,S}} a^{G}(S,\gamma) I_{\tilde{G}}(\gamma,\bar{f}),$$

by using the orbit map (1), then we must eliminate those $\tilde{\gamma}$ from the former sum such that $\tilde{\gamma} \neq \gamma^*$ for some $\gamma \in G(F)$. In doing this we could index the relevant conjugacy classes of $\tilde{G}(F)$ with conjugacy classes of $G(F)$. Unfortunately the orbit map is not injective. The aim of the following lemma is to measure the extent to which it is not injective on the elliptic set.

Lemma 9.1 *If γ_1 and γ_2 are F-elliptic in $M(F)$, and $\gamma_1^n = \gamma_2^n$, then $\gamma_1\gamma_2^{-1} \in \mu_n^M$.*

Proof. We restrict the proof to the case $M = G$ with assurances that the general case follows easily from this one. Suppose γ_1 and γ_2 are F-elliptic in $G(F)$. Then there exist elliptic tori T_1 and T_2, containing γ_1 and γ_2 respectively. There are isomorphisms $T_i \cong E_i^\times$, where E_i is a field extension of F for $i = 1,2$. We may therefore view γ_i as field elements of E_i, $i = 1,2$. Let $E = E_1 \cap E_2$ and let $f_i(X) \in E[X]$ be the minimal polynomial of γ_i for $i = 1,2$. Clearly $f_i(X)$ divides $X^n - \gamma_i^n$. Furthermore $[E_i : E]$ divides $[E_i : F] = r$. We may use the argument of Lemma 8.3, replacing E_1 with E_i and E with F to conclude that $E_1 = E_2 = E$. In E the equality $\gamma_1^n = \gamma_2^n$ implies $(\gamma_1\gamma_2^{-1})^n = 1$, and this clearly implies that $\gamma_1\gamma_2^{-1} \in \mu_n \subset F$. Translated back to the context of the group $G(F)$, this means $\gamma_1\gamma_2^{-1} \in \mu_n^G$. ☐

Suppose γ_1 and γ_2 belong to the same conjugacy class of $M(F)$. Then since μ_n^M is in the centre of $M(F)$, $\eta\gamma_1$ and $\eta\gamma_2$ also belong to the same conjugacy class of $M(F)$ for all $\eta \in \mu_n^M$. This fact allows us to define the quotient set $(M(F))_{M,S}/\mu_n^M$ in an obvious way.

Proposition 9.1 (5.1) *The expansion for $I(\tilde{f})$ may be expressed as*

$$n \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}/\mu_n^M} a^{\tilde{M}}(S, \gamma^*) I_{\tilde{M}}(\gamma^*, \tilde{f}).$$

Proof. According to Proposition 8.3, the distribution $I_{\tilde{M}}(\delta)$, with $\delta \in \mathrm{s}_0(M(F))$, vanishes unless $\mathbf{p}(\delta) = \gamma^n$ for some $\gamma \in M(F)$. From the previous lemma we see that the map $\gamma\mu_n^M \mapsto \gamma^n$ is injective on the F-elliptic set of $M(F)$. Thus the map of conjugacy classes,

$$(M(F))_{M,S}/\mu_n^M \xrightarrow{*} (\mathrm{s}_0(M(F)))_{\tilde{M},S},$$

given by

$$\gamma \mapsto (\gamma, 1)^n = (\gamma, 1)^n/(1, \kappa(\gamma))^n = \mathrm{s}_0(\gamma)^n = \mathrm{s}_0(\gamma^n),$$

is injective. The proposition now follows from the fact that $a^{\tilde{M}}(S, \delta)$ vanishes if $\delta \in \mathrm{s}_0(M(F))$ is not F-elliptic in $M(F)$.□

The trace formula for G, which we expect to match $I(f^*)$, is

$$\begin{aligned} I(f) &= \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}} a^M(S, \gamma) I_M(\gamma, f) \\ &= \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}/\mu_n^M} \sum_{\eta \in \mu_n^M} a^M(S, \eta\gamma) I_M(\eta\gamma, f). \end{aligned}$$

As earlier, we have the decomposition

$$a^M(S, \gamma) = i^M(S, \sigma) a^{M_\sigma}(S, u), \tag{23}$$

for $\gamma = \sigma u \in M(F)$. Since every element of μ_n^M is F-elliptic in M and also lies in $K_v \cap M(F_v)$ for all valuations v, it is not difficult to verify that $i^M(S, \eta\sigma) = i^M(S, \sigma)$ for all $\eta \in \mu_n^M$ and semisimple $\sigma \in M(F)$. This implies that $a^M(S, \eta\gamma) = a^M(S, \gamma)$ in the previous sum. Explicitly,

$$\begin{aligned} I(f) &= \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}/\mu_n^M} a^M(S, \gamma) \sum_{\eta \in \mu_n^M} I_M(\eta\gamma, f) \\ &= n \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}/\mu_n^M} a^M(S, \gamma) I_M^{\Sigma}(\gamma, f). \end{aligned}$$

The present forms of the two trace formulas and Theorem A of [10] suggest the following definition and theorem. We define

$$I_M^{\mathcal{M}}(\gamma, f) = I_{\check{M}}(\gamma^*, f^*),\ \gamma \in M(F_S).$$

Theorem 9.1 (A) *(i) Suppose that S is a finite set of valuations containing $S_{(n)}$. Then*

$$I_M^{\Sigma}(\gamma, f) = I_M^{\mathcal{M}}(\gamma, f),$$

for all $\gamma \in M(F_S)$.
(ii) Suppose $\gamma \in M(F)$. Then $a^{\check{M}}(S, \mathrm{s}_0(\gamma^n)) = a^M(S, \gamma)$ for any suitably large finite set S.

This theorem will be proved in §18.

Continuing in the same vein as [10], we make an induction hypothesis. Namely, that the theorem holds if G is replaced by G_1, where G_1 is a product of general linear groups over field extensions of F and $\dim_F G_1(F) < \dim_F G(F)$. The relevance of this induction hypothesis lies in the following observations. Given a semisimple element σ in $M(F)$, the centralizer $M_\sigma(F) \cong \prod_{i=1}^k \mathrm{GL}(b_i, F_i)$, where F_i/F is a finite field extension for $1 \le i \le k$, and $\sum_{i=1}^k b_i[F_i : F] = r$. Thus $M_\sigma(F)$ falls into the class of groups described in the induction hypothesis as long as $\dim_F M_\sigma < \dim_F G$. This last condition is satisfied as long as $M \ne G$ or σ is not in the centre of G.

By combining this induction hypothesis with the descent property ((16), Corollary 6.1) and splitting property ((17), Proposition 6.2), we obtain the following two lemmas. The reader is referred to pages 109-110 of [10] for the proofs.

Lemma 9.2 *Suppose M_1 and M are in $\mathcal{L}$ such that $M_1 \subsetneq M$. If $\gamma \in M_1(F_S)$ such that γ^n is G-regular, then $I_M^{\mathcal{M}}(\gamma, f) = I_M^{\Sigma}(\gamma, f)$.*

Lemma 9.3 *Suppose Theorem 9.1 (i) holds. Furthermore suppose S is a disjoint union of S_0 and S_1, where S_0 has the closure property and contains $S_{(n)}$, and S_1 consists of a single nonArchimedean valuation. If $f = f_0 f_1$ and $\gamma = \gamma_0\gamma_1 \in M(F_S)$ are decompositions corresponding to that of S, then $I_M^{\mathcal{M}}(\gamma_1, f_1) = I_M^{\Sigma}(\gamma_1, f_1)$.*

If we apply our induction hypothesis to (14) and the expansions (22) and (23), we obtain the following lemma.

Lemma 9.4 (5.2) *The distribution, $I^{\mathcal{M}}(f) - I(f)$ is the sum of*

$$n \sum_{M \in \mathcal{L}, M \neq G} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}/\mu_n^M} a^M(S,\gamma)(I_M^{\mathcal{M}}(\gamma, f) - I_M^{\mathcal{E}}(\gamma, f))$$

and

$$n \sum_{\delta \in A_G(F)} \sum_{u \in (\mathcal{U}_G(F))_{G,S}} (a^{\tilde{G}}(S,u) - a^G(S,u)) I_G^{\mathcal{E}}(\delta u, f).$$

One useful restriction that we may make in showing the first assertion of Theorem 9.1 is given by the following lemma.

Lemma 9.5 (3.6) *Suppose*

$$I_M^{\mathcal{E}}(\gamma, f) = I_M^{\mathcal{M}}(\gamma, f)$$

for every element $\gamma \in M(F_S)$ such that γ^n is G-regular and semisimple. Then the same formula holds for any element $\gamma \in M(F_S)$.

*Proof.*See Lemma 3.6 [10].□

In light of Lemma (9.5), we define the set $G_{*\text{reg}}(F_S)$ to be the set of elements $\gamma \in G(F_S)$ such that γ^n is G-regular. Clearly, the elements of $G_{*\text{reg}}(F_S)$ are the ones whose image under the transfer map (1) are $\tilde{G}$-regular.

10 Comparison of the Local Geometric Terms

The goal of this section is to establish a rough comparison between $I^{\mathcal{M}}(\gamma, f)$ and $I_M^{\mathcal{E}}(\gamma, f)$ under the assumption that Theorem 9.1 (i) is true, and then to compare their germ expansions. The comparison of germ expansions is a technical point whose ultimate purpose is to establish a comparison between $I^{\mathcal{M}}(\gamma, f)$ and $I_M^{\mathcal{E}}(\gamma, f)$ which no longer requires Theorem 9.1 (i) (Proposition 16.1).

The first assertion of Theorem 9.1 has a restriction on S, namely that S must contain $S_{(n)}$. If one were to assume this assertion to be true, one could still ask whether something like it would remain true for arbitrary S. This is the content of the next theorem.

Theorem 10.1 (6.1) *In the special case that $S \supset S_{(n)}$, we suppose that*

$$I_L^{\mathcal{M}}(\gamma, f) = I_L^{\mathcal{E}}(\gamma, f),$$

for any $\gamma \in L(F_S)$ and $L \in \mathcal{L}(M)$. Then there are unique constants

$$\varepsilon_L(S) = \varepsilon_L^G(S), \ L \in \mathcal{L}(M),$$

such that

$$I_M^{\mathcal{M}}(\gamma, f) = \sum_{L \in \mathcal{L}(M)} \hat{I}_M^{L,\mathcal{E}}(\gamma, \varepsilon_L(S) f_L), \ \gamma \in M(F_S).$$

The constants have the descent property

$$\varepsilon_M(S) = \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L) \varepsilon_{M_1}^L(S), \ M_1 \subset M,$$

and the splitting property

$$\varepsilon_M(S) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \varepsilon_M^{L_1}(S_1) \varepsilon_M^{L_2}(S_2), \ S = S_1 \cup S_2.$$

Proof. This theorem follows from the proof of Theorem 6.1 [10], with $I_M^{\mathcal{E}}$ replaced by $I_M^{\mathcal{M}}$ and I_M replaced by $I_M^{\mathcal{E}}$.□

Now we begin the comparison of germ expansions of $I^{\mathcal{M}}(\gamma, f)$ and $I_M^{\mathcal{E}}(\gamma, f)$. This is a local comparison in another sense of the word local. That is, it is a comparison of $I_{\mathcal{M}}(\gamma, f)$ and $I_M^{\mathcal{E}}(\gamma, f)$ over neighbourhoods of $\gamma \in M$. Lemma 10.4 and Proposition 10.1 will be the stepping stones used in Proposition 16.1.

Let $\mathcal{H}^{\mathrm{met}}(G(F_S))^0$ be the subspace of $\mathcal{H}^{\mathrm{met}}(G(F_S))$ spanned by functions

$$f = \prod_{v \in S} f_v, \ f_v \in \mathcal{H}^{\mathrm{met}}(G(F_v)),$$

which satisfy the following condition. For each finite valuation $v \in S$, $\delta \in A_G(F_v)$, and $u_v \in \mathcal{U}_G(F_v)$, we have

$$I_G(\delta_v u_v, f) = 0$$

unless $u_v = 1$. Such functions exist by §3.3 [31] and Corollary 27.3 [15]. The motivation for the definition of this subspace comes from the second sum of Lemma 9.4.

We will show that for any $f \in \mathcal{H}^{\mathrm{met}}(G(F_v))^0$, there exist germ expansions for $I_M^{\Sigma}(\gamma, f)$ and $I_M^{\mathcal{M}}(\gamma, f)$, and that these germ expansions are in some sense equivalent. First we make this notion of equivalence precise.

Let $\tilde{\sigma}$ be a semisimple element of $\tilde{M}(F_S)$, and let ϕ_1 and ϕ_2 be functions defined on an open subset Σ of $\tilde{\sigma}\tilde{M}_{\tilde{\sigma}}(F_S)$, whose closure contains an $\tilde{M}_{\tilde{\sigma}}$-invariant neighbourhood of $\tilde{\sigma}$. We say ϕ_1 is $(\tilde{M}, \tilde{\sigma})$-equivalent to ϕ_2 and write

$$\phi_1(\tilde{\gamma}) \overset{(\tilde{M},\tilde{\sigma})}{\sim} \phi_2(\tilde{\gamma}), \text{ for } \tilde{\gamma} \in \Sigma,$$

if there exists a compactly supported smooth function $\tilde{h}$ on $\tilde{M}(F_S)$ and a neighbourhood U of $\tilde{\sigma}$ in $\tilde{M}(F_S)$ such that

$$\phi_1(\tilde{\gamma}) - \phi_2(\tilde{\gamma}) = I_{\tilde{M}}^{\tilde{M}}(\tilde{\gamma}, \tilde{h}), \text{ for } \tilde{\gamma} \in \Sigma \cap U.$$

If $\tilde{M} = M$ then we make the additional stipulation that $\tilde{h} \in \mathcal{H}^{\mathrm{met}}(M(F_S))$.

For the remainder of this section v is a nonArchimedean valuation of F, and σ is a semisimple element in $M(F_v)$ which is also F_v-elliptic in $M(F_v)$.

Lemmas 10.1-10.3 allow us to derive a germ expansion for $I_M^{\Sigma}(\gamma, f)$ from the known germ expansion of $I_M(\gamma, f)$.

Lemma 10.1 *Let $L \in \mathcal{L}(M)$ and suppose that σ^n is F_v-elliptic in L. Then there exists $\eta_L \in \mu_n^M$ such that $L_{\eta_L\sigma} = L_{\sigma^n}$. In particular $\eta_L\sigma$ is F_v-elliptic in L.*

Proof. For the sake of convenience we suppose that $L = G$. Recall decomposition (2),

$$M(F_v) \cong \prod_{i=1}^{\ell} \mathrm{GL}(r_i, F_v).$$

For the duration of this lemma we will identify $M(F_v)$ with this direct product of general linear groups. Since σ is F_v-elliptic in $M(F_v)$, it has rational canonical form

$$\begin{pmatrix} \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_1 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} \sigma_\ell & & 0 \\ & \ddots & \\ 0 & & \sigma_\ell \end{pmatrix} \end{pmatrix},$$

where $\sigma_i \in \mathrm{GL}(m_i, F_v)$ generates a field extension F_i/F_v of degree m_i, m_i divides r_i, and σ_i appears r_i/m_i times. The rational canonical form of σ^n is

$$\begin{pmatrix} \begin{pmatrix} \sigma_1' & & 0 \\ & \ddots & \\ 0 & & \sigma_1' \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} \sigma_\ell' & & 0 \\ & \ddots & \\ 0 & & \sigma_\ell' \end{pmatrix} \end{pmatrix},$$

where σ_i' is the rational canonical form of σ_i^n in $\mathrm{GL}(m_i, F_v)$. Since σ^n is F_v-elliptic in $G(F_v)$, it also has rational canonical form

$$\begin{pmatrix} \sigma' & & 0 \\ & \ddots & \\ 0 & & \sigma' \end{pmatrix},$$

where $\sigma' \in \mathrm{GL}(m, F_v)$ generates a field extension F'/F_v of degree m, m divides r, and σ' appears r/m times. Since rational canonical form is unique up to permutation of the companion matrices, we must have $\sigma_i' = \sigma'$ for all $1 \le i \le \ell$.

Let us return to the elements $\sigma_i \in \mathrm{GL}(m_i, F_v)$. We may view them as elements of the fields F_i, $1 \le i \le \ell$. Let $E_i = F_v(\sigma_i^n)$. By replacing E_1 with F_i and E with E_i in Lemma 8.3, it follows that $[F_i : E_i] = 1$. That is, σ_i^n generates the same field extension of F_v as does σ_i. This implies that

$$F_1 = F_v(\sigma_1) = F_v(\sigma_1^n) = F_v(\sigma') = F_v(\sigma_i^n) = F_v(\sigma_i) = F_i,$$

for $1 \le i \le \ell$. In particular $m_i = m$, for $1 \le i \le \ell$. By viewing σ_1, σ_i and σ' as elements of F_1, we find $\sigma_1^n = \sigma_i^n = \sigma'$. Since F_1 contains μ_n, it follows that $\eta_i\sigma_1 = \sigma_i$ for some

$\eta_i \in \mu_n$. In other words σ is equal to

$$\begin{pmatrix} \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_1 \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \begin{pmatrix} \eta_2\sigma_1 & & 0 \\ & \ddots & \\ 0 & & \eta_2\sigma_1 \end{pmatrix} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \begin{pmatrix} \eta_\ell\sigma_1 & & 0 \\ & \ddots & \\ 0 & & \eta_\ell\sigma_1 \end{pmatrix} \end{pmatrix}.$$

Let

$$\eta_L = \begin{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \begin{pmatrix} \eta_2^{-1} & & 0 \\ & \ddots & \\ 0 & & \eta_2^{-1} \end{pmatrix} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \begin{pmatrix} \eta_\ell^{-1} & & 0 \\ & \ddots & \\ 0 & & \eta_\ell^{-1} \end{pmatrix} \end{pmatrix}.$$

Then

$$\eta_L\sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_1 \end{pmatrix},$$

and

$$G_{\eta_L\sigma}(F_v) \cong \mathrm{GL}(r/m, F_1) \cong G_{\sigma^n}(F_v).$$

Since $G_{\eta_L\sigma}(F_v) \subset G_{\sigma^n}(F_v)$, we have $G_{\eta_L\sigma}(F_v) = G_{\sigma^n}(F_v)$ as well.$\Box$

Lemma 10.2 *Suppose σ is F_v-elliptic in $L_1, L_2 \in \mathcal{L}(M)$. Then there exists $L \in \mathcal{L}(M)$ such that $L \supset L_1, L_2$ and σ is F_v-elliptic in L.*

Proof. Let L be the centralizer in G of the split torus $T = A_{L_1} \cap A_{L_2}$. Then $L \in \mathcal{L}(M)$ contains L_1 and L_2. Furthermore it may be verified that

$$A_L = A_{L_1} \cap A_{L_2} = A_{L_{1,\sigma}} \cap A_{L_{2,\sigma}} = A_{L_\sigma}.$$

It follows that σ is F_v-elliptic in L.□

A simple result of this lemma is the existence of a unique maximal Levi subgroup, $L' \in \mathcal{L}(M)$, such that σ^n is F_v-elliptic in L'. For the remainder of this section we assume that $\eta_{L'} \in \mu_n^M$ of Lemma 10.1 is the identity. As a consequence, σ is F_v-elliptic in $L \in \mathcal{L}(M)$ if and only if σ^n is F_v-elliptic in L.

Lemma 10.3 *Suppose that $L \in \mathcal{L}(M)$, σ is F_v-elliptic in L, and $\eta \in \mu_n^M$. Then $\eta\sigma$ is F_v-elliptic in L if and only if $\eta \in \mu_n^L$.*

Proof. If $\eta \in \mu_n^L$ then $L_{\eta\sigma} = L_\sigma$, so $\eta\sigma$ is clearly seen to be F_v-elliptic in L.

Conversely suppose $\eta\sigma$ is F_v-elliptic in L and, for the sake of simplicity that $L = G$. Since σ is F_v-elliptic in G and $\sigma \in M(F_v)$, by rational canonical form, it may be written as

$$\delta^{-1} \begin{pmatrix} \sigma' & & 0 \\ & \ddots & \\ 0 & & \sigma' \end{pmatrix} \delta,$$

where $\sigma' \in \mathrm{GL}(m, F_v)$ generates a field extension F'/F_v of degree m, m divides r, σ' appears r/m times and $\delta \in M(F_v)$. Consequently

$$\eta\sigma = \delta^{-1} \begin{pmatrix} \eta_1\sigma' & & 0 \\ & \ddots & \\ 0 & & \eta_{r/m}\sigma' \end{pmatrix} \delta,$$

where the scalar matrices of the form η_i are the projections of η into $\mathrm{GL}(m, F_v)$. In order for $\eta\sigma$ to be F_v-elliptic in G, we must have $\eta_i\sigma' = \eta_j\sigma'$ for $1 \leq i, j \leq r/m$. This implies that $\eta \in \mu_n^G$. □

The following lemma yields a germ expansion for $I_M^{\mathcal{E}}(\gamma, f)$.

Lemma 10.4 *Let $\gamma \in M_\sigma(F_v) \cap G_{\bullet\mathrm{reg}}$ and $f \in \mathcal{H}^{\mathrm{met}}(G(F_v))$. Then there exist functions $\gamma \mapsto g_M^L(\gamma,\delta)$ such that*

$$I_M^{\mathcal{E}}(\gamma, f) \overset{(M,\sigma)}{\sim} \sum_{L\in\mathcal{L}(M)} \sum_{\delta\in\sigma(\mathcal{U}_{L_\sigma}(F_v))} g_M^L(\gamma,\delta) I_L^{\mathcal{E}}(\delta, f).$$

Proof. Let $\eta \in \mu_n^M$. Then $\eta\sigma$ is a semisimple element and by formula 2.5 [6], we have

$$I_M(\gamma, f) \overset{(M,\eta\sigma)}{\sim} \sum_{L\in\mathcal{L}(M)} \sum_{\delta\in(\eta\sigma\mathcal{U}_{L_{\eta\sigma}}(F_v))} g_M^L(\gamma,\delta) I_L(\delta, f),\ \gamma \in M_{\eta\sigma}(F_v) \cap G_{\bullet\mathrm{reg}}.$$

It follows from a remark on p 272 [8], that $(M,\eta\sigma)$-equivalence of functions of γ is the same as (M_η,σ)-equivalence of functions of γ. As $M_\eta = M$, the above expansion may be written in the form

$$I_M(\gamma, f) \overset{(M,\sigma)}{\sim} \sum_{L\in\mathcal{L}(M)} \sum_{\delta\in(\eta\sigma\mathcal{U}_{L_{\eta\sigma}}(F_v))} g_M^L(\gamma,\delta) I_L(\delta, f),\ \gamma \in M_\sigma(F_v) \cap G_{\bullet\mathrm{reg}}.$$

Consequently

$$\begin{aligned} I_M^{\mathcal{E}}(\gamma, f) &= \sum_{\eta\in\mu_n^M/\mu_n^G} I_M(\eta\gamma, f) \\ &\overset{(M,\sigma)}{\sim} \sum_{L\in\mathcal{L}(M)} \sum_{\eta\in\mu_n^M/\mu_n^G} \sum_{\delta\in\eta\sigma(\mathcal{U}_{L_{\eta\sigma}}(F_v))} g_M^L(\eta\gamma,\delta) I_L(\delta, f). \end{aligned}$$

Now if $\gamma = \sigma\gamma_1$ and $\delta = \eta\sigma u$, where $u \in \mathcal{U}_{L_{\eta\sigma}}(F_v)$, then by 9.2 of [6] we have

$$g_M^L(\eta\gamma,\delta) = \begin{cases} g_{M_\sigma}^{L_{\eta\sigma}}(\gamma_1, u), & \text{if } \eta\sigma \text{ is } F_v\text{-elliptic in } L \\ 0, & \text{otherwise} \end{cases}.$$

By Lemma 10.1, if σ^n is not F_v-elliptic in L then neither is $\eta\sigma$ for any $\eta \in \mu_n^M/\mu_n^G$, so $g_M^L(\eta\gamma,\delta) = 0$. On the other hand if σ^n is F_v-elliptic in L then, by Lemma 10.3,

$$\begin{aligned} g_M^L(\eta\gamma,\delta) &= \begin{cases} g_{M_\sigma}^{L_{\eta\sigma}}(\gamma_1, u), & \text{if } \eta \in \mu_n^L/\mu_n^G \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} g_{M_\sigma}^{L_\sigma}(\gamma_1, u), & \text{if } \eta \in \mu_n^L/\mu_n^G \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} g_M^L(\gamma, \delta), & \text{if } \eta \in \mu_n^L/\mu_n^G \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

Combining these last two observations, we find that

$$I^{\mathcal{E}}_M(\gamma, f) \overset{(M,\sigma)}{\sim} \sum_{L\in\mathcal{L}(M)} \sum_{\delta\in\sigma(\mathcal{U}_{L_\sigma}(F_v))} g^L_M(\gamma,\delta) \sum_{\eta\in\mu^L_n/\mu^G_n} I_L(\eta\delta, f)$$
$$= \sum_{L\in\mathcal{L}(M)} \sum_{\delta\in\sigma(\mathcal{U}_{L_\sigma}(F_v))} g^L_M(\gamma,\delta) I^{\mathcal{E}}_L(\delta, f).\square$$

The next lemma shows that there is a similar germ expansion for the distribution $I^{\mathcal{M}}_M(\gamma, f)$.

Lemma 10.5 *There exist functions* $\tilde{\gamma} \mapsto g^{\tilde{L}}_{\tilde{M}}(\tilde{\gamma}, \delta^*)$ *such that*

$$I^{\mathcal{M}}_M(\gamma, f) \overset{(M,\sigma)}{\sim} \sum_{L\in\mathcal{L}(M)} \sum_{\delta\in\sigma(\mathcal{U}_{L_\sigma}(F_v))} g^{\tilde{L}}_{\tilde{M}}(\gamma^*, \delta^*) I^{\mathcal{M}}_L(\delta, f)$$

for $\gamma \in M_\sigma(F_v) \cap G_{*\mathrm{reg}}$ *and* $f \in \mathcal{H}^{\mathrm{met}}(G(F_v))$.

Proof. Proposition 9.1 [8] translates into the metaplectic context as

$$I_{\tilde{M}}(\tilde{\gamma}, f^*) \overset{(\tilde{M},\sigma^*)}{\sim} \sum_{L\in\mathcal{L}(M)} \sum_{\tilde{\delta}\in\sigma^*(\mathcal{U}_{\tilde{L}_{\sigma^*}}(F_v))} g^{\tilde{L}}_{\tilde{M}}(\tilde{\gamma}, \tilde{\delta}) I_{\tilde{L}}(\tilde{\delta}, f^*)$$

for $\tilde{\gamma} \in \sigma^*\tilde{M}^0_{\sigma^*}(F_v) \cap \tilde{G}_{\mathrm{reg}}$. Lemma 9.2 [8] translates as

$$g^{\tilde{L}}_{\tilde{M}}(\tilde{\gamma}, \tilde{\delta}) = \begin{cases} g^{\tilde{L}_{\sigma^*}}_{\tilde{M}_{\sigma^*}}(\gamma_1, u), & \text{if } \sigma^* \text{ is } F_v\text{-elliptic in } \tilde{L} \\ 0, & \text{otherwise} \end{cases},$$

where $\tilde{\gamma} = \sigma^*\gamma_1$, and $\tilde{\delta} = \sigma^* u$, for $u \in \mathcal{U}_{\tilde{L}^0_{\sigma^*}}(F_v)$. By definition, σ^* is F_v-elliptic in $\tilde{L}$ if $\mathbf{p}(\sigma^*) = \sigma^n$ is F_v-elliptic in L. It is easily shown that $\widetilde{L_\sigma} \subset \tilde{L}_{\sigma^*} \subset \widetilde{L_{\sigma^n}}$. Therefore, if σ^n is F_v-elliptic in L, we have that $L_\sigma = L_{\sigma^n}$ by our assumption from Lemma 10.1 and in turn that

$$\tilde{L}_{\sigma^*} = \widetilde{L_\sigma} = \widetilde{L_{\sigma^n}}.$$

Consequently,

$$g^{\tilde{L}}_{\tilde{M}}(\tilde{\gamma}, \tilde{\delta}) = \begin{cases} g^{\widetilde{L_\sigma}}_{\widetilde{M_\sigma}}(\gamma_1, u), & \text{if } \sigma \text{ is } F_v\text{-elliptic in } L \\ 0, & \text{otherwise} \end{cases}.$$

By taking these facts into consideration, we obtain the expansion

$$I_{\tilde{M}}(\tilde{\gamma}, f^*) \overset{(\tilde{M},\sigma^*)}{\sim} \sum_{L\in\mathcal{L}(M)} \sum_{\tilde{\delta}\in\sigma^*(\mathcal{U}_{\widetilde{L_\sigma}}(F_v))} g^{\tilde{L}}_{\tilde{M}}(\tilde{\gamma}, \tilde{\delta}) I_{\tilde{L}}(\tilde{\delta}, f^*)$$

for $\tilde{\gamma} \in \sigma^* \tilde{M}^0_{\sigma^*}(F_v) \cap \tilde{G}_{\text{reg}}$. Proposition 8.2 and Proposition 8.1 together tell us that $I_{\tilde{L}}(\tilde{\delta}, f^*)$ vanishes unless $\mathbf{p}(\tilde{\delta}) = \delta^n$ for some $\delta \in G(F_v)$. The set $\sigma(\mathcal{U}_{L_\sigma}(F_v))$ maps bijectively onto the set $\sigma^*(\mathcal{U}_{\tilde{L}_\sigma^-}(F_v))$ under the map $*$. This can be deduced from

$$\tau_v(\sigma u, \sigma u) = \tau_v(u\sigma, \sigma u) = \tau_v(\sigma, \sigma),\ u \in \mathcal{U}_{L_\sigma}(F_v)$$

(see (2.2) [15]). Hence

$$I_{\tilde{M}}(\tilde{\gamma}, f^*) \overset{(\tilde{M},\sigma^*)}{\sim} \sum_{L \in \mathcal{L}(M)} \sum_{\delta \in \sigma(\mathcal{U}_{L_\sigma}(F_v))} g^{\tilde{L}}_{\tilde{M}}(\tilde{\gamma}, \delta^*) I_{\tilde{L}}(\delta^*, f^*)$$

for $\tilde{\gamma} \in \sigma^* \tilde{M}^0_{\sigma^*}(F_v) \cap \tilde{G}_{\text{reg}}$. Once again, by the local vanishing property, $I_{\tilde{M}}(\tilde{\gamma}, f^*)$ vanishes unless $\mathbf{p}(\tilde{\gamma}) = \gamma^n$ for some $\gamma \in G_{*\text{reg}}$. We claim that for each $L \in \mathcal{L}(M)$ and $\delta \in \sigma(\mathcal{U}_{L_\sigma}(F_v))$, the function $g^{\tilde{L}}_{\tilde{M}}(\tilde{\gamma}, \delta^*)$ has the same vanishing property. We may assume inductively that this is true for $L \neq G$. Fix $\delta_1 \in \sigma(\mathcal{U}_{L_\sigma}(F_v))$. According to §3.3 [31] we may choose $\tilde{f}_1 \in \mathcal{H}^{\text{met}}(\tilde{G}(F_v))$ such that

$$I_{\tilde{G}}(\tilde{\delta}, \tilde{f}_1) = \begin{cases} 1, & \text{if } \tilde{\delta} = \delta_1^* \\ 0, & \text{otherwise} \end{cases},$$

for $\tilde{\delta} \in \sigma^*(\mathcal{U}_{\tilde{G}_{\sigma^*}}(F_v))$. In particular

$$I_{\tilde{G}}(\delta^*, \tilde{f}_1) = \begin{cases} 1, & \text{if } \delta^* = \delta_1^* \\ 0, & \text{otherwise} \end{cases},$$

for $\delta \in \sigma(\mathcal{U}_{G_\sigma}(F_v))$. It is easily shown in this instance that $\delta^* = \delta_1^*$ if and only if $\delta = \delta_1$. Thus

$$I_{\tilde{G}}(\delta^*, \tilde{f}_1) = \begin{cases} 1 & \delta = \delta_1 \\ 0 & \text{otherwise} \end{cases},$$

for $\delta \in \sigma(\mathcal{U}_{G_\sigma}(F_v))$. If we substitute $\tilde{f}_1$ into our last germ expansion, the desired vanishing property for $g^{\tilde{G}}_{\tilde{M}}(\gamma^*, \delta^*)$ follows. Our germ expansion now has the form

$$I_{\tilde{M}}(\gamma^*, f^*) \overset{(\tilde{M},\sigma^*)}{\sim} \sum_{L \in \mathcal{L}M} \sum_{\delta \in \sigma(\mathcal{U}_{L_\sigma}(F_v))} g^{\tilde{L}}_{\tilde{M}}(\gamma^*, \delta^*) I_{\tilde{L}}(\delta^*, f^*)$$

for $\gamma \in \tilde{M}_\sigma(F_v) \cap G_{*\text{reg}}$.

Finally, as noted in §3, the orbital integral of any function in $\mathcal{H}(\tilde{M}(F_v))$ is equal to the orbital integral of a matching function in $\mathcal{H}^{\text{met}}(M(F_v))$. Therefore $(\tilde{M}, \sigma^*)$-equivalence may be taken to be (M,σ)-equivalence. The proof of the lemma now follows from this observation.□

In what is left of this section we show the (M,σ)-equivalence of $g_M^L(\gamma,\delta)$ and $g_{\tilde{M}}^{\tilde{L}}(\gamma,\delta)$.

Lemma 10.6 (7.1) *Suppose Theorem 9.1 holds for G. Then for each $u \in \mathcal{U}_G(F_v)$ we have*

$$g_M^G(\gamma,u) \overset{(M,1)}{\sim} g_{\tilde{M}}^{\tilde{G}}(\gamma^*,u^*), \gamma \in M(F_v) \cap G_{*\mathrm{reg}}.$$

Proof. We may assume by induction that

$$g_M^L(\gamma,u) \overset{(M,1)}{\sim} g_{\tilde{M}}^{\tilde{L}}(\gamma^*,u^*), \gamma \in M(F_v) \cap G_{*\mathrm{reg}},$$

for all $L \in \mathcal{L}(M)$ such that $L \neq G$. We may equate the germ expansions of Lemmas 10.4 and 10.5 since we are assuming Theorem 9.1 to hold. Together with the induction assumption, this yields

$$\sum_{u \in (\mathcal{U}_G(F_v))} (g_{\tilde{M}}^{\tilde{G}}(\gamma^*,u^*) - g_M^G(\gamma,u)) I_{\tilde{G}}(u^*,f^*) \overset{(M,1)}{\sim} 0,$$

for $\gamma \in M(F_v) \cap G_{*\mathrm{reg}}$. As in Lemma 10.5, for a a fixed element $u_1 \in (\mathcal{U}_G(F_v))$, we may choose $\tilde{f}_1 \in \mathcal{H}(\tilde{G}(F_v))$ such that

$$I_{\tilde{G}}(u^*,\tilde{f}_1) = \begin{cases} 1, & u = u_1 \\ 0, & \text{otherwise} \end{cases},$$

for $u \in (\mathcal{U}_G(F_v))$. The lemma now follows by replacing f^* with $\tilde{f}_1$ in the last $(M,1)$-equivalence. □

Lemma 10.7 (7.2) *Assume that the main theorem of [5] holds for $\tilde{G}(F_v)$. Then*

$$g_{\tilde{M}}^{\tilde{G}}(\gamma^*,1) \overset{(M,1)}{\sim} g_M^G(\gamma,1),\ \gamma \in M(F_v) \cap G_{*\mathrm{reg}}.$$

Proof. Choose $\pi \in \Pi_{\mathrm{temp}}^{\mathrm{met}}(G(F_v))$ such that π is supercuspidal. By Theorem 26.1 [15] and Corollary 26.1 [15], we know then that π^* exists and is supercuspidal. Let f be a matrix coefficient of the contragredient representation of π. Since f has compact support modulo the centre of G, it may easily be shown that $f \in \mathcal{H}_{\mathrm{ac}}^{\mathrm{met}}(G(F_v))$. For a definition of

$\mathcal{H}_{\mathrm{ac}}(G(F_v))$ see §11 [9]. We may assume that f^* is a matrix coefficient of π^*. The main theorem of [5] asserts that

$$I_M(\gamma, f) = (-1)^{\dim(A_M/A_G)}\mathrm{vol}(G_\gamma(F_v)/A_M(F_v))^{-1}\mathrm{tr}(\pi(f))|D(\gamma)|_v^{1/2}\Theta_\pi(\gamma),$$

where $\gamma \in M(F_v)\cap G_{\mathrm{reg}}$ and Θ_π is the character of π. The right hand side of this equation is taken to be zero if γ is not F_v-elliptic in M. The Haar measure on $A_M(F_v)$ is normalized by

$$\mathrm{vol}(A_M(F_v)\cap K_v) = \mathrm{vol}(\mathfrak{a}_M/H_M(A_M(F_v)))$$

(*cf.* §2 [5]). If $\gamma \in M(F_v)\cap G_{*\mathrm{reg}}$ then

$$G_{\eta\gamma}(F_v) = G_{\gamma^n}(F_v) = G_\gamma(F_v)$$

for all $\eta \in \mu_n^M$, and so it follows that

(24) $I_M^{\Sigma}(\gamma, f)$

$$= (-1)^{\dim(A_M/A_G)}\mathrm{vol}(G_\gamma(F_v)/A_M(F_v))^{-1}\mathrm{tr}(\pi(f))\sum_{\eta\in\mu_n^M/\mu_n^G}|D(\eta\gamma)|_v^{1/2}\Theta_\pi(\eta\gamma),$$

for $\gamma \in M(F_v)\cap G_{*\mathrm{reg}}$.

In the metaplectic context the Theorem of [5] becomes

(25) $nI_{\tilde{M}}^{\mathcal{M}}(\gamma, f)$

$$= (-1)^{\dim(A_M^n/A_G^n)}\mathrm{vol}(\tilde{G}_{\gamma^*}(F_v)/\widetilde{A_M}(F_v))^{-1}\mathrm{tr}(\pi^*(f^*))|D(\gamma^*)|_v^{1/2}\Theta_{\pi^*}(\gamma^*),$$

where $\gamma \in M(F_v)\cap G_{*\mathrm{reg}}$, and the Haar measure on $\widetilde{A_M}$ is normalized by

$$\mathrm{vol}((\widetilde{A_M}(F_v)\cap\tilde{K}_v)/\mathbf{i}(\mu_n)) = \mathrm{vol}(\mathfrak{a}_{\tilde{M}}/H_{\tilde{M}}(\widetilde{A_M}(F_v))).$$

Once again, the right hand side of equation (25) is interpreted to vanish if γ is not F_v-elliptic in M. In order to establish equality between the left hand sides of equations (24) and (25), we may therefore restrict our attention to the case that $\gamma \in M(F_v)\cap G_{*\mathrm{reg}}$ is F_v-elliptic in M.

The first step towards establishing this equality is to show that

(26) $$\mathrm{vol}(G_\gamma(F_v)/A_M(F_v))^{-1} = c'\mathrm{vol}(\tilde{G}_{\gamma^*}(F_v)/\widetilde{A_M^n}(F_v))^{-1},$$

where $c' = |n|_v^{r/2}$ is the normalizing factor of §24 [15]. We have already normalized the Haar measures $d\tilde{x}/d\tilde{z}$ on $\tilde{G}(F_v)/\widetilde{A_G^n}(F_v)$ and dx/dz on $G(F_v)/A_G(F_v)$ so that

$$c'\mathrm{vol}(\tilde{K}_v/(\widetilde{A_G^n}(F_v) \cap \tilde{K}_v)) = \mathrm{vol}(K_v/(A_G(F_v) \cap K_v)).$$

The Haar measures on $\tilde{G}(F_v)/\tilde{G}_{\gamma^*}(F_v)$ and $G(F_v)/G_\gamma(F_v)$ are normalized so that

$$\mathrm{vol}(\tilde{K}_v/(\tilde{G}_{\gamma^*}(F_v) \cap \tilde{K}_v)) = \mathrm{vol}(K_v/(G_\gamma \cap K_v)).$$

Taking the quotient of these last two equalities yields Haar measures on $\tilde{G}_{\gamma^*}(F_v)/\widetilde{A_G^n}(F_v)$ and $G(F_v)/A_G(F_v)$ which are normalized so that

$$(27) \qquad c'\mathrm{vol}\left(\frac{\tilde{G}_{\gamma^*}(F_v) \cap \tilde{K}_v}{\widetilde{A_G^n}(F_v) \cap \tilde{K}_v}\right) = \mathrm{vol}\left(\frac{G_\gamma(F_v) \cap K_v}{A_G(F_v) \cap K_v}\right).$$

The Haar measure on $\widetilde{A_M}(F_v)/\widetilde{A_G^n}(F_v)$ is normalized by (*cf.* §2 [5])

$$\begin{aligned} \mathrm{vol}\left(\frac{\widetilde{A_M}(F_v) \cap \tilde{K}_v}{\widetilde{A_G^n}(F_v) \cap \tilde{K}_v}\right) &= \mathrm{vol}((\mathfrak{a}_{\tilde{M}}/H_{\tilde{M}}(\widetilde{A_M}(F_v))) + \mathfrak{a}_{\tilde{G}}) \\ &= \mathrm{vol}((\mathfrak{a}_M/H_M(A_M(F_v))) + \mathfrak{a}_G) \\ &= \mathrm{vol}\left(\frac{A_M(F_v) \cap K_v}{A_G(F_v) \cap K_v}\right). \end{aligned}$$

By taking the quotient of this last equality with (27), we obtain equation (26). Thus,

$$(28) I_M^M(\gamma, f) = (\frac{c'}{n})(-1)^{\dim(A_M/A_G)}\mathrm{vol}(G_\gamma(F_v)/A_M(F_v))^{-1}\mathrm{tr}(\pi^*(f^*))|D(\gamma^*)|_v^{1/2}\Theta_{\pi^*}(\gamma^*).$$

The representations π and π^* are related by the equalities

$$\mathrm{tr}\pi(f) = \mathrm{tr}\pi^*(f^*)$$

and (Definition 26.1 [15])

$$(29) \qquad |D(\gamma^*)|_v^{1/2}\Theta_{\pi^*}(\gamma^*) = (\frac{n}{c'}) \sum_{\{\delta \in G_\gamma(F_v)/A_G(F_v) : \delta^*\zeta = \gamma^*\}} \zeta|D(\delta)|_v^{1/2}\Theta_\pi(\delta),$$

where $\zeta \in \widetilde{A_G^n}(F_v)/A_G^*(F_v)$ and $A_G^* = \{\delta_1^* : \delta_1 \in A_G(F_v)\}$. Observe that $\widetilde{A_G^n}(F_v)/A_G^*(F_v)$ may be identified with a subset of μ_n. Now let $\delta \in G_\gamma(F_v)$, $\zeta \in \mathrm{i}(\mu_n)$ and suppose that

$\delta^*\zeta = \gamma^*$. This implies that $\delta^n = \gamma^n$. In this case both δ and γ are F_v-elliptic in M, so according to Lemma 9.1 there exists some $\eta \in \mu_n^M$ such that $\delta = \eta\gamma$. Conversely, if $\delta = \eta\gamma$ for some $\eta \in \mu_n^M$ then $\delta^* = \gamma^*$ by Lemma 6.1. In other words

$$\{\delta \in G_\gamma(F_v)/A_G(F_v) : \delta^*\zeta = \gamma^*\} = \{\eta\gamma \in A_G(F_v) : \eta \in \mu_n^M/\mu_n^G\}.$$

A simple consequence of this and equations (28) and (29) is the equality

$$I_M^{\Sigma}(\gamma, f) = I_M^{\mathcal{M}}(\gamma, f), \tag{30}$$

for $\gamma \in M(F_v) \cap G_{*\mathrm{reg}}$. By Lemma 9.5, equation (30) is true for arbitrary $\gamma \in M(F_v)$. In particular,

$$I_M^{\Sigma}(1, f) = I_M^{\mathcal{M}}(1, f). \tag{31}$$

Let γ be an element in $M(F_v) \cap G_{*\mathrm{reg}}$ which is close to the identity. Then

$$I_M^{\Sigma}(\gamma, f) \overset{(M,1)}{\sim} \sum_{L \in \mathcal{L}(M)} \sum_{u \in (\mathcal{U}_L(F_v))} g_M^L(\gamma, u) I_L^{\Sigma}(u, f)$$

and

$$I_M^{\mathcal{M}}(\gamma, f) \overset{(M,1)}{\sim} \sum_{L \in \mathcal{L}(M)} \sum_{u \in (\mathcal{U}_L(F_v))} g_{\bar{M}}^{\bar{L}}(\gamma^*, u^*) I_L^{\mathcal{M}}(u, f)$$

by Lemmas 10.4 and 10.5. Now suppose that $u \in (\mathcal{U}_L(F_v))$ such with $u \neq 1$. Then u can be represented as an induced unipotent conjugacy class u_1^L, where $u_1 \in (\mathcal{U}_{L_1}(F_v))$ and L_1 is a proper Levi subgroup of L. The descent property of Proposition 6.1 and the descent property for the metaplectic group respectively yield

$$I_L^{\Sigma}(u, f) = \sum_{L_2 \in \mathcal{L}(L_1)} d_{L_1}^G(L, L_2) \hat{I}_{L_1}^{L_2, \Sigma}(u_1, f_{L_2})$$

and

$$I_L^{\mathcal{M}}(u, f) = \sum_{L_2 \in \mathcal{L}(L_1)} d_{L_1}^G(L, L_2) \hat{I}_{L_1}^{L_2, \mathcal{M}_i}(u_1, f_{L_2}).$$

However, f is a supercusp form on $G(F_v)$ so $f_{L_2} = 0$ for any proper Levi subgroup L_2 of G. Hence both $I_L^{\Sigma}(u, f)$ and $I_L^{\mathcal{M}}(u, f)$ vanish. Equation (30) may now be written as

$$\sum_{L \in \mathcal{L}(M)} g_M^L(\gamma, 1) I_L^{\Sigma}(1, f) \overset{(M,1)}{\sim} \sum_{L \in \mathcal{L}(M)} g_{\bar{M}}^{\bar{L}}(\gamma^*, 1) I_L^{\mathcal{M}}(1, f).$$

We assume inductively that $g_M^L(\gamma, 1) = g_{\tilde{M}}^{\tilde{L}}(\gamma^*, 1)$ for all $L \in \mathcal{L}(M)$ such that $L \neq G$. Observe that equation (31) implies that $I_L^{\mathcal{E}}(1, f) = I_L^{\mathcal{M}}(1, f)$ for $L \in \mathcal{L}(M)$. As a result, the previous $(M, 1)$-equivalence reduces to

$$g_M^G(\gamma, 1) I_G(1, f) \overset{(M,1)}{\sim} g_{\tilde{M}}^{\tilde{G}}(\gamma^*, 1) I_G(1, f).$$

The lemma now follows from the fact that $I_G(1, f) \neq 0$.□

Proposition 10.1 (7.3) *Let $f \in \mathcal{H}^{\mathrm{met}}(G(F_v))^0$. Then*

$$I_M^{\mathcal{M}}(\gamma, f) \overset{(M,\sigma)}{\sim} \sum_{L \in \mathcal{L}(M)} \sum_{\delta \in \sigma(\mathcal{U}_{L_\sigma}(F_v))} g_M^L(\gamma, \delta) I_L^{\mathcal{M}}(\delta, f),$$

for $\gamma \in M_\sigma(F_v) \cap G_{*\mathrm{reg}}$.

Proof. Suppose $\delta = \sigma u$, where $u \in \mathcal{U}_{L_\sigma}(F_v)$, and $\gamma = \sigma\gamma_1$, where $\gamma_1 \in M_\sigma(F_v)$. Then according to Lemma 9.2 [8] and the proof of Lemma 10.5,

$$g_M^L(\gamma, \delta) = \begin{cases} g_{M_\sigma}^{L_\sigma}(\gamma_1, u), & \text{if } \sigma \text{ is } F_v\text{-elliptic in } L \\ 0, & \text{otherwise} \end{cases}$$

and

$$g_{\tilde{M}}^{\tilde{L}}(\gamma^*, \delta^*) = \begin{cases} g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\gamma_1^*, u^*), & \text{if } \sigma \text{ is } F_v\text{-elliptic in } L \\ 0, & \text{otherwise} \end{cases}.$$

By the germ expansion of Lemma 10.5, it suffices to show that

$$g_{M_\sigma}^{L_\sigma}(\gamma, u) \overset{(M_\sigma,1)}{\sim} g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\gamma^*, u^*),$$

when σ is F_v-elliptic in L, and $L \in \mathcal{L}(M)$.

Suppose that σ is F_v-elliptic in L and that $L_\sigma \neq G$. Then, by the induction hypothesis at the end of §9, Theorem 9.1 holds for L_σ. We may therefore apply Lemma 10.6 with L_σ in place of G to obtain

$$g_{M_\sigma}^{L_\sigma}(\gamma_1, u) \overset{(M_\sigma,1)}{\sim} g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\gamma_1^*, u^*).$$

By Lemma 2.1 of [8], it follows that $(M_\sigma, 1)$-equivalence of these germs as functions of γ_1 is that same as (M, σ)-equivalence as functions of γ. That is

$$g_M^L(\gamma, \delta) = g_{M_\sigma}^{L_\sigma}(\gamma_1, u) \overset{(M,\sigma)}{\sim} g_{\widetilde{M_\sigma}}^{\widetilde{L_\sigma}}(\gamma_1^*, u^*) = g_{\tilde{M}}^{\tilde{L}}(\gamma^*, \delta^*).$$

The remaining possibility is that $\sigma \in A_G(F_v)$ and $L = G$. Suppose this is the case. Then

$$\sum_{\delta \in \sigma(\mathcal{U}_G(F_v))} g_{\tilde{M}}^{\tilde{G}}(\gamma^*, \delta^*) I_G^{\mathcal{M}}(\delta, f) = \sum_{\delta \in \sigma(\mathcal{U}_G(F_v))} g_{\tilde{M}}^{\tilde{G}}(\gamma^*, \delta^*) I_G(\delta, f).$$

Since $f \in \mathcal{H}^{\text{met}}(G(F_v))^0$, we have that $I_G(\delta, f)$ vanishes in the above sum unless $\delta = \sigma$. Since σ is central

$$g_{\tilde{M}}^{\tilde{G}}(\gamma^*, \sigma^*) = g_{\tilde{M}}^{\tilde{G}}(\gamma^*, 1) \overset{(M,1)}{\sim} g_M^G(\gamma, 1) = g_M^G(\gamma, \sigma)$$

by Lemma 9.2 [8] and Lemma 10.7. Hence

$$\begin{aligned} \sum_{\delta \in \sigma(\mathcal{U}_G(F_v))} g_{\tilde{M}}^{\tilde{G}}(\gamma^*, \delta^*) I_G^{\mathcal{M}}(\delta, f) &= g_M^G(\gamma, \sigma) I_G(\sigma, f) \\ &= \sum_{\delta \in \sigma(\mathcal{U}_G(F_v))} g_M^G(\gamma, \delta) I_G^{\mathcal{M}}(\delta, f) \end{aligned}$$

and the lemma is completed.□

11 The Local Spectral Terms

Leaving the terms of the geometric side of the trace formulas behind, we take an excursion to the spectral sides of the trace formulas. The spectral sides are partially composed of invariant distributions,

$$I_{\tilde{M}}(\tilde{\pi}, f), \ \tilde{\pi} \in \Pi(\tilde{M}(F_S)),$$

which are sums of traces that are weighted by normalized intertwining operators. These distributions are introduced in §3 [6]. We assume that the reader is to some degree familiar with this introduction and recall some of it below.

Given $\rho \in \Sigma^{\text{met}}(M(F_S))$, we define

$$I_M^{\mathcal{M}}(\rho, X, f) = I_{\tilde{M}}(\rho^*, X^*, f^*),$$

for all $X \in \mathfrak{a}_{M,S}$ and $f \in \mathcal{H}^{\text{met}}(G(F_S))$. If $L \in \mathcal{L}(M)$ and $\lambda \in \mathfrak{a}_{M,S}^*$ is in general position, then the induced representation ρ_λ^L belongs to $\Sigma^{\text{met}}(L(F_S))$. When ρ_λ^L appears as an argument of $I_L(\cdot)$ or $I_L^{\mathcal{M}}(\cdot)$, we will often suppress the superscript L. For $\pi \in \Pi^{\mathrm{met}}(M(F_S))$, we define

$$I_M^{\mathcal{M}}(\pi, X, f) = \sum_P \omega_P \sum_L \sum_\rho \int_{\epsilon_P + i\mathfrak{a}_{M,S}^*/i\mathfrak{a}_{L,S}^*} r_M^L(\pi_\lambda, \rho_\lambda) I_L^{\mathcal{M}}(\rho_\lambda, h_L(X), f) e^{-\lambda(X)} d\lambda,$$

where P, L and ρ are summed over $\mathcal{P}(M)$, $\mathcal{L}(M)$ and $\Sigma^{\mathrm{met}}(M(F_S))$ respectively. For definitions of r_M^L and the constant ω_P see §6 [9] and §3 [6].

As on p 127 of [10], we identify representations π in $\Pi^{\mathrm{met}}(M(F_S)^1)$ with orbits $\{\pi_\lambda : \lambda \in \mathfrak{a}_M\}$ in $\Pi^{\mathrm{met}}(M(F_S))$ if π is not unitary. If π belongs to $\Pi^{\mathrm{met}}_{\mathrm{unit}}(M(F_S)^1)$, then we identify it with the orbit $\{\pi_\lambda : \lambda \in i\mathfrak{a}_M^*\}$ in $\Pi^{\mathrm{met}}_{\mathrm{unit}}(M(F_S))$. We make similar identifications for representations in $\Pi^{\mathrm{met}}(M(\mathbf{A})^1)$ and $\Pi^{\mathrm{met}}_{\mathrm{unit}}(M(\mathbf{A})^1)$. If $\pi \in \Pi^{\mathrm{met}}_{\mathrm{unit}}(M(F_S)^1)$, set

$$I_M(\pi, f) = I_M(\pi_\lambda, 0, f),$$

and

$$I_M^{\mathcal{M}}(\pi, f) = I_M^{\mathcal{M}}(\pi_\lambda, 0, f),$$

for any $\lambda \in i\mathfrak{a}_M^*$. It may be verified that these definitions are indeed well defined. Both of these definitions are independent of S, if S is large, and therefore may be extended to representations in $\Pi^{\mathrm{met}}_{\mathrm{unit}}(M(\mathbf{A})^1)$. In complete analogy with the expressions of the geometric side of the trace formula, we hope to identify $I_M(\pi, f)$ with the terms $I_{\tilde{M}}(\tilde{\pi}, f^*)$ occurring in the spectral side of the trace formula of $\tilde{G}$.

We may draw analogies between the local geometric and the local spectral terms of the trace formulas. In order to compare the local geometric terms of the trace formulas for G and $\tilde{G}$, we use the transfer map (1). One might surmise that the analogous transfer map for the local spectral terms might be (8). Unfortunately, this map does not intrinsically relate the traces of the representations to each other. In what follows we define certain constants which relate representations in $\Pi(\tilde{M}(F_S))$ to representations in $\Pi^{\mathrm{met}}(M(F_S))$ in a fashion that is compatible with their traces.

By the Langlands quotient theorem and §5 [9] there exist constants $\Delta(\tilde{\pi}, \tilde{\rho})$ and $\Gamma(\tilde{\rho}, \tilde{\pi})$ for arbitrary $\tilde{\rho} \in \Sigma(\tilde{M}(F_S))$ and $\tilde{\pi} \in \Pi(\tilde{M}(F_S))$, such that

$$\mathrm{tr}(\tilde{\rho}) = \sum_{\tilde{\pi}' \in \Pi(\tilde{M}(F_S))} \Gamma(\tilde{\rho}, \tilde{\pi}')\mathrm{tr}(\tilde{\pi}'),$$

and

$$\mathrm{tr}(\tilde\pi) = \sum_{\tilde\rho' \in \Sigma(\tilde M(F_S))} \Delta(\tilde\pi, \tilde\rho')\mathrm{tr}(\tilde\rho').$$

Two consequences of Lemma 3.1 are the identities

$$\mathrm{tr}(\rho) = \sum_{\pi' \in \Pi^{\mathrm{met}}(M(F_S))} \Gamma(\rho, \pi')\mathrm{tr}(\pi')$$

and

$$\mathrm{tr}(\pi) = \sum_{\rho' \in \Sigma^{\mathrm{met}}(M(F_S))} \Delta(\pi, \rho')\mathrm{tr}(\rho'),$$

for any $\pi \in \Pi^{\mathrm{met}}(M(F_S))$ and $\rho \in \Sigma^{\mathrm{met}}(M(F_S))$.

Suppose $\tilde\pi \in \Pi(\tilde M(F_S))$, $\rho \in \Sigma^{\mathrm{met}}(M(F_S))$ and set

$$\Delta(\tilde\pi, \rho) = \Delta(\tilde\pi, \rho^*).$$

For $\pi \in \Pi^{\mathrm{met}}(M(F_S))$ we set

$$\delta(\tilde\pi, \pi) = \sum_{\rho \in \Sigma^{\mathrm{met}}(M(F_S))} \Delta(\tilde\pi, \rho)\Gamma(\rho, \pi).$$

As can be seen from the next proposition, the map,

$$\tilde\pi \mapsto \sum_{\pi \in \Pi^{\mathrm{met}}(M(F_S))} \delta(\tilde\pi, \pi), \tag{32}$$

is the transfer map which allows us to compare traces of representations.

Proposition 11.1 (8.2) *For any $f \in \mathcal{H}^{\mathrm{met}}(M(F_S))$ and $\tilde\pi \in \Pi(\tilde M(F_S))$ we have*

$$\mathrm{tr}\tilde\pi(f^*) = \sum_{\pi \in \Pi^{\mathrm{met}}(M(F_S))} \delta(\tilde\pi, \pi)\mathrm{tr}\pi(f).$$

Proof. By our assumptions

$$\mathrm{tr}\tilde\pi(f^*) = \sum_{\tilde\rho \in \Sigma(\tilde M(F_S))} \Delta(\tilde\pi, \tilde\rho)\mathrm{tr}\tilde\rho(f^*)$$

$$= \sum_{\rho \in \Sigma^{\mathrm{met}}(M(F_S))} \Delta(\tilde{\pi}, \rho^*) \mathrm{tr}\rho(f)$$
$$= \sum_{\rho \in \Sigma^{\mathrm{met}}(M(F_S))} \sum_{\pi \in \Pi^{\mathrm{met}}(M(F_S))} \Delta(\tilde{\pi}, \rho^*) \Gamma(\rho, \pi) \mathrm{tr}\pi(f)$$
$$= \sum_{\pi \in \Pi^{\mathrm{met}}(M(F_S))} \delta(\tilde{\pi}, \pi) \mathrm{tr}\pi(f). \square$$

Corollary 11.1 (8.3) *Suppose S consists of one place v for which $|n|_v = 1$, and that $\pi \in \Pi^{\mathrm{met}}(M(F_v))$ is an unramified representation. Then for any $\tilde{\pi} \in \Pi(\tilde{M}(F_v))$*

$$\delta(\tilde{\pi}, \pi) = \begin{cases} 1, & \text{if } \tilde{\pi} = \pi^* \\ 0, & \text{otherwise} \end{cases}.$$

Proof. Take f to be an arbitrary function in $\mathcal{H}^{\mathrm{met}}(M(F_v))$ which is bi-invariant under $K_v \cap M(F_v)$. Theorem 16 [15] tells us that $\mathrm{tr}\pi^*(f^*) = \mathrm{tr}\pi(f)$. The corollary now follows from the proposition and the linear independence of characters. $\square$

This corollary allows us to define an map

$$\delta(\tilde{\pi}, \pi) = \prod_v \delta(\tilde{\pi}_v, \pi_v)$$

for adèlic representations $\tilde{\pi} = \otimes_v \tilde{\pi}_v \in \Pi(\tilde{M}(\mathbf{A}))$ and $\pi = \otimes_v \pi_v \in \Pi^{\mathrm{met}}(M(\mathbf{A}))$. All of the above formulas remain valid in the adèlic formulation as well. If $\tilde{\pi} \in \Pi^{\mathrm{met}}(M(\mathbf{A})^1)$ and $\pi \in \Pi^{\mathrm{met}}(M(\mathbf{A})^1)$ we define

$$\delta(\tilde{\pi}, \pi) = \sum_{\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}} \delta(\tilde{\pi}_{\tilde{\lambda}}, \pi_\lambda)$$

for arbitrary $\tilde{\lambda} \in \mathfrak{a}^*_{M,\mathbf{C}}$. This definition may be verified to be well-defined.

12 The Spectral Side of the Trace Formula

Now that we have a spectral transfer map, we can compare the spectral sides of the trace formulas of G and $\tilde{G}$. Recall that the spectral side of the trace formula for $\tilde{G}$ is of the form

$$I(\tilde{f}) = \sum_{t \geq 0} I_t(\tilde{f}),$$

where

(33) $$I_t(\tilde{f}) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(\tilde{M},t)} a^{\tilde{M}}(\tilde{\pi}) I_{\tilde{M}}(\tilde{\pi},\tilde{f}) d\tilde{\pi}.$$

Once again, it is convenient to denote $I_t(f^*)$ as $I_t^{\mathcal{M}}(f)$. A detailed discussion of the terms occurring in this formula may be found in §4 [7]. The definition of $\Pi(\tilde{M},t)$ is repeated here (p 132 [10]), as we will often have recourse to it in the sequel. Let $M_1 \in \mathcal{L}$ and t be a positive real number. We are obliged to first define two other sets, $\Pi(\tilde{M}(\mathbf{A})^1,t)$ and $\Pi_{\mathrm{disc}}(\tilde{M}_1,t)$, before we define $\Pi(\tilde{M},t)$. Given a representation $\tilde{\pi} \in \Pi(\tilde{M}(\mathbf{A}))$, let $\nu_{\tilde{\pi}}$ be the infinitesimal character of the Archimedean factor of $\tilde{\pi}$. The set $\Pi(\tilde{M}(\mathbf{A})^1,t)$ is defined to be the set of (equivalence classes) of representations $\tilde{\pi} \in \Pi(\tilde{M}(\mathbf{A}))$ such that $\|\mathrm{Im}(\nu_{\tilde{\pi}})\| = t$. Similarly, we set

$$\Pi^{\mathrm{met}}(M(\mathbf{A})^1,t) = \Pi(M(\mathbf{A})^1,t) \cap \Pi^{\mathrm{met}}(M(\mathbf{A})^1).$$

We write $\Pi_{\mathrm{disc}}(\tilde{M}_1,t)$ for the subset of $\Pi_{\mathrm{unit}}(\tilde{M}(\mathbf{A})^1,t)$ consisting of irreducible constituents of induced representations

$$\tilde{\sigma}_\lambda^{\tilde{M}_1},\ L \in \mathcal{L}^{M_1},\ \tilde{\sigma} \in \Pi_{\mathrm{unit}}(\tilde{L}(\mathbf{A})^1,t),\ \lambda \in i\mathfrak{a}_{\tilde{M}}^*/i\mathfrak{a}_{\tilde{M}_1}^*,$$

in which $\tilde{\sigma}_\lambda$ satisfies the following two conditions:

1. $a_{\mathrm{disc}}^{\tilde{L}}(\tilde{\sigma}) \neq 0$.

2. There is an element $s \in W^{M_1}(\mathfrak{a}_L)_{\mathrm{reg}}$ such that $s\tilde{\sigma}_\lambda = \tilde{\sigma}_\lambda$.

Then $\Pi(\tilde{M},t)$ is defined as the disjoint union over $M_1 \in \mathcal{L}^M$ of the sets

$$\Pi_{\tilde{M}_1}(\tilde{M},t) = \{\tilde{\pi} = \tilde{\pi}_{1,\lambda} : \tilde{\pi}_1 \in \Pi_{\mathrm{disc}}(\tilde{M}_1,t),\ \lambda \in i\mathfrak{a}_{\tilde{M}_1}^*/i\mathfrak{a}\tilde{M}^*\}.$$

We define the sets $\Pi_{\mathrm{disc}}^{\mathrm{met}}(M,t)$, $\Pi_{M_1}(M,t)$ and $\Pi^{\mathrm{met}}(M,t)$ as above, except that $\Pi_{\mathrm{unit}}(M(\mathbf{A})^1,t)$ and $\Pi_{\mathrm{unit}}(L(\mathbf{A})^1,t)$ are replaced by $\Pi_{\mathrm{unit}}^{\mathrm{met}}(M(\mathbf{A})^1,t)$ and $\Pi_{\mathrm{unit}}^{\mathrm{met}}(L(\mathbf{A})^1,t)$.

Let us recollect §9, where we examined the geometric sides of the trace formulas. In Proposition 9.1, the geometric side of the trace formula for $\tilde{G}$ was expressed in a manner that was compatible with the orbit map. We will be compelled to follow suit and express the spectral side of the trace formula for $\tilde{G}$ in a manner that is compatible with the spectral transfer map (32). This will be carried out in §15.

For the the time being, we set up the appropriate grouping of representations for the global datum, $a^{\tilde{M}}(\tilde{\pi})$. In other words, we define the global datum, $a^{M,\mathcal{M}}(\pi)$, which ought to correspond to the global datum, $a^{\tilde{M}}(\tilde{\pi})$, occurring in the trace formula for $\tilde{G}$. This is similar in spirit to the grouping of the local geometric terms in §6.

Set

$$\Pi^{\mathrm{met}}(M(\mathbb{A})^1) = \{\pi \in \Pi^{\mathrm{met}}(M(\mathbb{A})^1) : \pi_\lambda \in \Pi^{\mathrm{met}}(M(\mathbb{A})) \text{ for some } \lambda \in \mathfrak{a}^*_{M,\mathbb{C}}\}.$$

We first define $a^{M_1,\mathcal{M}}_{\mathrm{disc}}$ for $M_1 \in \mathcal{L}^M$. Set

$$a^{M_1,\mathcal{M}}_{\mathrm{disc}}(\pi_1) = \sum_{\tilde{\pi} \in \Pi(\tilde{M}_1(\mathbb{A})^1)} a^{\tilde{M}_1}_{\mathrm{disc}}(\tilde{\pi}) \delta(\tilde{\pi}, \pi_1)$$

for any $\pi_1 \in \Pi^{\mathrm{met}}(M_1(\mathbb{A})^1)$. This sum may be shown to be finite using the arguments of Lemma 9.1 [10]. For $\pi = \pi_{1,\lambda}$, where $\lambda \in \mathfrak{a}^*_{M_1,\mathbb{C}}/\mathfrak{a}^*_{M,\mathbb{C}}$, we set

$$a^{M,\mathcal{M}}(\pi) = a^{M_1,\mathcal{M}}_{\mathrm{disc}}(\pi_1) r^M_{M_1}(\pi_{1,\lambda}).$$

The function $r^M_{M_1}(\pi_{1,\lambda})$ is not defined for arbitrary $\pi_1 \in \Pi^{\mathrm{met}}(M_1(\mathbb{A})^1)$, and so the definition of $a^{M_1,\mathcal{M}}(\pi)$ is not valid as it now stands. The obstacle stems from the fact that the global map $r^M_{M_1}$ is derived from the adèlic version of the normalizing factors of intertwining operators (*cf.* §4 and §14). As such it is defined in terms of an infinite product, indexed by the valuations of F, and might not converge. One expects such normalizing factors to converge and have analytic continuation for automorphic representations. This borne out from the theory of Eisenstein series (*cf.* §4 [9]).

In order to rectify this situation, we make the following induction hypothesis. We assume that for any $M_1 \in \mathcal{L}$ with $M_1 \neq G$, that

$$a^{M_1,\mathcal{M}}_{\mathrm{disc}}(\pi_1) = a^{M_1}_{\mathrm{disc}}(\pi_1),$$

for all $\pi_1 \in \Pi^{\mathrm{met}}(M_1(\mathbb{A})^1)$. In this case $a^{M_1,\mathcal{M}}_{\mathrm{disc}}(\pi_1)$ vanishes unless π_1 belongs to $\Pi^{\mathrm{met}}_{\mathrm{disc}}(M_1, t)$. If $M_1 \in \mathcal{L}^M$, $\pi_1 \in \Pi^{\mathrm{met}}_{\mathrm{disc}}(M_1, t)$ and $\lambda \in i\mathfrak{a}^*_{M_1,\mathbb{C}}/i\mathfrak{a}^*_{M,\mathbb{C}}$ then the function $r^M_{M_1}$ is defined and the earlier definition of $a^{M,\mathcal{M}}(\pi_{1,\lambda})$ makes sense.

The global datum, $a^{M,\mathcal{M}}(\pi_1)$, suggests the definitions of new sets of representations along the same lines as the definitions of $\Pi_{\mathrm{disc}}(M_1,t)$, $\Pi_{M_1}(M,t)$ and $\Pi(M,t)$ above. We define the sets of (equivalence classes of) representations $\Pi^{\mathcal{M}}_{\mathrm{disc}}(M_1,t)$, $\Pi^{\mathcal{M}}_{M_1}(M,t)$ and $\Pi^{\mathcal{M}}(M,t)$ as above, except that $a^{\tilde{M}}$ is replaced with $a^{M,\mathcal{M}}$.

We are now in the position to state the spectral analogue of Theorem 9.1.

Theorem 12.1 (B) *(i) Suppose that S is a finite set of valuations which has the closure property and contains $S_{(n)}$. Then*

$$I^{\mathcal{M}}_{M}(\pi,f)=I_M(\pi,f),\ \ \pi\in\Pi^{\mathrm{met}}_{\mathrm{unit}}(M(\mathbb{A})^1),\ \ f\in\mathcal{H}^{\mathrm{met}}(G(F_S)).$$

(ii) For any given

$$\pi=\pi_{1,\lambda},\ \ \pi_1\in\Pi^{\mathrm{met}}(M(\mathbb{A})^1),\ \ \lambda\in\mathfrak{a}^*_{M_1,\mathbb{C}}/\mathfrak{a}^*_{M,\mathbb{C}},$$

we have

$$a^{M,\mathcal{M}}(\pi)=a^M(\pi).$$

This proof of this theorem will be completed at the end of §18.

13 Comparison of the Local Spectral Terms

The purpose of this section is to show that (i) of Theorem 9.1 implies (i) of Theorem 12.1. We achieve this with help from the maps θ^L_M and ${}^c\theta^L_M$ defined in §4 [6]. These maps are defined on $\tilde{\mathcal{H}}_{\mathrm{ac}}(L(F_S))$ and take values in $\tilde{\mathcal{I}}_{\mathrm{ac}}(M(F_S))$ for every pair of Levi subgroups $M\subset L$ in $\mathcal{L}$. The spaces $\tilde{\mathcal{H}}_{\mathrm{ac}}(L(F_S))$ and $\tilde{\mathcal{I}}_{\mathrm{ac}}(M(F_S))$ contain $\mathcal{H}(L(F_S))$ and $\mathcal{I}(M(F_S))$ respectively and are defined in §11 [9]. They also satisfy the following properties:

(34) $$\sum_{L\in\mathcal{L}(M)}\hat{\theta}^L_M({}^c\theta_L(f))=\sum_{L\in\mathcal{L}(M)}{}^c\hat{\theta}^L_M(\theta_L(f))=0,$$

(35) $$I_M(\gamma,f)=\sum_{L\in\mathcal{L}(M)}{}^c\hat{I}^L_M(\gamma,\theta_L(f)),$$

and

$$ {}^{c}I_M(\gamma, f) = \sum_{L\in\mathcal{L}(M)} \hat{I}^L_M(\gamma, {}^{c}\theta_L(f)), \tag{36} $$

for $\gamma \in M(F_S)$ and $f \in \tilde{\mathcal{H}}_{\mathrm{ac}}(G(F_S))$. For the definition of ${}^{c}I_M$ see §4 [6]. Set

$$ \theta^{\Sigma}_M(f) = \sum_{\eta\in\mu^M_n/\mu^G_n} {}^{\eta}\theta_M(f). $$

$$ {}^{c}\theta^{\Sigma}_M(f) = \sum_{\eta\in\mu^M_n/\mu^G_n} {}^{\eta c}\theta_M(f). $$

Here ${}^{\eta}\theta_M(f)$ is defined as ${}^{\eta}h_M$ if $h \in \mathcal{H}(M(F_S))$ such that $h_M = \theta(f)$ (*cf.* §3).

Properties (35) and (36) may be adapted to the distributions of the form I^{Σ}_M. We may mimic the arguments of §6 to arrive at the equality ${}^{c}I_M(\eta\gamma, f) = {}^{c}I_M(\gamma, f)$ for all $\gamma \in M(F_S)$ and $\eta \in \mu^G_n$. Thus, imitating the definition of I^{Σ}_M, we set

$$ {}^{c}I^{\Sigma}_M(\gamma, f) = \sum_{\eta\in\mu^M_n/\mu^G_n} {}^{c}I_M(\eta\gamma, f), $$

for all $\gamma \in M(F_S)$ and $f \in \tilde{\mathcal{H}}^{\mathrm{met}}_{\mathrm{ac}}(G(F_S))$. The analogue of property (36) for ${}^{c}I^{\Sigma}_M$ is then seen to be

$$ \begin{aligned} {}^{c}I^{\Sigma}_M(\gamma, f) &= \sum_{\eta\in\mu^M_n/\mu^G_n} \sum_{L\in\mathcal{L}(M)} \hat{I}^L_M(\eta\gamma, {}^{c}\theta_L(f)) \\ &= \sum_{L\in\mathcal{L}(M)} \sum_{\eta\in\mu^L_n/\mu^G_n} \hat{I}^{L,\Sigma}_M(\eta\gamma, {}^{c}\theta_L(f)) \\ &= \sum_{L\in\mathcal{L}(M)} \sum_{\eta\in\mu^L_n/\mu^G_n} \hat{I}^{L,\Sigma}_M(\gamma, {}^{\eta c}\theta_L(f)) \\ &= \sum_{L\in\mathcal{L}(M)} \hat{I}^{L,\Sigma}_M(\gamma, {}^{c}\theta^{\Sigma}_L(f)). \end{aligned} \tag{37} $$

After a similar computation we may conclude that the analogue of property (35) is

$$ I^{\Sigma}_M(\gamma, f) = \sum_{L\in\mathcal{L}(M)} {}^{c}\hat{I}^{L,\Sigma}_M(\gamma, \theta^{\Sigma}_L(f)). $$

Part of our assumption concerning the existence of the invariant trace formula for $\tilde{G}$ (*cf.* §5) is the existence of maps, $\theta^{\tilde{L}}_{\tilde{M}}$, ${}^{c}\theta^{\tilde{L}}_{\tilde{M}}$ and ${}^{c}I^{\tilde{L}}_{\tilde{M}}(\tilde{\gamma})$, satisfying properties corresponding to (34) - (36). Let $f \in \tilde{\mathcal{H}}_{\mathrm{ac}}^{\mathrm{met}}(G(F_S))$, $\pi \in \Pi_{\mathrm{temp}}^{\mathrm{met}}(M(F_S))$, $X \in \mathfrak{a}_{M,S}$ and $\gamma \in G_{\ast\mathrm{reg}} \cap M(F_S)$. Define

$$^{c}I_M^{\mathcal{M}}(\gamma, f) = {}^{c}I_{\tilde{M}}(\gamma^*, f^*),$$

$$\theta_M^{\mathcal{M}}(f, \pi, X) = n^{\dim A_M}\theta_{\tilde{M}}(f^*, \pi^*, X^*),$$

and

$$^{c}\theta_M^{\mathcal{M}}(f, \pi, X) = n^{\dim A_M}\, {}^{c}\theta_{\tilde{M}}(f^*, \pi^*, X^*).$$

Lemma 13.1 (10.1) *Let $\gamma \in M(F_S)$ and $f \in \tilde{\mathcal{H}}_{\mathrm{ac}}^{\mathrm{met}}(G(F_S))$. Then the following properties hold.*

(38)
$$\sum_{L \in \mathcal{L}(M)} \hat{\theta}_M^{L,\mathcal{M}}({}^{c}\theta_L^{\mathcal{M}}(f)) = \sum_{L \in \mathcal{L}(M)} {}^{c}\hat{\theta}_M^{L,\mathcal{M}}(\theta_L^{\mathcal{M}}(f)) = 0,$$

(39)
$$I_M^{\mathcal{M}}(\gamma, f) = \sum_{L \in \mathcal{L}(M)} {}^{c}\hat{I}_M^{L,\mathcal{M}}(\gamma, \theta_L^{\mathcal{M}}(f)),$$

(40)
$$^{c}I_M^{\mathcal{M}}(\gamma, f) = \sum_{L \in \mathcal{L}(M)} \hat{I}_M^{L,\mathcal{M}}(\gamma, {}^{c}\theta_L^{\mathcal{M}}(f)).$$

Proof. Let $\pi \in \Pi_{\mathrm{temp}}^{\mathrm{met}}(M(F_S))$, $X \in \mathfrak{a}_{M,S}$, $L \in \mathcal{L}(M)$ and $h \in \tilde{\mathcal{H}}_{\mathrm{ac}}^{\mathrm{met}}(L(F_S))$. Observe that

$$\begin{aligned}
\theta_M^{L,\mathcal{M}}(h)^*(\pi^*, X^*) &= n^{-\dim A_M}\theta_M^{L,\mathcal{M}}(h, \pi, X) \\
&= n^{-\dim A_M} n^{\dim A_M}\theta_{\tilde{M}}^{\tilde{L}}(h^*, \pi^*, X^*) \\
&= \theta_{\tilde{M}}^{\tilde{L}}(h^*)(\pi^*, X^*).
\end{aligned}$$

That is, $\theta_{\tilde{M}}^{\tilde{L}}(h^*) = n^{\dim(A_M/A_L)}\theta_M^{L,\mathcal{M}}(h)^*$.

We may now apply this equation to prove (38).

$$\begin{aligned}
\sum_{L \in \mathcal{L}(M)} {}^{c}\theta_M^{L,\mathcal{M}}(\theta_L^{\mathcal{M}}(f), \pi, X) &= \sum_{L \in \mathcal{L}(M)} n^{\dim A_M}\, {}^{c}\theta_{\tilde{M}}^{\tilde{L}}(\theta_L^{\mathcal{M}}(f)^*, \pi^*, X^*) \\
&= \sum_{L \in \mathcal{L}(M)} n^{\dim A_M}\, {}^{c}\theta_{\tilde{M}}^{\tilde{L}}(\theta_{\tilde{L}}(f^*), \pi^*, X^*) \\
&= n^{\dim A_M} \sum_{L \in \mathcal{L}(M)} {}^{c}\theta_{\tilde{M}}^{\tilde{L}}(\theta_{\tilde{L}}(f^*), \pi^*, X^*) \\
&= 0.
\end{aligned}$$

The second half of (38) follows in a similar fashion.

Property (40) is also easily deduced.

$$\begin{aligned} {}^cI_M^{\mathcal{M}}(\gamma, f) &= {}^cI_{\tilde{M}}(\gamma^*, f^*) \\ &= \sum_{L \in \mathcal{L}(M)} \hat{I}_{\tilde{M}}^{\tilde{L}}(\gamma^*, {}^c\theta_{\tilde{L}}(f^*)) \\ &= \sum_{L \in \mathcal{L}(M)} \hat{I}_{\tilde{M}}^{\tilde{L}}(\gamma^*, {}^c\theta_L^{\mathcal{M}}(f)^*) \\ &= \sum_{L \in \mathcal{L}(M)} \hat{I}_M^{L,\mathcal{M}}(\gamma, {}^c\theta_L^{\mathcal{M}}(f)). \end{aligned}$$

Property (39) follows accordingly.□

Theorem 13.1 (10.2) *Assume that*

$$I_L^{\mathcal{M}}(\gamma, f) = I_L^{\mathcal{E}}(\gamma, f)$$

for each $L \in \mathcal{L}(M)$, $\gamma \in L(F_S)$ *and* $f \in \mathcal{H}^{\mathrm{met}}(G(F_S))$, *in the special case that* $S \supset S_{(n)}$. *Then for any* $f \in \tilde{\mathcal{H}}_{\mathrm{ac}}^{\mathrm{met}}(G(F_S))$ *and* $X \in \mathfrak{a}_{M,S}$ *we have*

$$\theta_M^{\mathcal{M}}(f) = \theta_M^{\mathcal{E}}(f),$$

$${}^c\theta_M^{\mathcal{M}}(f) = {}^c\theta_M^{\mathcal{E}}(f),$$

$$I_M^{\mathcal{M}}(\rho, X, f) = I_M(\rho^*, X^*, f^*),\ \rho \in \Sigma^{\mathrm{met}}(M(F_S)),$$

and

$$I_M^{\mathcal{M}}(\pi, X, f) = I_M(\pi^*, X^*, f^*),\ \pi \in \Pi^{\mathrm{met}}(M(F_S)).$$

Proof. The proof of this theorem is identical to Theorem 10.2 [10], if one replaces $I_M^{\mathcal{E}}$, $I_M(\gamma)$, θ_M and $\theta_M^{\mathcal{E}}$ of that proof with $I_M^{\mathcal{M}}$, $I_M^{\mathcal{E}}(\gamma)$, $\theta_M^{\mathcal{E}}$ and $\theta_M^{\mathcal{M}}$ respectively. The only portion of the proof where there is a difference worth noting in on p 140, where it is shown that

$${}^c\theta_M^{\mathcal{E}}(f, \pi, X) - {}^c\theta_M(f, \pi, X)$$

is compactly supported in $X \in \mathfrak{a}_{M,S}$. We therefore take the trouble to write out the analogue of this part of the proof, which is to show that

$${}^c\theta_M^{\mathcal{M}}(f, \pi, X) - {}^c\theta_M^{\mathcal{E}}(f, \pi, X)$$

is compactly supported in $X \in \mathfrak{a}_{M,S}$.

We assume inductively that the theorem holds if G is replaced by $L \in \mathcal{L}$ such that $L \neq G$ or if M is replaced by $L \in \mathcal{L}$ such that $L \supsetneq M$. Let $\gamma \in M(F_S)$ and consider the expression

$$(41) \qquad {}^cI_M^{\mathcal{M}}(\gamma, f) - \sum_{L_1 \in \mathcal{L}(M)} \varepsilon_M^{L_1}(S)\, {}^cI_{L_1}^{\Sigma}(\gamma, f).$$

By properties (40) and (37), we may write this as

$$(42) \qquad \begin{aligned} &\sum_{L \in \mathcal{L}(M)} \hat{I}_M^{L,\mathcal{M}}(\gamma, {}^c\theta_L^{\mathcal{M}}(f)) - \sum_{L_1 \in \mathcal{L}(M)} \sum_{L_2 \in \mathcal{L}(L_1)} \varepsilon_M^{L_1}(S) \hat{I}_{L_1}^{L_2,\Sigma}(\gamma, {}^c\theta_{L_2}^{\Sigma}(f)) \\ &= \sum_{L \supsetneq M} \hat{I}_M^{L,\mathcal{M}}(\gamma, {}^c\theta_L^{\mathcal{M}}(f)) - \sum_{L_1 \in \mathcal{L}^L(M)} \varepsilon_M^{L_1}(S) \hat{I}_{L_1}^{L,\Sigma}(\gamma, {}^c\theta_L^{\Sigma}(f)) \\ &+ \hat{I}_M^M(\gamma, {}^c\theta_M^{\mathcal{M}}(f) - {}^c\theta_M^{\Sigma}(f)) \end{aligned}$$

By Theorem 10.1 and the induction hypothesis, we have

$$\sum_{L_1 \in \mathcal{L}^L(M)} \varepsilon_M^{L_1}(S) \hat{I}_{L_1}^{L,\Sigma}(\gamma, {}^c\theta_L^{\Sigma}(f)) = \hat{I}_M^{L,\mathcal{M}}(\gamma, {}^c\theta_L^{\Sigma}(f)) = \hat{I}_M^{L,\mathcal{M}}(\gamma, {}^c\theta_L^{\mathcal{M}}(f)).$$

Therefore the sum over $L \supsetneq M$ on the right-hand side of (42) vanishes and we are left with

$$(43) \qquad {}^cI_M^{\mathcal{M}}(\gamma, f) - \sum_{L_1 \in \mathcal{L}(M)} \varepsilon_M^{L_1}(S)\, {}^cI_{L_1}^{\Sigma}(\gamma, f) = \hat{I}_M^M(\gamma, {}^c\theta_M^{\mathcal{M}}(f) - {}^c\theta_M^{\Sigma}(f)).$$

Since f is taken to belong to $\mathcal{H}^{\text{met}}(G(F_S))$, Lemma 4.4 [6] tells us that the left-hand side of (43) has bounded support as a function of γ in the space of conjugacy classes of $M(F_S)$. The same is therefore true for the right-hand side of (43). For a given $X \in \mathfrak{a}_{M,S}$, the restriction of (43) to

$$\{\gamma \in M(F_S) : H_M(\gamma) = X\}$$

is the orbital integral of a function, $h^X \in \mathcal{H}_{\text{ac}}^{\text{met}}(M(F_S))$, with support in

$$M(F_S)^X = \{\delta \in M(F_S) : H_M(\delta) = X\}.$$

The function h^X vanishes for X outside of a compact set of $\mathfrak{a}_{M,S}$, as the support of (43) is bounded in γ. Furthermore, the equality,

$$h^X(\pi, X) = {}^c\theta_M^{\mathcal{M}}(f, \pi, X) - {}^c\theta_M(f, \pi, X), \ \pi \in \Pi_{\text{temp}}^{\text{met}}(M(F_S)),$$

follows from an application of Fourier inversion on $\mathfrak{a}_{M,S}$ (*cf.* (7.3) [9]).□

Corollary 13.1 (10.3) *Under the assumption of Theorem 13.1, we have*

$$^{c}I_M^{\mathcal{M}}(\gamma) = \sum_{L \in \mathcal{L}(M)} \varepsilon_M^L(S)\, ^{c}I_L^{\mathcal{E}}(\gamma, f)$$

for any $\gamma \in G_{*\mathrm{reg}}$ *and* $f \in \tilde{\mathcal{H}}_{\mathrm{ac}}^{\mathrm{met}}(G(F_S))$.

Proof. In the proof of Theorem 13.1, which follows by the same arguments as that of Theorem 10.2 [10], one shows that (41) is equal to (43) and that (43) vanishes. The corollary follows.□

With Theorem 13.1 in place, the proof of (i) of Theorem 12.1 follows *mutatis mutandis* from the argument on p 145 of [10]. We include it here for the sake of continuity. We wish to show that

$$I_M^{\mathcal{M}}(\pi, 0, f) = I_M(\pi, 0, f), \; \pi \in \Pi_{\mathrm{unit}}^{\mathrm{met}}(M(F_S)).$$

The above distributions are defined by

$$I_{\tilde{M}}(\tilde{\pi}, X, \tilde{f}) = \sum_P \omega_P \sum_L \sum_{\tilde{\rho}} \int_{\varepsilon_P + i\mathfrak{a}_{\tilde{M},S}^*/i\mathfrak{a}_{\tilde{L},S}^*} r_{\tilde{M}}^{\tilde{L}}(\tilde{\pi}_\lambda, \tilde{\rho}_\lambda) I_{\tilde{L}}(\tilde{\rho}_\lambda, h_{\tilde{L}}(X), \tilde{f}) e^{-\lambda(X)} d\lambda,$$

where P, L and $\tilde{\rho}$ are summed over $\mathcal{P}(M)$, $\mathcal{L}(M)$ and $\Sigma(\tilde{M}(F_S))$ respectively. By our assumptions on the normalization of intertwining operators in §5, we have

$$r_{\tilde{M}}^{\tilde{L}}(\pi_{\lambda^*}^*, \rho_{\lambda^*}^*) = r_M^L(\pi_\lambda, \rho_\lambda)$$

(see §6 [9]). Therefore it suffices to prove

$$I_L^{\mathcal{M}}(\rho_\lambda, h_L(X), f) = I_L(\rho_\lambda, h_L(X), f) \tag{44}$$

for all $L \in \mathcal{L}(M)$, $X \in \mathfrak{a}_M$, $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ and $\rho \in \Sigma^{\mathrm{met}}(M(F_S))$ with $\Delta(\rho, \pi) \neq 0$. By using the splitting property for these distributions (Proposition 9.4 [6]) it suffices to prove (44) for $S = \{v\}$. Suppose then that $\rho \in \Sigma^{\mathrm{met}}(M(F_v))$ and that $\Delta(\rho, \pi) \neq 0$ for some $\pi \in \Pi_{\mathrm{unit}}^{\mathrm{met}}(M(F_v))$. Then the central character of ρ must be unitary. From the definition of standard representations, it follows that ρ is either tempered or induced from a proper parabolic subgroup of $M(F_v)$. Suppose first that ρ is tempered. Then by the proof of Lemma 3.1 [6], we have

$$I_L^{\mathcal{M}}(\rho_\lambda, h_L(X), f) = I_L(\rho_\lambda, h_L(X), f) = \begin{cases} 0, & L \neq G \\ f_G(\rho_\lambda, h_G(X)), & L = G \end{cases}.$$

Now suppose that $\rho = \rho_1^M$, where $\rho_1 \in \Sigma^{\mathrm{met}}(M_1(F_v))$ and M_1 is a proper Levi subgroup of M. We apply the descent property (Corollary 8.5 [6]) to (44) and find that it suffices to show

$$\hat{I}_{M_1}^{L_1,\mathcal{M}}(\rho_{1,\lambda}, X_1, f_{L_1}) = \hat{I}_{M_1}^{L_1}(\rho_{1,\lambda}, X_1, f_{L_1}), \tag{45}$$

for $X_1 \in \mathfrak{a}_{M_1}$ and $L_1 \in \mathcal{L}(M_1)$ with $L_1 \neq G$. The induction hypothesis of §9 allows us to apply Theorem 13.1, with G replaced by L_1, in order to obtain (45). The proof is now complete. □

14 More on Normalizing Factors

This section is devoted to the construction of a few additional (G, M) families which we will need in order to compare the spectral sides of the trace formulas. We will return to the actual comparison of the trace formulas in the following section.

The normalizing factors $r_{\tilde{Q}|\tilde{P}}$ of §4 are meromorphic functions of $\mathfrak{a}_{M,\mathbf{C}}^*$ defined in terms of functions

$$r_{\alpha^*} : \Pi(\tilde{M}(F_S)) \times \mathbf{C} \to \mathbf{C},\ \alpha \in \Sigma_Q \cap \Sigma_{\bar{P}}$$

which are meromorphic in $\mathbf{C}$. In fact the following equalities hold:

$$r_{\alpha^*}(\pi^*, s) = r_\alpha(\pi, s),\ \pi \in \Pi^{\mathrm{met}}(M(F_S)),\ s \in \mathbf{C};$$

$$r_{\tilde{Q}|\tilde{P}} = \prod_{\alpha \in \Sigma_Q \cap \Sigma_{\bar{P}}} r_{\alpha^*}(\pi^*, \lambda^*(\alpha^{\vee *})),\ \lambda \in \mathfrak{a}_{M,\mathbf{C}}^*.$$

Let $\tilde{\sigma} = \otimes_v \tilde{\sigma}_v$ and $\pi = \otimes_v \pi_v$ be representations in $\Pi(\tilde{M}(\mathbf{A}))$ and $\Pi^{\mathrm{met}}(M(\mathbf{A}))$ respectively, and assume that

$$\delta(\tilde{\sigma}, \pi) = \prod_v \delta(\tilde{\sigma}_v, \pi_v)$$

does not vanish. Define

$$\tilde{r}_\alpha(\tilde{\sigma}_v, \pi_v, s) = r_{\alpha^*}(\tilde{\sigma}_v, s)^{-1} r_\alpha(\pi_v, s).$$

If $\tilde{\sigma}_v$ and π_v are unramified representations and $|n|_v = 1$ then $\tilde{\sigma}_v = \pi_v^*$ by Corollary 11.1 and so

$$\begin{aligned} r_{\alpha^*}(\tilde{\sigma}_v, s)^{-1} r_\alpha(\pi_v, s) &= r_{\alpha^*}(\pi_v^*, s)^{-1} r_\alpha(\pi_v, s) \\ &= r_\alpha(\pi_v, s)^{-1} r_\alpha(\pi_v, s) \\ &= 1. \end{aligned}$$

We may thus define the infinite product

$$\tilde{r}_\alpha(\tilde{\sigma}, \pi, s) = \prod_v \tilde{r}_\alpha(\tilde{\sigma}_v, \pi_v, s).$$

Set

$$\tilde{r}_{P_1|P_2}(\tilde{\sigma}_{\lambda^*}, \pi_\lambda) = \prod_{\alpha \in \Sigma_{P_1} \cap \Sigma_{\overline{P}_2}} \tilde{r}_\alpha(\tilde{\sigma}, \pi, \lambda(\alpha^\vee)).$$

We define a (G, M) family

$$\tilde{r}_P(\nu, \tilde{\sigma}_{\lambda^*}, \pi_\lambda, P_0) = \tilde{r}_{P|P_0}(\tilde{\sigma}_{\lambda^*}, \pi_\lambda)^{-1} \tilde{r}_{P|P_0}(\tilde{\sigma}_{\lambda^* + \nu^*}, \pi_{\lambda+\nu}),$$

where $P \in \mathcal{P}(M)$ and $\nu \in i\mathfrak{a}_M^*$.

Lemma 14.1 (11.3) *(a) Take $\tilde{\sigma}$ and π as above. Then for each $L \in \mathcal{L}(M)$, $\tilde{r}_M^L(\tilde{\sigma}_{\lambda^*}, \pi_\lambda)$ is independent of P_0 and is also a rational function of the variables $\{\lambda(\alpha^\vee), q_v^{-\lambda(\alpha^\vee)}\}_{v \in S}$, where S is a finite set of valuations outside of which $\tilde{\sigma}$ and π are unramified, and q_v is the order of the residue field of F_v.*
(b) Suppose in addition that $\tilde{\sigma} \in \Pi_{\mathrm{disc}}(\tilde{M}, t)$ and $\pi \in \Pi_{\mathrm{disc}}^{\mathrm{met}}(M, t)$. Then

$$\tilde{r}_{P_1|P_2}(\tilde{\sigma}_{\lambda^*}, \pi_\lambda) = r_{P_1|P_2}(\tilde{\sigma}_{\lambda^*})^{-1} r_{P_1|P_2}(\pi_\lambda), \ P_1, P_2 \in \mathcal{P}(M).$$

In particular, for each $L \in \mathcal{L}(M)$, the function $\tilde{r}_M^L(\tilde{\sigma}_{\lambda^*}, \pi_\lambda)$ is regular for $\lambda \in i\mathfrak{a}_M^*$. Moreover,

$$r_M^L(\pi_\lambda) = \sum_{L_1 \in \mathcal{L}^L(M)} n^{\dim(A_M/A_{L_1})} r_{\tilde{M}}^{\tilde{L}_1}(\tilde{\sigma}_{\lambda^*}) \tilde{r}_{L_1}^L(\tilde{\sigma}_{\lambda^*}, \pi_\lambda).$$

Proof. Part (a) of the lemma follows from the computations on page 149 of [10]. Under the hypotheses of part (b), $r_{\tilde{P}_1|\tilde{P}_2}(\tilde{\sigma}_{\lambda^*})$ and $r_{P_1|P_2}(\pi_\lambda)$ are regular functions in $\lambda \in i\mathfrak{a}_M^*$ (§6 [2]). Thus, if one unravels the definition of $\tilde{r}_{P_1|P_2}(\tilde{\sigma}_{\lambda^*}, \pi_\lambda)$, one obtains the first equality and the the regularity on $i\mathfrak{a}_M^*$. The last equality follows from an application of Lemma 6.5 [1] to

$$r_P(\nu, \pi_\lambda, P_0) = \tilde{r}_P(\nu, \tilde{\sigma}_{\lambda^*}, \pi_\lambda, P_0) r_{\tilde{P}}(\nu/n, \tilde{\sigma}_{\lambda^*}, \tilde{P}_0). \square$$

We may define further (G, M) families along the same lines as the definition of the previous one. If we replace π in the above discussion with some $\rho \in \Sigma^{\mathrm{met}}(M(\mathbf{A}))$ such that $\Delta(\tilde{\sigma}, \rho) \neq 0$, we obtain the (G, M) family

$$\tilde{r}_P(\nu, \tilde{\sigma}_{\lambda^*}, \rho_\lambda, P_0) = \tilde{r}_{P|P_0}(\tilde{\sigma}_{\lambda^*}, \rho_\lambda)^{-1} \tilde{r}_{P|P_0}(\tilde{\sigma}_{\lambda^*+\nu^*}, \rho_{\lambda+\nu}).$$

We define yet another (G, M) family for representations $\tilde{\sigma}_1, \tilde{\sigma}_2 \in \Pi(\tilde{M}(\mathbf{A}))$ such that $\delta(\tilde{\sigma}_i, \pi) \neq 0$ for $i = 1, 2$ and some $\pi \in \Pi^{\mathrm{met}}(M(\mathbf{A}))$. Set

$$\tilde{r}_P(\nu, \tilde{\sigma}_{1,\lambda^*}, \tilde{\sigma}_{2,\lambda^*}, P_0) = \tilde{r}_P(\nu, \tilde{\sigma}_{1,\lambda^*}, \pi, P_0) \tilde{r}_P(\nu, \tilde{\sigma}_{2,\lambda^*}, \pi_\lambda, P_0)^{-1}.$$

This (G, M) family is independent of π. Lemma 6.5 of [1] applied to this last (G, M) family yields

$$\tilde{r}_M^L(\tilde{\sigma}_{1,\lambda^*}, \pi_\lambda) = \sum_{L_1 \in \mathcal{L}^L(M)} \tilde{r}_M^{L_1}(\tilde{\sigma}_{1,\lambda^*}, \tilde{\sigma}_{2,\lambda^*}) \tilde{r}_{L_1}^L(\tilde{\sigma}_{2,\lambda^*}, \pi_\lambda). \tag{46}$$

For arbitrary $\tilde{\sigma} \in \Pi(\tilde{M}(\mathbf{A}))$, $\rho = \otimes_v \rho_v \in \Sigma^{\mathrm{met}}(M(\mathbf{A}))$ and $\pi \in \Pi^{\mathrm{met}}(M(\mathbf{A}))$. Set

$$r_P(\pi_\lambda, \rho_\lambda) = \Delta(\pi, \rho) \tilde{r}_P(\pi_\lambda, \rho_\lambda),$$

$$r_P(\nu, \tilde{\sigma}_{\lambda^*}, \pi_\lambda) = \delta(\tilde{\sigma}, \pi) \tilde{r}_P(\nu, \tilde{\sigma}_{\lambda^*}, \pi_\lambda),$$

and

$$r_P(\nu, \tilde{\sigma}_{\lambda^*}, \rho_\lambda) = \Delta(\tilde{\sigma}, \rho) \tilde{r}_P(\nu, \tilde{\sigma}_{\lambda^*}, \rho_\lambda).$$

In the following lemma we compare the last (G, M) family to another via map (8).

Lemma 14.2 (11.4) *For each $L \in \mathcal{L}(M)$ we have*

$$r_M^L(\tilde{\sigma}_{\lambda^*}, \rho_\lambda) = n^{-\dim(A_M/A_L)} r_{\tilde{M}}^{\tilde{L}}(\tilde{\sigma}_{\lambda^*}, \rho_{\lambda^*}^*).$$

Proof. According to Lemma 6.2 [1],

$$
\begin{aligned}
&r_{\tilde{M}}^{\tilde{L}}(\tilde{\sigma}_{\lambda^*}, \rho^*_{\lambda^*}) \\
&= \Delta(\tilde{\sigma}, \pi) \prod_v \lim_{\nu^* \to 0} \sum_{P \in \mathcal{P}^L(M)} \tilde{r}_{\tilde{P}}^{\tilde{L}}(\nu^*, \tilde{\sigma}_{v,\lambda^*}, \pi^*_{v,\lambda^*}) \mathrm{vol}\left(\mathfrak{a}_{\tilde{M}}^{\tilde{L}} / \mathbf{Z}((\Delta_M^L)^{\vee *})\right) \prod_{\alpha \in \Delta_M^L} (\nu^*(\alpha^{\vee *})^{-1}) \\
&= \Delta(\tilde{\sigma}, \pi) \prod_v \lim_{\nu \to 0} \sum_{P \in \mathcal{P}^L(M)} \tilde{r}_P^L(\nu, \tilde{\sigma}_{v,\lambda^*}, \pi_{v,\lambda}) \mathrm{vol}\left(\mathfrak{a}_{\tilde{M}}^{\tilde{L}} / \mathbf{Z}((\Delta_M^L)^{\vee *})\right) \prod_{\alpha \in \Delta_M^L} (\nu^*(\alpha^{\vee *})^{-1}) \\
&= \Delta(\tilde{\sigma}, \pi) \prod_v \lim_{\nu \to 0} \sum_{P \in \mathcal{P}^L(M)} \tilde{r}_P^L(\nu, \tilde{\sigma}_{v,\lambda^*}, \pi_{v,\lambda}) n^{\dim(A_M/A_L)} \mathrm{vol}\left(\mathfrak{a}_M^L / \mathbf{Z}((\Delta_M^L)^{\vee})\right) \prod_{\alpha \in \Delta_M^L} \nu(\alpha^\vee)^{-1} \\
&= n^{\dim(A_M/A_L)} \Delta(\tilde{\sigma}, \pi) \tilde{r}_M^L(\tilde{\sigma}_{\lambda^*}, \pi_\lambda) \\
&= n^{\dim(A_M/A_L)} r_M^L(\tilde{\sigma}_{\lambda^*}, \pi_\lambda). \square
\end{aligned}
$$

The other (G, M) families defined in this section satisfy versions of Lemma 14.2 as well. These versions are proven similarly.

15 A formula for $I_t^{\mathcal{M}}$

As announced in §12, the object of this section is to express the spectral side of the trace formula for $\tilde{G}$ in a manner that is compatible with (32). This amounts to expressing the spectral side in terms of the global datum $a^{M,\mathcal{M}}$ and the set of representations $\Pi^{\mathcal{M}}(M, t)$. At the end of this section, we apply these results to a comparison of the spectral sides of the trace formulas.

We first deal with the local spectral terms of $\tilde{G}$.

Lemma 15.1 (12.1) *Suppose that $\tilde{\sigma} \in \Pi_{\mathrm{unit}}(\tilde{M}(\mathbf{A})^1)$. Then the distribution*

$$(47) \qquad I_{\tilde{M}}(\tilde{\sigma}, f^*) = \sum_{L \in \mathcal{L}(M)} \sum_{\pi \in \Pi^{\mathrm{met}}(M(\mathbf{A})^1)} \int_{\epsilon_M + i\mathfrak{a}_M^*/i\mathfrak{a}_L^*} r_M^L(\tilde{\sigma}_{\lambda^*}, \pi_\lambda) I_L^{\mathcal{M}}(\pi_\lambda, f) d\lambda.$$

Proof. This proof is almost identical to the proof of Lemma 12.1 [10]. It is included so that the reader may feel a sense of continuity. Any statements which seem unjustified may be compared to the analogous statements in Lemma 12.1, where the detail are given. To begin, relabel the summation index L by L_1. We then replace $I_{L_1}^{\mathcal{M}}(\pi_\lambda, f)$ with the expression

$$\sum_{Q\in\mathcal{P}(L_1)} \omega_Q \sum_{L\in\mathcal{L}(L_1)} \sum_{\rho\in\Pi^{\mathrm{met}}(M(\mathbf{A})^1)} \int_{\epsilon_Q+i\mathfrak{a}^*_{L_1}/i\mathfrak{a}^*_L} r^L_{L_1}(\pi_{\lambda+\mu},\rho_{\lambda+\mu}) I^{\mathcal{M}}_L(\rho_{\lambda+\mu},f)d\mu.$$

We deform the contour of integration in μ so that (47) is equal to the sum over $L_1, L \in \mathcal{L}(M)$, with $L_1 \subset L$, of

$$\sum_{\pi\in\Pi^{\mathrm{met}}(M(\mathbf{A})^1)} \sum_{\rho\in\Sigma^{\mathrm{met}}(M(\mathbf{A})^1)} \int_{\epsilon_M+i\mathfrak{a}^*_M/i\mathfrak{a}^*_L} r^L_M(\tilde\sigma_{\lambda^*},\pi_\lambda) r^L_{L_1}(\pi_\lambda,\rho_\lambda) I^{\mathcal{M}}_L(\rho_\lambda,f)d\lambda.$$

Taking the sums over L_1 and π inside the integral we find that

$$\begin{aligned}
&\sum_{\pi\in\Pi^{\mathrm{met}}(M(\mathbf{A})^1)} \sum_{L_1\in\mathcal{L}^L(M)} r^{L_1}_M(\tilde\sigma_{\lambda^*},\pi_\lambda) r^L_{L_1}(\pi_\lambda,\rho_\lambda)\\
&= \tilde r^L_M(\tilde\sigma_{\lambda^*},\rho_\lambda) \sum_{\pi\in\Pi^{\mathrm{met}}(M(\mathbf{A})^1)} \delta(\tilde\sigma,\pi)\Delta(\pi,\rho)\\
&= \tilde r^L_M(\tilde\sigma_{\lambda^*},\rho_\lambda) \sum_{\rho_1\in\Sigma^{\mathrm{met}}(M(\mathbf{A})^1)} \Delta(\tilde\sigma,\rho_1) \sum_{\pi\in\Pi^{\mathrm{met}}(M(\mathbf{A})^1)} \Gamma(\rho_1,\pi)\Delta(\pi,\rho)\\
&= \tilde r^L_M(\tilde\sigma_{\lambda^*},\rho_\lambda)\Delta(\tilde\sigma,\rho)\\
&= r^L_M(\tilde\sigma_{\lambda^*},\rho_\lambda).
\end{aligned}$$

The right-hand side of equation (47) is now of the form

$$\sum_{L\in\mathcal{L}(M)} \sum_{\rho\in\Sigma^{\mathrm{met}}(M(\mathbf{A})^1)} \int_{\epsilon_M+i\mathfrak{a}^*_M/i\mathfrak{a}^*_L} r^L_M(\tilde\sigma_{\lambda^*},\rho_\lambda) I^{\mathcal{M}}_L(\rho_\lambda,f)d\lambda. \tag{48}$$

By Lemma 14.2 we have

$$\begin{aligned}
&\sum_{\rho\in\Sigma^{\mathrm{met}}(M(\mathbf{A})^1)} r^L_M(\tilde\sigma_{\lambda^*},\rho_\lambda) I^{\mathcal{M}}_L(\rho_\lambda,f)\\
&= n^{-\dim(A_M/A_L)} \sum_{\rho\in\Sigma^{\mathrm{met}}(M(\mathbf{A})^1)} r^{\bar L}_{\bar M}(\tilde\sigma_{\lambda^*},\rho^*_{\lambda^*}) I_{\bar L}(\rho^*_{\lambda^*},f^*)\\
&= \sum_{\bar\rho\in\Sigma(\bar M(\mathbf{A})^1)} r^{\bar L}_{\bar M}(\tilde\sigma_{\lambda^*},\bar\rho_{\lambda^*},f^*) I_{\bar L}(\bar\rho_{\lambda^*},f^*).
\end{aligned}$$

Substituting back into (48) and noting that $d\lambda = n^{\dim(A_M/A_L)}d\lambda^*$, we obtain

$$\sum_{L\in\mathcal{L}(M)} \sum_{\bar\rho\in\Sigma(\bar M(\mathbf{A})^1)} \int_{\epsilon_M+i\mathfrak{a}^*_M/i\mathfrak{a}^*_L} r^{\bar L}_{\bar M}(\tilde\sigma_{\lambda^*},\bar\rho_{\lambda^*}) I_{\bar L}(\rho^*_{\lambda^*},f^*)d\lambda^*.$$

Since $\tilde{\sigma}$ is unitary this last expression is equal to $I_{\tilde{M}}(\tilde{\sigma}, f^*)$.□

Before considering the spectral side of the trace formula for $\tilde{G}$, we define

$$r_M^L(\tilde{\sigma}_{\lambda^*}) = n^{\dim(A_M/A_L)} r_{\tilde{M}}^{\tilde{L}}(\tilde{\sigma}_{\lambda^*})$$

for $L \in \mathcal{L}(M)$, $\tilde{\sigma} \in \Pi_{\text{disc}}^{\text{met}}(M,t)$ and $\lambda \in \mathfrak{a}_{M,\mathbf{C}}^*$. Observe that Lemma 14.1 part (b) may be rewritten as

$$r_M^L(\pi_\lambda) = \sum_{L_1 \in \mathcal{L}^L(M)} r_M^{L_1}(\tilde{\sigma}_{\lambda^*}) \tilde{r}_{L_1}^L(\tilde{\sigma}_{\lambda^*}, \pi_\lambda).$$

This definition will make some impending computations tidier.

Proposition 15.1 (12.2) *Suppose that $t \geq 0$ and $f \in \mathcal{H}^{\text{met}}(G(F_S))$. Then*

$$I_t^{\mathcal{M}}(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi^{\mathcal{M}}(M,t)} a^{M,\mathcal{M}}(\pi) I_M^{\mathcal{M}}(\pi, f) d\pi. \tag{49}$$

Proof. From previous formulas we know that $I_t^{\mathcal{M}}(f)$ equals

$$\sum_{\{M_1, M \in \mathcal{L}: M \supset M_1\}} |W_0^{M_1}||W_0^G|^{-1} \sum_{\tilde{\sigma} \in \Pi_{\text{disc}}(\tilde{M}_1, t)} \int_{i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_M^*} a_{\text{disc}}^{\tilde{M}}(\tilde{\sigma}) r_{\tilde{M}_1}^{\tilde{M}}(\tilde{\sigma}_{\lambda^*}) I_{\tilde{M}}(\tilde{\sigma}_{\lambda^*}, f^*) d\lambda^*.$$

We replace $r_{\tilde{M}_1}^{\tilde{M}}(\tilde{\sigma}_{\lambda^*})$ with $n^{\dim(A_{M_1}/A_M)} r_{M_1}^M(\tilde{\sigma}_{\lambda^*})$ and $I_{\tilde{M}}(\tilde{\sigma}_{\lambda^*}, f^*)$ with the expression derived in Lemma 15.1 in order to obtain the equality of $r_{M_1}^M(\tilde{\sigma}_{\lambda^*}) I_{\tilde{M}}(\tilde{\sigma}_{\lambda^*}, f^*)$ with

$$n^{\dim(A_{M_1}/A_M)} \sum_{L \in \mathcal{L}(M)} \sum_{\pi_1 \in \Pi^{\text{met}}(M_1(\mathbf{A})^1, t)} \int_{\epsilon_M + i\mathfrak{a}_M^*/i\mathfrak{a}_L^*} r_{M_1}^M(\tilde{\sigma}_{\lambda^*}) r_M^L(\tilde{\sigma}_{\lambda^* + \mu^*}, \pi_{1,\lambda+\mu}) I_L^{\mathcal{M}}(\pi_{1,\lambda+\mu}, f) d\mu.$$

Using the identity $d\lambda^* = n^{-\dim(A_{M_1}/A_M)} d\lambda$, and deforming the contour of integration appropriately we find that $I_t^{\mathcal{M}}(f)$ equals the sum over $M_1 \subset M \subset L \in \mathcal{L}$, and $\pi_1 \in \Pi^{\text{met}}(M_1(\mathbf{A})^1, t)$ of

$$|W_0^{M_1}||W_0^G|^{-1} \int_{\epsilon_M + i\mathfrak{a}_{M_1}^*/i\mathfrak{a}_L^*} \sum_{\tilde{\sigma} \in \Pi_{\text{disc}}(\tilde{M}, t)} a_{\text{disc}}^{\tilde{M}_1}(\tilde{\sigma}) r_{M_1}^M(\tilde{\sigma}_{\lambda^*}) r_M^L(\tilde{\sigma}_{\lambda^*}, \pi_{1,\lambda}) I_L^{\mathcal{M}}(\pi_{1,\lambda}, f) d\lambda. \tag{50}$$

The term $r_M^L(\tilde{\sigma}_{\lambda^*}, \pi_{1,\lambda})$ in the above sum vanishes unless $\delta(\tilde{\sigma}, \pi_1) \neq 0$. Fix some $\tilde{\sigma}_1 \in \Pi_{\text{disc}}(\tilde{M}_1(\mathbf{A})^1, t)$ such that $\delta(\tilde{\sigma}_1, \pi_1) \neq 0$. Then for any other $\tilde{\sigma}$ with $\delta(\tilde{\sigma}, \pi_1) \neq 0$ we may write

$$r_{M_1}^M(\tilde{\sigma}_{\lambda^*}) = \sum_{\{L_1: M_1 \subset L \subset M\}} r_{M_1}^{L_1}(\tilde{\sigma}_{1,\lambda^*}) \tilde{r}_{L_1}^M(\tilde{\sigma}_{1,\lambda^*}, \tilde{\sigma}_{\lambda^*}).$$

We substitute this expression into (50) and deform the contour of integration from $\varepsilon_M + i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_L$ to $\varepsilon_{L_1} + i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_L$ for some small regular point ε_{L_1} in $\mathfrak{a}^*_{L_1}$. We then bring the sum over M inside the integral. Notice that

$$\sum_{\{M:L_1\subset M\subset L\}} \tilde{r}^M_{L_1}(\tilde{\sigma}_{1,\lambda^*}, \tilde{\sigma}_{\lambda^*}) r^L_M(\tilde{\sigma}_{\lambda^*}, \pi_{1,\lambda}) = \delta(\tilde{\sigma}, \pi_1)\tilde{r}^L_{L_1}(\tilde{\sigma}_{\lambda^*}, \pi_{1,\lambda})$$

by equation (46). Observe also that

$$\sum_{\tilde{\sigma}\in\Pi_{\mathrm{disc}}(\tilde{M}_1,t)} a^{\tilde{M}_1}_{\mathrm{disc}}(\tilde{\sigma})\delta(\tilde{\sigma}, \pi_1) = a^{M_1,\mathcal{M}}_{\mathrm{disc}}(\pi_1).$$

Now (50) may be written in the form

$$(51) \qquad \sum_{\{L_1:M_1\subset L_1\subset L\}} \int_{\varepsilon_{L_1}+i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_L} a^{M_1,\mathcal{M}}_{\mathrm{disc}}(\pi_1) r^{L_1}_{M_1}(\bar{\sigma_{\lambda^*}})\tilde{r}^L_{L_1}(\tilde{\sigma}_{\lambda^*}, \pi_{1,\lambda}) I^{\mathcal{M}}_L(\pi_{1,\lambda}, f)d\lambda.$$

If $\tilde{M_1} = G$ then (51) reduces to

$$(52) \qquad \sum_{\pi_1\in\Pi^{\mathcal{M}}_{\mathrm{disc}}(G,t)} a^{\mathcal{M}}_{\mathrm{disc}}(\pi_1) I^{\mathcal{M}}_G(\pi_1, f).$$

If $M_1 \neq G$ then the induction hypothesis stated after the definition of $a^{M_1,\mathcal{M}}_{\mathrm{disc}}$ yields

$$a^{M_1,\mathcal{M}}_{\mathrm{disc}}(\pi_1) = a^{M_1}_{\mathrm{disc}}(\pi_1)$$

for $\pi_1 \in \Pi^{\mathrm{met}}(M_1, t)$. It is immediate from this equality that $a^{M_1,\mathcal{M}}_{\mathrm{disc}}(\pi_1)$ vanishes unless $\pi \in \Pi^{\mathrm{met}}_{\mathrm{disc}}(M_1, t)$. By a variant of Lemma 14.1, the integrand of (51) is analytic for λ near $i\mathfrak{a}^*_{M_1}$, we may deform the contour of integration from $\varepsilon_{L_1} + i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_L$ to $i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_L$. This allows us to take the sum over L_1 inside the integral. It is a simple exercise in (G, M) families to show that

$$\sum_{\{L_1:M_1\subset L_1\subset L\}} r^{L_1}_{M_1}(\tilde{\sigma}_{1,\lambda^*})\tilde{r}^L_{L_1}(\tilde{\sigma}_{1,\lambda^*}, \pi_{1,\lambda}) = r^L_{M_1}(\pi_{1,\lambda}).$$

Thus (51) is equal to

$$\int_{i\mathfrak{a}^*_{M_1}/i\mathfrak{a}^*_L} a^{M_1,\mathcal{M}}_{\mathrm{disc}}(\pi_1) r^L_{M_1}(\pi_{1,\lambda}) I^{\mathcal{M}}_L(\pi_{1,\lambda}, f)d\lambda,$$

for $M_1 \neq G$. Combining this expression with (52) we obtain

$$I_t^{\mathcal{M}}(f) = \sum_{L \in \mathcal{L}} |W_0^L||W_0^G|^{-1} \int_{\Pi^{\mathcal{M}}(L,t)} a^{L,\mathcal{M}}(\pi) I_L^{\mathcal{M}}(\pi, f) d\pi. \square$$

If we turn our attention to the spectral side of the trace formula of G, then we must show that it may be expressed exclusively in terms of representations belonging to $\Pi^{\mathrm{met}}(M(F_S))$. This is essential if we wish to compare it to the expansion of Lemma 15.1 and may be regarded as a spectral vanishing property.

Lemma 15.2 *Suppose $f \in \mathcal{H}^{\mathrm{met}}(G(F_S))$, $X \in \mathfrak{a}_{M,S}$ and $\rho \in \Sigma(M(F_S))$, such that ρ has unitary central character. Then $I_M(\rho, X, f) = 0$ unless $\rho \in \Sigma^{\mathrm{met}}(M(F_S))$.*

Proof. Suppose $I_M(\rho_\lambda, X, f) \neq 0$. We first prove the lemma for the case $M = G$. In this case

$$I_G(\rho, X, f) = f_G(\rho, X) = \int_{i\mathfrak{a}_{G,S}^*} \mathrm{tr}(\rho_\lambda(f)) e^{-\lambda(X)} d\lambda.$$

Hence $\mathrm{tr}(\rho_\lambda(f)) \neq 0$ for some $\lambda \in i\mathfrak{a}_{G,S}^*$, and so by definition $\rho_\lambda \in \Pi_{\mathrm{temp}}^{\mathrm{met}}(G(F_S)) \subset \Sigma^{\mathrm{met}}(G(F_S))$. Since $\Sigma^{\mathrm{met}}(G(F_S))$ is stable under twists by elements in $i\mathfrak{a}_{G,S}^*$, ρ also belongs to $\Sigma^{\mathrm{met}}(G(F_S))$.

We may now assume inductively that $I_M^L(\rho, X, h) = 0$ unless $\rho \in \Sigma^{\mathrm{met}}(M(F_S))$, given $\rho \in \Sigma(M(F_S))$, $X \in \mathfrak{a}_{M,S}$, $h \in \mathcal{H}^{\mathrm{met}}(L(F_S))$ and $L \in \mathcal{L}(M)$ with $L \neq G$. Suppose $I_M(\rho, X, f) \neq 0$ and $\rho = \pi_\lambda^M$ for some $\pi \in \Pi_{\mathrm{temp}}(M_1(F_S))$, $\lambda \in \mathfrak{a}_{M_1,\mathbb{C}}^*$ and Levi subgroup M_1 of M. If $M_1 = M$, then ρ is tempered, since it has unitary central character. Moreover by Lemma 3.1 of [6],

$$I_M(\rho, X, f) = \begin{cases} f_G(\rho, X), & \text{if } M = G \\ 0, & \text{otherwise} \end{cases}.$$

This case has already been taken care of. Now suppose that M_1 is a proper Levi subgroup of M. According to Corollary 8.5 [6], with $\Pi(M(F_S))$ replaced by $\Sigma(M(F_S))$, we have

$$I_M(\rho, X, f) = \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L) \hat{I}_{M_1}^L(\pi_\lambda, X, f_L).$$

This implies that $\hat{I}_M^{L_1}(\pi_\lambda, X, f_{L_1}) \neq 0$ for some $L_1 \in \mathcal{L}(M_1)$. By the induction assumption, $\hat{I}_{M_1}^L(\pi_\lambda, X, f_{L_1})$ vanishes unless $\pi_\lambda \in \Sigma^{\mathrm{met}}(M_1(F_S))$. Hence $\rho = \pi_\lambda^M \in \Sigma^{\mathrm{met}}(M(F_S))$ by transitivity of induction.$\square$

Corollary 15.1 *Let $\pi \in \Pi_{\text{unit}}(M(F_S))$, $f \in \mathcal{H}^{\text{met}}(G(F_S))$ and $X \in \mathfrak{a}_{M,S}$. Then $I_M(\pi, X, f) = 0$ unless $\pi \in \Pi^{\text{met}}_{\text{unit}}(M(F_S))$.*

Proof. Suppose $I_M(\pi, X, f) \neq 0$. Then by definition

$$I_M(\pi, X, f) = \sum_P \omega_P \sum_L \sum_{\rho}{}' r^L_M(\pi_\lambda, \rho_\lambda) I_L(\rho_\lambda, h_L(X), f) e^{-\lambda(X)} d\lambda \neq 0,$$

where P, L and ρ are summed over $\mathcal{P}(M)$, $\mathcal{L}(M)$ and $\Sigma^{\text{met}}(M(F_S))$ respectively. In particular

$$r^L_M(\pi_\lambda, \rho'_\lambda) I_L(\rho'_\lambda, h_L(X), f) \neq 0,$$

for some $\rho' \in \Sigma^{\text{met}}(M(F_S))$. It follows from

$$r^L_M(\pi_\lambda, \rho'_\lambda) \neq 0$$

that $\Delta(\pi, \rho') \neq 0$, where

$$\text{tr}\pi = \sum_{\rho \in \Sigma(M(F_S))} \Delta(\pi, \rho) \text{tr}\rho$$

(*cf.* 6.4 [9]). Since π is unitary, ρ' must have unitary central character. As a result ρ'^L also has unitary central character. Since $I_L(\rho'_\lambda, h_L(X), f) \neq 0$, Lemma 15.2 implies that $\rho' \in \Sigma^{\text{met}}(M(F_S))$. Thus π also belongs to $\Pi^{\text{met}}_{\text{unit}}(M(F_S))$ by Lemma 3.1.□

Proposition 15.2 *Suppose that $t \geq 0$ and $f \in \mathcal{H}^{\text{met}}(G(\mathbb{A}))$. Then*

$$I_t(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi^{\text{met}}(M,t)} a^M(\pi) I_M(\pi, f) d\pi.$$

Proof. As $\Pi(M, t) \subset \Pi_{\text{unit}}(M(\mathbb{A})^1)$, we know from Corollary 15.1 that $I_M(\pi, f)$ vanishes for any $\pi \in \Pi(M, t)$ unless $\pi \in \Pi^{\text{met}}(M(\mathbb{A})^1)$. The lemma now follows as $\Pi^{\text{met}}(M, t) = \Pi(M, t) \cap \Pi^{\text{met}}(M(\mathbb{A})^1)$.□

We can now apply Proposition 15.1 and Proposition 15.2 to obtain a striking simplification in the comparison between the spectral sides of the trace formulas.

Lemma 15.3 (12.3) *Suppose that $t \geq 0$ and $f \in \mathcal{H}^{\text{met}}(G(\mathbb{A}))$. Then*

$$I_t^{\mathcal{M}}(f) - I_t(f) = \sum_{\pi \in \tilde{\Pi}^{\text{met}}(G(\mathbb{A})^1, t)} (a^{\mathcal{M}}_{\text{disc}}(\pi) - a_{\text{disc}}(\pi)) \text{tr}(f^1),$$

where f^1 is the restriction of f to $G(\mathbb{A})^1$.

Proof. This proof is almost identical to Lemma 12.3 [10] and is included solely for the sake of continuity. Consider the difference of (49) and

$$I_t(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi^{\mathrm{met}}(M,t)} a^M(\pi) I_M(\pi, f) d\pi.$$

If $M_1 \in \mathcal{L}$ and $M_1 \subsetneq M \subset G$ then the induction hypothesis of §12 implies that

$$a^{M,\mathcal{M}}(\pi) = a^M(\pi), \;\; \pi \in \Pi_{M_1}^{\mathrm{met}}(M, t),$$

and $\Pi_{M_1}^{\mathcal{M}}(M,t) = \Pi_{M_1}^{\mathrm{met}}(M,t)$. If π is not unitary both $a^{M,\mathcal{M}}(\pi)$ and $a^M(\pi)$ vanish. When π is unitary we know from §13 that

$$I_M^{\mathcal{M}}(\pi, f) = I_M(\pi, f).$$

Therefore, the only terms which remain in the difference, $I_t^{\mathcal{M}}(f) - I_t(f)$, are the ones indexed by $M_1 = M = G$. These terms are of the form

$$I_G^{\mathcal{M}}(\pi, f) = \mathrm{tr}\pi^*((f^1)^*) = \mathrm{tr}\pi(f^1) = I_G(\pi, f),$$

where $\pi \in \Pi_{\mathrm{disc}}^{\mathrm{met}}(G, t)$.□

16 The map ε_M

Having simplified the comparison of the spectral sides of the trace formulas, we attempt to do the same for the geometric sides. We may lighten the burden of this task considerably by making further restrictions on f and by adding yet another induction hypothesis.

Let $\mathcal{H}^{\mathrm{met}}(G(\mathbf{A}), M)$ be the subspace of $\mathcal{H}^{\mathrm{met}}(G(\mathbf{A}))$ spanned by functions

$$f = \prod_v f_v, \;\; f_v \in \mathcal{H}^{\mathrm{met}}(G(F_v)),$$

which have the following property. For two finite places v_1 and v_2, which are not in $S_{(n)}$,

$$f_{v_i, L} = 0, \;\; L \in \mathcal{L}, \; i = 1, 2,$$

unless L contains a conjugate of M. If S is a finite set of places with the closure property, which contains $S_{(n)}$ and at least two other finite places, we define $\mathcal{H}^{\mathrm{met}}(G(F_S), M)$ the same way.

From this point on we fix $M \in \mathcal{L}$ such that $M \neq G$. The additional induction hypothesis is that

$$I_L^{\mathcal{M}}(\gamma, f) = I_L^{\mathcal{E}}(\gamma, f),$$

for all $\gamma \in L(F_S)$, S containing S' and $L \in \mathcal{L}(M)$ with $L \neq M$.

The proof of Lemma 13.1 [10] may be imitated to obtain the following lemma.

Lemma 16.1 (13.1) *For $f \in \mathcal{H}^{\mathrm{met}}(G(\mathbb{A}), M)$, the distribution*

$$I^{\mathcal{M}}(f) - I(f)$$

equals the sum of

$$n|W(\mathfrak{a}_M)|^{-1} \sum_{\gamma \in (M(F))_{M,S}/\mu_n^M} a^M(S,\gamma)(I_M^{\mathcal{M}}(\gamma, f) - I_M^{\mathcal{E}}(\gamma, f))$$

and

$$\sum_{\delta \in A_G(F)} \sum_{u \in (\mathcal{U}_G(F))_{G,S}} (a^{\mathcal{M}}(S,u) - a(S,u)) I_G(\delta u, f).$$

Proof. The lemma follows from the splitting properties (Lemma 6.2, (17)) and the properties of f. See Lemma 13.1 [10] for details.□

It was indicated in §10 that a version of Theorem 10.1 would be proven. This new version is the content of the following proposition. It is more general in that the hypothesis,

$$I_L^{\mathcal{M}}(\gamma, f) = I_L^{\mathcal{E}}(\gamma, f), \text{ for } S \supset S_{(n)},$$

is omitted. However, the map replacing $f \mapsto \varepsilon_M(S)f$ of Theorem 10.1 is more complicated.

Proposition 16.1 (13.2) *There are unique maps*

$$\varepsilon_L : \mathcal{H}^{\mathrm{met}}(G(F_S))^0 \to \mathcal{I}_{\mathrm{ac}}(\tilde{L}(F_S)),\ L \in \mathcal{L}(M),$$

such that

$$I_M^{\mathcal{M}}(\gamma, f) = \sum_{L \in \mathcal{L}(M)} I_M^{L,\mathcal{E}}(\gamma, \varepsilon_L(f)),\ \gamma \in M(F_S),\ f \in \mathcal{H}^{\mathrm{met}}(G(F_S))^0.$$

The maps have the descent property

$$\varepsilon_M(f)_{M_1} = \sum_{L\in\mathcal{L}(M_1)} d^G_{M_1}(M,L)\hat{\varepsilon}^L_{M_1}(f_L),\ M_1 \subset M,$$

and the splitting property

$$\varepsilon_M(f) = \sum_{L_1,L_2\in\mathcal{L}(M)} d^G_M(L_1,L_2)\hat{\varepsilon}^{L_1}_M(f_{1,L_1})\hat{\varepsilon}^{L_2}_M(f_{2,L_2}),$$

for $f = f_1f_2$ as in Proposition 6.2.

Remark. It follows from the first equality of Proposition 16.1, the induction hypothesis of §9, and the induction hypothesis of this section that

$$I^{\mathcal{M}}_M(\gamma,f) - I^{\mathcal{E}}_M(\gamma,f) = \hat{I}^{M,\mathcal{E}}_M(\gamma,\varepsilon_M(f)),$$

for $\gamma \in M(F_S))$ and $f \in \mathcal{H}^{\mathrm{met}}(G(F_S))^0$. See (13.1**) [10] for details.

Proof. Define

$$\varepsilon_M(\gamma,f) = I^{\mathcal{M}}_M(\gamma,f) - \sum_{L\supsetneq M} \hat{I}^{L,\mathcal{E}}_M(\gamma,\varepsilon_L(S)f_L),\ \gamma\in M(F_S).$$

Following the proof of Proposition 13.2 [10], we need only show that $\varepsilon_M(\gamma,f)$ is the orbital integral of a function in $\mathcal{H}^{\mathrm{met}}_{\mathrm{ac}}(M(F_S))$ for $\gamma\in M(F_S)\cap G_{*\mathrm{reg}}$ and $S=\{v\}$. Define

$${}^c\varepsilon_M(\gamma,f) = {}^cI^{\mathcal{M}}_M(\gamma,f) - \sum_{L\supsetneq M}\varepsilon^L_M(S)\,{}^cI^{\mathcal{E}}_L(\gamma,f)$$

for $\gamma\in M(F_S)\cap G_{*\mathrm{reg}}$. By Lemma 4.4 [6], ${}^c\varepsilon_M(\gamma,f)$ has bounded support as a function of γ in the space of conjugacy classes in $M(F_S)$. The following lemma relates ${}^c\varepsilon_M(\gamma,f)$ to $\varepsilon_M(\gamma,f)$.

Lemma 16.2 (13.3) *Suppose that γ belongs to $M(F_S)\cap G_{\mathrm{reg}}$. Then*

$$\varepsilon_M(\gamma,f) - {}^c\varepsilon_M(\gamma,f) = \hat{I}^M_M(\gamma,\theta^{\mathcal{M}}_M(f)-\theta^{\mathcal{E}}_M(f)) = \hat{I}^M_M(\gamma,{}^c\theta^{\mathcal{E}}_M(f)-{}^c\theta^{\mathcal{M}}_M(f)).$$

Proof. We may apply the descent properties of both $I_M^{L,\Sigma}$ and $\varepsilon_M(S)$ to the sum

$$\sum_{L\supsetneq M} \hat{I}_M^{L,\Sigma}(\gamma, \varepsilon_L(S)f_L)$$

occurring in the definition of $\varepsilon_M(\gamma, f)$ to obtain

$$\varepsilon_M(\gamma, f) = I_M^{\mathcal{M}}(\gamma, f) - \sum_{\{L_1\in\mathcal{L}(M):L_1\neq G\}} \varepsilon_M^{L_1}(S) I_{L_1}^{\Sigma}(\gamma, f).$$

It follows from (37) and (39) that

$$\begin{aligned} &\varepsilon_M(\gamma, f) - {}^c\varepsilon_M(\gamma, f) \\ &= (I_M^{\mathcal{M}}(\gamma, f) - {}^cI_M^{\mathcal{M}}(\gamma, f)) - \sum_{\{L_1\in\mathcal{L}(M):L_1\neq G\}} \varepsilon_M^{L_1}(S)(I_{L_1}^{\Sigma}(\gamma, f) - {}^cI_{L_1}^{\Sigma}(\gamma, f)) \\ &= \sum_{L\neq G}({}^c\hat{I}_M^{L,\mathcal{M}}(\gamma, \theta_L^{\mathcal{M}}(f)) - \sum_{L_1\in\mathcal{L}^L(M)} \varepsilon_M^{L_1}(S)\, {}^c\hat{I}_{L_1}^{L,\Sigma}(\gamma, \theta_L^{\Sigma}(f))). \end{aligned}$$

We may apply the induction hypothesis at the beginning of this section to the terms in the above sum for which $L \neq M$. Theorem 13.1 and Corollary 13.1 hold in this case and so

$${}^c\hat{I}_M^{L,\mathcal{M}}(\gamma, \theta_L^{\mathcal{M}}(f)) = {}^c\hat{I}_M^{L,\mathcal{M}}(\gamma, \theta_L^{\Sigma}(f)) = \sum_{L_1\in\mathcal{L}^L(M)} \varepsilon_M^{L_1}(S)\, {}^c\hat{I}_{L_1}^{L,\Sigma}(\gamma, \theta_L^{\Sigma}(f)).$$

Thus, the summands indexed by $L \supsetneq M$ vanish.

The remaining term is

$$\hat{I}_M^M(\gamma, \theta_M^{\mathcal{M}}(f) - \theta_M^{\Sigma}(f)).$$

The second equality of this lemma follows from

$${}^c\theta_M^{\Sigma}(f) - {}^c\theta_M^{\mathcal{M}}(f) = \theta_M^{\mathcal{M}}(f) - \theta_M^{\Sigma}(f),$$

which in turn follows from (34), (38), Theorem 13.1 and our induction hypothesis.□

We now return to the proof of Proposition 16.1. Suppose v is a nonArchimedean valuation of F. Let $f \in \mathcal{H}^{\text{met}}(G(F_v))^0$ and σ be a semisimple element in $M(F_v)$. We will show that

(53) $$\varepsilon_M(\gamma, f) \overset{(M,\sigma)}{\sim} 0,$$

for $\gamma \in M_\sigma(F_v) \cap G_{\text{reg}}$. Notice that we may assume that $\eta_{L'}$, as defined in §10 for σ, may be taken to be the identity. Indeed

$$\varepsilon_M(\eta\gamma, f) = \varepsilon_M(\gamma, f)$$

and

$$I_M^M(\eta\sigma, f_1) = I_M^M(\sigma, f_1)$$

for any $\eta \in \mu_n^M$, $\gamma \in M(F_v)$ and $f_1 \in \mathcal{H}^{\text{met}}(M(F_v))$. Since $\eta_{L'} \in \mu_n^M$, it follows that (53) holds for arbitrary semisimple $\sigma \in M(F_v)$ if it holds for those σ with $\eta_{L'} = 1$.

If $\mathfrak{a}_{M_\sigma} \neq \mathfrak{a}_M$ then $M_\sigma(F_v)$ is contained in a proper Levi subgroup M_1 of M. We may then apply descent to $\varepsilon_M(\gamma, f)$ to obtain

$$\varepsilon_M(\gamma, f) = \varepsilon_{M,M_1}(v)\hat{I}_{M_1}^{M_1}(\gamma, f_{M_1})$$

(*cf.* (13.2)* [10]). Clearly $\varepsilon_M(\gamma, f)$ is the orbital integral of a function in $\mathcal{H}^{\text{met}}(M(F_v))$ in this case.

On the other hand, if $\mathfrak{a}_{M_\sigma} = \mathfrak{a}_M$ then the germ expansions of Lemma 10.4 and Proposition 10.1 yield the (M, σ)-equivalence of

$$\varepsilon_M(\gamma, f) = I_M^M(\gamma, f) - \sum_{L_1 \supsetneq M} \hat{I}_M^{L_1, \mathcal{E}}(\gamma, \varepsilon_{L_1}(f))$$

with

$$\sum_{L \in \mathcal{L}(M)} \sum_{\delta \in \sigma(\mathcal{U}_{L_\sigma}(F_v))} g_M^L(\gamma, \delta) \left(I_L^M(\delta, f) - \sum_{\{L_1 \in \mathcal{L}(L): L_1 \neq M\}} \hat{I}_L^{L_1, \mathcal{E}}(\delta, \varepsilon_{L_1}(f)) \right).$$

We may disregard the summand for which $L = M$ as $g_M^M(\gamma, \delta) = 0$. If $L \supsetneq M$ we may take the sum over all subgroups $L_1 \in \mathcal{L}(L)$ and the difference in parentheses vanishes as well. As a result (53) holds in general.

By Lemma 16.2, $\varepsilon_M(\gamma, f)$ is the sum of ${}^c\varepsilon_M(\gamma, f)$ and

$$\hat{I}_M^M(\gamma, {}^c\theta_M^{\mathcal{E}}(f) - {}^c\theta_M^M(f)).$$

The latter term is an orbital integral of a function in $\mathcal{I}_{ac}^{\text{met}}(M(F_v))$. The former term, ${}^c\varepsilon_M(\gamma, f)$, has bounded support as a function of conjugacy classes in $M(F_v)$. By using this fact, equivalence (53) and a partition of unity argument on the space of semisimple conjugacy classes under the quotient topology, we may conclude that ${}^c\varepsilon_M(\gamma, f)$ is everywhere an orbital integral. This means that there exists a function ${}^c\varepsilon_M(f)$ in $\mathcal{I}^{\mathrm{met}}(M(F_v))$ such that

$$ {}^c\varepsilon_M(\gamma, f) = \hat{I}_M^M(\gamma, {}^c\varepsilon_M(f)). $$

Hence the function

$$ \varepsilon_M(f) = {}^c\varepsilon_M(f) + {}^c\theta_M(f) - {}^c\theta_M^{\mathcal{M}}(f) $$

satisfies the requirements of the proposition.

In order to complete the proof of Proposition 16.1, it remains to be shown that it holds when S consists of a single Archimedean, and hence complex, valuation. This will not be very taxing, as $\tilde{G}(\mathbf{C})$ splits over $G(\mathbf{C})$. For any $\tilde{f} \in \mathcal{H}(\tilde{G}(\mathbf{C}))$ and $\gamma \in G_{*\mathrm{reg}}(\mathbf{C})$ set

$$ f(\gamma) = |D(\gamma^n)/D(\gamma)|_{\mathbf{C}}^{1/2} \tilde{f}(\gamma^*). $$

It should be noted that the complex norm is taken to be $|z|_{\mathbf{C}} = z\bar{z}$ for $z \in \mathbf{C}$. The function f extends to a smooth function on $G(\mathbf{C})$ with compact support. Moreover

$$ \begin{aligned} & n^{\dim(A_M/A_G)} J_M(\gamma, f) \\ = \; & n^{\dim(A_M/A_G)} |D(\gamma)|_{\mathbf{C}}^{1/2} \int_{G(\mathbf{C})/G_\gamma(\mathbf{C})} f(x^{-1}\gamma x) v_M(x)dx \\ = \; & |D(\gamma)|_{\mathbf{C}}^{1/2} \int_{G(\mathbf{C})/G_\gamma(\mathbf{C})} |D(x^{-1}\gamma^n x)/D(x^{-1}\gamma x)|_{\mathbf{C}}^{1/2} \tilde{f}(x^{-1}\gamma^* x) v_{\tilde{M}}(x)dx \\ = \; & |D(\gamma^n)|_{\mathbf{C}}^{1/2} \int_{\tilde{G}(\mathbf{C})/\tilde{G}_{\gamma^*}(\mathbf{C})} \tilde{f}(x^{-1}\gamma^* x) v_{\tilde{M}}(x)dx \\ = \; & J_{\tilde{M}}(\gamma^*, \tilde{f}), \end{aligned} $$

where $\gamma \in M(\mathbf{C})$ such that $M_\gamma(\mathbf{C}) = G_\gamma(\mathbf{C})$. In particular, f and $\tilde{f}$ have matching orbital integrals. Consequently $f \in \mathcal{H}^{\mathrm{met}}(G(\mathbf{C}))$, and f^* may be taken to be equal to $\tilde{f}$.

We wish to show that

(54) $$ I_M^{\mathcal{M}}(\gamma, f) = I_M^{\mathcal{E}}(\gamma, f), \; \gamma \in M(\mathbf{C}). $$

By the inductive definition of $I_M(\gamma, f)$ ((2.1) [6]),

$$\begin{aligned} I_M^{\Sigma}(\gamma, f) &= \sum_{\eta \in \mu_n^M/\mu_n^G} J_M(\eta\gamma, f) - \sum_{L \in \mathcal{L}_0(M)} \hat{I}_M^{L,\Sigma}(\eta\gamma, \phi_L(f)) \\ &= \sum_{\eta \in \mu_n^M/\mu_n^G} J_M(\eta\gamma, f) - \sum_{L \in \mathcal{L}_0(M)} \hat{I}_M^{L,\Sigma}(\gamma, \sum_{\eta_1 \in \mu_n^L/\mu_n^G} {}^{\eta_1}\phi_L(f)). \end{aligned}$$

It is easy to show that

$$\sum_{\eta \in \mu_n^M/\mu_n^G} J_M(\eta\gamma, f) = J_{\tilde{M}}(\gamma^*, f^*), \ \gamma \in M(\mathbf{C}).$$

Indeed, by our previous calculation for $J_M(\gamma, f)$ and Lemma 6.1,

$$\begin{aligned} \sum_{\eta \in \mu_n^M/\mu_n^G} J_M(\eta\gamma, f) &= \sum_{\eta \in \mu_n^M/\mu_n^G} n^{-\dim(A_M/A_G)} J_{\tilde{M}}((\eta\gamma)^*, f^*) \\ &= \sum_{\eta \in \mu_n^M/\mu_n^G} n^{-\dim(A_M/A_G)} J_{\tilde{M}}(\gamma^*, f^*) \\ &= J_{\tilde{M}}(\gamma^*, f^*). \end{aligned}$$

Thus equation (54) follows from a simple induction argument if we can show that $\sum_{\eta \in \mu_n^L/\mu_n^G} {}^{\eta}\phi_L(f)$ matches $\phi_{\tilde{L}}(f^*)$ for $L \in \mathcal{L}_0(M)$. Let us focus on $\sum_{\eta \in \mu_n^L/\mu_n^G} {}^{\eta}\phi_L(f)$ first. According to the Archimedean trace Paley-Wiener theorem ([3]), there exists a function $h \in \mathcal{H}(L(\mathbf{C}))$ such that $h_L = \phi_L(f)$. By definition (§3),

$$\begin{aligned} \sum_{\eta \in \mu_n^L/\mu_n^G} ({}^{\eta}\phi_L(f))(\pi) &= \sum_{\eta \in \mu_n^L/\mu_n^G} \operatorname{tr} \int_{L(\mathbf{C})} \pi(x)\, {}^{\eta}h(x)dx \\ &= \sum_{\eta \in \mu_n^L/\mu_n^G} \operatorname{tr} \int_{L(\mathbf{C})} \pi(x)h(\eta x)dx \\ &= \left(\sum_{\eta \in \mu_n^L/\mu_n^G} \pi(\eta^{-1}) \right) \operatorname{tr} \int_{L(\mathbf{C})} \pi(x)h(x)dx \\ &= \left(\sum_{\eta \in \mu_n^L/\mu_n^G} \pi(\eta^{-1}) \right) (\phi_L(f))(\pi), \end{aligned}$$

for $\pi \in \Pi_{\text{temp}}(L(\mathbf{C}))$. Now $\pi \in \Pi_{\text{temp}}^{\text{met}}(L(\mathbf{C}))$ if and only if its central character is trivial on μ_n^L (This follows from the fact that tempered representations are principal series representations (Theorem 14.91 [20]) and the arguments of §2.1 [13].). Consequently

$$\sum_{\eta\in\mu_n^L/\mu_n^G} ({}^\eta\phi_L(f))(\pi) = \begin{cases} n^{\dim(A_L/A_G)}(\phi_L(f))(\pi), & \text{if } \pi\in\Pi_{\text{temp}}^{\text{met}}(L(\mathbf{C})) \\ 0, & \text{otherwise} \end{cases} .$$

As a result, in order to show (54) by induction, it suffices to show that $n^{\dim(A_L/A_G)}\phi_L(f) \in \mathcal{I}^{\text{met}}(L(\mathbf{C}))$ matches $\phi_{\tilde{L}}(f^*) \in \mathcal{I}(\tilde{L}(\mathbf{C}))$ for $L\in\mathcal{L}_0(M)$. In other words, it suffices to show

$$n^{\dim(A_L/A_G)}\phi_L(f,\pi) = \phi_{\tilde{L}}(f^*,\pi^*),\ \ \pi\in\Pi_{\text{temp}}^{\text{met}}(L(\mathbf{C})).$$

By definition (§7 [9]), this equality is one and the same as

(55) $$n^{\dim(A_L/A_G)}\operatorname{tr}\int_{G(\mathbf{C})}\mathcal{R}_L(\pi,P)\pi^G(x)f(x)dx = \operatorname{tr}\int_{\tilde{G}(\mathbf{C})}\mathcal{R}_{\tilde{L}}(\pi^*,\tilde{P})(\pi^*)^{\tilde{G}}(x)f^*(x)dx,$$

where $\mathcal{R}_L(\cdot,P)$, $P\in\mathcal{P}(L)$ is a (G,M) family obtained from normalized intertwining operators. Since all irreducible admissible tempered representations of $G(\mathbf{C})$ are principal series representations (Theorem 14.91 [20]), it suffices to prove (55) for $L=M_0$. In this case, π^* may be represented by $\otimes_{i=1}^r\omega_i$, where each ω_i is a quasi-character of $\mathbf{C}^\times$, and $\pi=\otimes_{i=1}^r\omega_i^n$ (*cf.* §2.1 [13]). It is a tedious, but straightforward, task to justify the following computation.

$$\begin{aligned}
&n^{\dim(A_{M_0}/A_G)}\operatorname{tr}\int_{G(\mathbf{C})}\mathcal{R}_{M_0}(\pi,P_0)\pi^G(x)f(x)dx\\
&= \operatorname{tr}\int_{G(\mathbf{C})}\mathcal{R}_{\tilde{M}_0}(\pi^*,\tilde{P}_0)(\pi^*)^{\tilde{G}}(x^*)f^*(x^*)|D(x^n)/D(x)|_{\mathbf{C}}^{1/2}dx\\
&= \operatorname{tr}\int_{A_G(\mathbf{C})}\int_{G(\mathbf{C})/A_G(\mathbf{C})}\mathcal{R}_{\tilde{M}_0}(\pi^*,\tilde{P}_0)(\pi^*)^{\tilde{G}}(x^*z^*)f^*(x^*z^*)|D(x^n)/D(x)|_{\mathbf{C}}^{1/2}dxdz\\
&= \operatorname{tr}\int_{A_G(\mathbf{C})}\int_{\tilde{G}(\mathbf{C})/\widetilde{A_G^n}(\mathbf{C})}\mathcal{R}(\pi^*,\tilde{P}_0)(\pi^*)^{\tilde{G}}(x^*z^*)f^*(x^*z^*)dx^*dz\\
&= \operatorname{tr}\int_{\widetilde{A_G^n}(\mathbf{C})}\int_{\tilde{G}(\mathbf{C})/\widetilde{A_G^n}(\mathbf{C})}\mathcal{R}_{\tilde{M}_0}(\pi^*,\tilde{P}_0)(\pi^*)^{\tilde{G}}(x^*z)f^*(x^*z)dx^*dz\\
&= \operatorname{tr}\int_{\tilde{G}(\mathbf{C})}\mathcal{R}_{\tilde{M}_0}(\pi^*,\tilde{P}_0)(\pi^*)^{\tilde{G}}(x)f^*(x)dx.
\end{aligned}$$

The coefficient $n^{\dim(A_{M_0}/A_G)}$ in the first equality is absorbed by $\mathcal{R}_{\tilde{M}_0}(\pi^*,\tilde{P}_0)$ (*cf.* Lemma 14.2). In the third equality we claim that the change of variable produced by the map

$$G(\mathbf{C})/A_G(\mathbf{C})\to\tilde{G}(\mathbf{C})/\widetilde{A_G^n}(\mathbf{C}),$$

$$xA_G(\mathbf{C}) \mapsto x^{*}\widetilde{A_G^n}(\mathbf{C}),$$

is $|D(x^n)/D(x)|_{\mathbf{C}}^{1/2}$. We give an idea of the proof of this claim. Let $\mathfrak{g}$ be the complex Lie algebra of $G(\mathbf{C}) = \mathrm{GL}(r, \mathbf{C})$. Recall that the exponential map

$$\exp : \mathfrak{g} \to G(\mathbf{C}),$$

is surjective and a local diffeomorphism. The Haar measure on $G(\mathbf{C})$ is equal to the inverse of the differential of the exponential map applied to a differential form on $\mathfrak{g}$ which yields Lebesgue measure on $\mathfrak{g}$. Suppose $x \in G(\mathbf{C})$ and $X \in \mathfrak{g}$ such that $\exp(X) = x$. Then the following diagram commutes.

$$\begin{array}{ccc} X & \mapsto & nX = \sum_{k=0}^{n-1} \mathrm{Ad}(x)^k X \\ \downarrow & & \downarrow \\ x & \mapsto & x^n = \prod_{k=0}^{n-1} x^{-k} x x^k \end{array}$$

From the earlier remarks, it follows that the change of variable resulting from the lower map is equal to the Jacobian of the upper map. The Jacobian of the upper map is $|\det \sum_{k=0}^{n-1} \mathrm{Ad}(x)^k|_{\mathbf{C}}^{1/2}$. The claim now follows from the equality

$$1 - \mathrm{Ad}(x)^n = (1 - \mathrm{Ad}(x)) \sum_{k=0}^{n-1} \mathrm{Ad}(x)^k$$

and the definition of $D(x)$ (§3).

We find that for $L \in \mathcal{L}(M)$

$$\varepsilon_M^L(f) = \begin{cases} f, & \text{if } L = G \\ 0, & \text{otherwise} \end{cases} . \tag{56}$$

The proof of Proposition 16.1 now follows as in the proof of Proposition 13.2 [10].□

Corollary 16.1 *Let S be a set of Archimedean valuations of F. Then*

$$I_M^M(\gamma, f) - I_M^E(\gamma, f) = 0, \ f \in \mathcal{H}^{\mathrm{met}}(G(F_S)).$$

Proof. This follows from Proposition 16.1 and (56). □

17 Comparison for $f \in \mathcal{H}^{\text{met}}(G(\mathbf{A}), M)$

We now give a sketch of the proof that $I^{\mathcal{M}}(f) - I(f) = 0$, for a certain subset $\mathcal{H}^{\text{met}}(G(\mathcal{A}), M)$ of functions in $\mathcal{H}^{\text{met}}(G(\mathbf{A})$. The train of reasoning in this section relies entirely on §15 and §16 [10]. We will outline the arguments found there and leave it to the reader to confirm the details.

There are a few definitions in [10] which must be transcribed before we may appeal to the arguments of §15 [10].

Let $\mathcal{H}^{\text{met}}(G(\mathbf{A}), M)^0$ be the space of functions f in

$$\mathcal{H}^{\text{met}}(G(\mathbf{A}), M) \cap \mathcal{H}^{\text{met}}(G(\mathbf{A}))^0$$

which satisfy one additional condition. Namely f vanishes at any element in $G(\mathbf{A})$ whose component at each finite place v belongs to $A_G(F_v)$.

The remark after Proposition 16.1 is used to great effect in the following lemma.

Lemma 17.1 (15.1) *Suppose that $f \in \mathcal{H}^{\text{met}}(G(\mathbf{A}), M)^0$. Then*

$$I^{\mathcal{M}}(f) - I(f) = |W(\mathfrak{a}_M)|^{-1} \hat{I}^M(\varepsilon_M(f)),$$

where I^M is the analogue for M of $I = I^G$.

Proof. By the properties of f and the remark after Proposition 16.1, we see that $I^{\mathcal{M}}(f) - I(f)$ equals

$$|W(\mathfrak{a}_M)|^{-1} \sum_{\gamma \in (M(F))_{M,S}} a^M(S,\gamma) \hat{I}_M^{M,\Sigma}(\gamma, \varepsilon_M(f)),$$

for a large set of valuations S. By the cuspidality conditions on f (*cf.* Lemma 15.1 [10]), it follows that

$$\hat{I}_{M_1}^{M,\Sigma}(\gamma, \varepsilon_M(f)),\ \gamma \in M_1(F),$$

for any M_1 such that $M \supsetneq M_1$. Thus by (14),

$$\begin{aligned} &\sum_{\gamma \in (M(F))_{M,S}} a^M(S,\gamma) \hat{I}_M^{M,\Sigma}(\gamma, \varepsilon_M(f)) \\ &= \sum_{M_1 \in \mathcal{L}^M} |W_0^{M_1}||W_0^M|^{-1} \sum_{\gamma \in (M_1(F))_{M_1,S}} a^{M_1}(S,\gamma) \hat{I}_{M_1}^{M,\Sigma}(\gamma, \varepsilon_M(f)) \\ &= \hat{I}^M(\varepsilon_M(f)). \square \end{aligned}$$

It is our intention to apply the method of separation of variables to Lemma 17.1. This method relies on the Archimedean factors of $f \in \mathcal{H}^{\mathrm{met}}(G(\mathbf{A}), M)^0$. Some necessary notation is given before this method is sketched.

Let S_∞ denote the set of Archimedean valuations of F. Then $G(F_{S_\infty})$ may be regarded as a real Lie group. Let $\mathfrak{h}_{\mathbf{C}}$ denote the standard Cartan subalgebra of its complexified Lie algebra. Let $\mathfrak{h}$ be the real form of $\mathfrak{h}_{\mathbf{C}}$ associated to the split real form of $G(F_{S_\infty})$. Then $\mathfrak{h}$ is isomorphic to $\mathfrak{a}_{M_0}$ as a vector space and therefore contains all vector spaces of the form $\mathfrak{a}_M$. Let $\mathfrak{h}^1$ be the orthogonal complement of $\mathfrak{a}_G$ in $\mathfrak{h}$. We recall the theory of multipliers. Let α belong to $\mathcal{E}(\mathfrak{h}^1)^W$, the convolution algebra of compactly supported, W^G-invariant distributions on $\mathfrak{h}^1$. Then there is an action, $f \mapsto f_\alpha$, on $\mathcal{H}^{\mathrm{met}}(G(\mathbf{A}))$ such that $f_{\alpha,M}(\pi) = \hat{\alpha}(\nu_\pi) f_M(\pi)$ for all $\pi \in \Pi^{\mathrm{met}}(M(\mathbf{A}))$. As usual, ν_π is taken to be the infinitesimal character of the Archimedean factor of π. This action of $\mathcal{E}(\mathfrak{h}^1)^W$ on $\mathcal{H}^{\mathrm{met}}(G(\mathbf{A}))$ affects only the Archimedean factor of f.

Lemma 17.2 *Let $\alpha \in \mathcal{E}(\mathfrak{h}^1)^W$ and $f \in \mathcal{H}^{\mathrm{met}}(G(F_S))$. Set $\alpha^*(\nu) = n^{\dim(\mathfrak{h}^1)}\alpha(n\nu)$. Then $f_\alpha \in \mathcal{H}^{\mathrm{met}}(G(F_S))$ and $(f_\alpha)^* = f^*_{\alpha^*}$.*

Proof. Let $\alpha \in \mathcal{E}(\mathfrak{h}^1)^W$ and $\pi \in \Pi^{\mathrm{met}}_{\mathrm{temp}}(G(F_S)$. Then

$$\begin{aligned} f^*_{\alpha^*,\tilde{G}}(\pi^*) &= \widehat{\alpha^*}(\nu_{\pi^*}) f^*_{\tilde{G}}(\pi^*) \\ &= \widehat{\alpha^*}(\nu_\pi/n) f_G(\pi) \\ &= \hat{\alpha}(\nu_\pi) f_G(\pi) \\ &= f_{\alpha,G}(\pi) \end{aligned}$$

It follows from Proposition 27.3 [15] that f_α and $f^*_{\alpha^*}$ match. In particular we may take $f_\alpha \in \mathcal{H}^{\mathrm{met}}(G(F_S))$ and set $(f_\alpha)^* = f^*_{\alpha^*}$.□

Corollaries 14.2 and 14.3 [10] follow without difficulty in the metaplectic context. Corollary 14.4 has a much simpler proof in our case.

Lemma 17.3 (14.4) *Let $f \in \mathcal{H}^{\mathrm{met}}(G(F_S))$, and $\alpha \in \mathcal{E}(\mathfrak{h}^1)^W$. Then $\varepsilon_M(f_\alpha) = \varepsilon_M(f)_\alpha$.*

Proof. Let S be the disjoint union of S_∞ and S_0. Let $f = f_\infty f_0$ be the corresponding decomposition of f. Note that by the splitting property of Theorem 10.1,

$$\varepsilon_M(S) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2)\varepsilon_M^{L_1}(S_\infty)\varepsilon_M^{L_2}(S_0).$$

It follows from the fact that $\tilde{G}(F_{S_\infty})$ splits over $G(F_{S_\infty})$, that $\hat{\varepsilon}_M^{L_2}(f_{\infty,L_2}) = 0$ unless $L_2 = M$. Also,

$$d_M^G(L_1, M) = \begin{cases} 1, & L_1 = G \\ 0, & \text{otherwise} \end{cases}.$$

Consequently the splitting formula above reduces to

$$\begin{aligned} \varepsilon_M(f) &= \varepsilon_M(f_0)\hat{\varepsilon}_M^M(f_{\infty,M}) \\ &= \varepsilon_M(f_0)f_{\infty,M}. \end{aligned}$$

Therefore

$$\begin{aligned} \varepsilon_M(f)_\alpha &= (\varepsilon_M^G(f_0)f_{\infty,M})_\alpha \\ &= \varepsilon_M^G(f_0)(f_{\infty,M,\alpha}) \\ &= \varepsilon_M^G(f_0)(f_{\infty,\alpha,M}) \\ &= \varepsilon_M(f_\alpha). \square \end{aligned}$$

Let $\mathfrak{h}_u^*$ be the set of points ν in $\mathfrak{h}_{\mathbb{C}}^*/i\mathfrak{a}_G^*$ such that $\bar{\nu} = -s\nu$ for some element $s \in W^G$ of order two. The Archimedean infinitesimal character of ν_π associated to any $\pi \in \Pi_{\text{unit}}^{\text{met}}(G(\mathbb{A})^1)$ belongs to $\mathfrak{h}_u^*$.

The background for the method of separation by infinitesimal characters is settled. The object of the application of this method is the following lemma. There is but one bit of notation left to give. For $\nu_1 \in \mathfrak{h}_u^*$ define

$$\Pi_{\nu_1}^{\text{met}}(G(\mathbb{A})^1) = \{\pi \in \Pi^{\text{met}}(G(\mathbb{A})^1) : \nu_\pi = s\nu_1 \text{ for some } s \in W_G\}.$$

Lemma 17.4 (15.4) *For each $f \in \mathcal{H}^{\text{met}}(G(\mathbb{A}), M)^0$ and $\nu_1 \in \mathfrak{h}_u^*$, we have*

$$\sum_{\pi \in \Pi_{\nu_1}^{\text{met}}(G(\mathbb{A})^1)} (a_{\text{disc}}^{\mathcal{M}}(\pi) - a_{\text{disc}}(\pi))\text{tr}\pi(f^1) = 0.$$

Now, on to the description of the method of separation by infinitesimal characters. First we choose $\nu_1 \in \mathfrak{h}_u^*$ and $\alpha_1 \in \mathcal{E}(\mathfrak{h}^1)^W$ as in Lemma 15.2 [10]. Given $T \geq 0$ it follows from Lemma 6.3 [7] and Lemma 15.2 [10] that

$$|\sum_{t \leq T} I_t^{\mathcal{M}}(f_{\alpha_1^m}) - I_t(f_{\alpha_1^m}) - |W(\mathfrak{a}_M)|^{-1}\hat{I}^M(\varepsilon_M(f)_{\alpha_1^m})| \leq Ce^{-kNm}$$

for some positive constants C, k and N. Thus

$$\sum_{t \leq T} I_t^{\mathcal{M}}(f_{\alpha_1^m}) - I_t(f_{\alpha_1^m}) - |W(\mathfrak{a}_M)|^{-1}\hat{I}^M(\varepsilon_M(f)_{\alpha_1^m}) \tag{57}$$

approaches zero as m approaches infinity. On the other hand, we may write

$$\sum_{t \leq T} I_t^{\mathcal{M}}(f_{\alpha_1^m}) - I_t(f_{\alpha_1^m}) \tag{58}$$

as

$$\sum_{t \leq T} \sum_{\pi \in \Pi^{\mathrm{met}}(G(\mathbb{A})^1, t)} (a_{\mathrm{disc}}^{\mathcal{M}}(\pi) - a_{\mathrm{disc}}(\pi)) \mathrm{tr}(f^1)\alpha_1^m$$

by Lemma 15.3 and the theory of multipliers. Moreover by Lemma 15.3 [10], we may write

$$\sum_{t \leq T} |W(\mathfrak{a}_M)|^{-1}\hat{I}^M(\varepsilon_M(f)_{\alpha_1^m}) \tag{59}$$

as a finite sum,

$$|W(\mathfrak{a}_M)|^{-1} \sum_{t \leq T} \sum_{\pi \in \Pi_{\mathrm{disc}}^{\mathrm{met}}(M, t)} a_{\mathrm{disc}}^M(\pi) \int_{i\mathfrak{a}_M^*/i\mathfrak{a}_G^*} \varepsilon_M(f^1, \pi, \lambda)\hat{\alpha}_1(\nu_\pi + \lambda)^m d\lambda,$$

for some Schwartz function

$$\lambda \mapsto \varepsilon_M(f^1, \pi, \lambda), \ \lambda \in i\mathfrak{a}_M^*/i\mathfrak{a}_G^*.$$

The multiplier α_1 was chosen so that

$$0 \leq \hat{\alpha}_1(\nu_\pi + \lambda) < 1,$$

for all but finitely many $\lambda \in i\mathfrak{a}_M^*/i\mathfrak{a}_G^*$ in the above integral. Thus, by the dominated convergence theorem, the integral approaches zero as m approaches infinity. Thus (59) and (57) both approach zero as m approaches infinity. Since (57) is the difference of (58) and (59), we see that (58) has this property as well. Once again by our choice of α_1, the inequality,

$$0 \leq \hat{\alpha}_1(\nu_\pi) < 1,$$

holds for $\pi \in \Pi^{\mathrm{met}}(G(\mathbf{A})^1, t)$ unless $\pi \in \Pi^{\mathrm{met}}_{\nu_1}(G(\mathbf{A})^1)$, in which case $\hat{\alpha}_1(\nu_\pi) = 1$. This yields Lemma 17.4 and ends our discussion of the method of separation by infinitesimal characters.

The following Proposition is essentially an extension of Lemma 17.4 to functions in $\mathcal{H}^{\mathrm{met}}(G(\mathbf{A}), M)$, and an application of Lemma 15.3. If follows *mutatis mutandis* from §16 [10].

Proposition 17.1 (16.2) *For any $f \in \mathcal{H}^{\mathrm{met}}(G(\mathbf{A}), M)$, we have*

$$I^{\mathcal{M}}(f) = I(f).$$

18 Proofs of Theorem 9.1 and Theorem 12.1

At long last, we prove Theorem 9.1 and Theorem 12.1. We begin with Theorem 9.1 (i). First we establish a certain degree of freedom at the Archimedean valuations.

Lemma 18.1 *Suppose S_1 is a finite set of valuations with the closure property and $S_1 \supset S_{(n)}$. Suppose further that v is a valuation of F not contained in S and set $S = S_1 \cup \{v\}$. If $f = f_1 f_v \in \mathcal{H}^{\mathrm{met}}(G(F_{S_1}))$ and $\gamma = \gamma_1\gamma_v \in M(F_S) \cap G_{*\mathrm{reg}}$ are corresponding decompositions, then*

$$\begin{aligned} I_M^{\mathcal{M}}(\gamma, f) - I_M^{\mathcal{E}}(\gamma, f) &= (I_M^{\mathcal{M}}(\gamma_1, f_1) - I_M^{\mathcal{E}}(\gamma_1, f_1)) I_G(\gamma_v, f_v) \\ &+ (I_M^{\mathcal{M}}(\gamma_v, f_v) - I_M^{\mathcal{E}}(\gamma_v, f_v)) I_G(\gamma_1, f_1). \end{aligned}$$

If v is Archimedean, then

$$I_M^{\mathcal{M}}(\gamma, f) - I_M^{\mathcal{E}}(\gamma, f) = (I_M^{\mathcal{M}}(\gamma_1, f_1) - I_M^{\mathcal{E}}(\gamma_1, f_1)) I_G(\gamma_v, f_v).$$

Proof. We may apply the splitting properties ((17), Lemma 6.2) to obtain

$$\sum_{L_0,L_1\in\mathcal{L}(M)} d_M^G(L_0,L_1)\left(\hat{I}_M^{L_0,\mathcal{M}}(\gamma_1,f_{1,L_0})\hat{I}_M^{L_1,\mathcal{M}}(\gamma_v,f_{v,L_1}) - \hat{I}_M^{L_0,\mathcal{E}}(\gamma_1,f_{1,L_0})\hat{I}_M^{L_1,\mathcal{E}}(\gamma_v,f_{v,L_1})\right).$$

According to the induction hypothesis of §9, the summands above, for which either L_0 or L_1 is not equal to G, vanish. By the properties of $d_M^G(L_0,L_1)$ (*cf.* Lemma, §7 [6]), the lemma follows.

For v Archimedean, the second equality follows from the first and Corollary 16.1.□

The following lemma is a tool which allows us to derive local results from the global result of Proposition 17.1.

Lemma 18.2 *Suppose w is an Archimedean valuation and S is a large, finite set of valuations containing $S_{(n)}$ and the Archimedean valuations of F. Furthermore, suppose*

$$f' = f'_w \prod_{v\neq w} f'_v \in \mathcal{H}^{\text{met}}(G(\mathbf{A}),M)$$

such that $I^{\mathcal{M}}(f') - I(f')$ equals the sum of

$$n|W(\mathfrak{a}_M)|^{-1} \sum_{\gamma\in(M(F))_{M,S}/\mu_n^M} a^M(S,\gamma)(I_M^{\mathcal{M}}(\gamma,f') - I_M^{\mathcal{E}}(\gamma,f'))$$

and

$$\sum_{\delta\in A_G(F)}\sum_{u\in(\mathcal{U}_G(F))_{G,S}} (a^{\mathcal{M}}(S,u) - a(S,u))I_G(\delta u,f')$$

as in Lemma 16.1. Then for each $\gamma\in M(F)\cap G_{*\text{reg}}$, which is F-elliptic in $M(F)$, there exists a function $f_w\in\mathcal{H}^{\text{met}}(G(\mathbf{C}))$ such that $f = f_w\prod_{v\neq w} f'_v$ belongs to $\mathcal{H}^{\text{met}}(G(\mathbf{A}),M)$ and

$$\begin{aligned} I^{\mathcal{M}}(f) - I(f) &= \text{vol}(M_\gamma(F)\backslash M_\gamma(\mathbf{A})^1)(I_M^{\mathcal{M}}(\gamma,f_S) - I_M^{\mathcal{E}}(\gamma,f_S)) \\ &= 0. \end{aligned}$$

Proof. If the image of γ in $M(F_w)$ does not belong to the support of f'_w, then by the second equation of Lemma 18.1, $I^{\mathcal{M}}(f') - I(f')$ must vanish and we are done. Suppose then that the image of γ above does lie in the support of f'_w. The set of valuations S has been chosen so that $f' \in \mathcal{H}^{\mathrm{met}}(G(F_S), M)$. In other words,

$$f' = f'_w \prod_{v \in S - \{w\}} f'_v \prod_{v \notin S} f^0_v,$$

where f^0_v are the functions defined in §7. The support of f' is contained in a compact open subset, $U'_w \times \prod_{v \in S-\{w\}} U_v \times \prod_{v \notin S} K_v$, of $G(\mathbf{A})$. Since $G(F)$ is a discrete subgroup of $G(\mathbf{A})$, we may choose $f_w \in \mathcal{H}^{\mathrm{met}}(G(\mathbf{C}))$ to be supported on a sufficiently small neighbourhood $U_w \subset U'_w$ such that γ is the only element of $G(F)$ contained in $U_w \times \prod_{v \in S-\{w\}} U_v \times \prod_{v \notin S} K_v$. The function f, obtained by replacing f'_w with f_w, is clearly also in $\mathcal{H}^{\mathrm{met}}(G(F_S), M)$. It is also not necessary to increase the size of S in the expansion of $I^{\mathcal{M}}(f) - I(f)$ as in Lemma 16.1 (*cf.* §4 [7]). By construction then, the only summand of this expansion which survives is

$$a^M(S,\gamma)(I^{\mathcal{M}}_M(\gamma, f_S) - I^{\Sigma}_M(\gamma, f_S)).$$

Proposition 17.1 tells us that this expression vanishes. By Theorem 8.2 [4] we may conclude that

$$a^M(S,\gamma) = \mathrm{vol}(M_\gamma(F)\backslash M_\gamma(\mathbf{A})^1),$$

for S sufficiently large.□

Lemma 18.2 has an immediate application. Namely,

$$I^{\mathcal{M}}_M(\gamma, f) - I^{\Sigma}_M(\gamma, f), \ \gamma \in M(F_S) \cap G_{*\mathrm{reg}},$$

may be compared to the orbital integral $I_G(\gamma, f)$.

Lemma 18.3 *Suppose that V is a finite set of valuations with the closure property such that either $V \supset S_{(n)}$ or V consists of a single valuation not contained in $S_{(n)}$. Then for every function $f_V \in \mathcal{H}^{\mathrm{met}}(G(F_V))$ there exists a smooth complex-valued function ε_M such that*

$$I^{\mathcal{M}}_M(\gamma_V, f_V) - I^{\Sigma}_M(\gamma_V, f_V) = \varepsilon_M(\gamma_V) I_G(\gamma_V, f_V), \ \gamma_V \in M(F_V).$$

Proof. Suppose first that V contains $S_{(n)}$. Lemma 18.1 allows us to restrict to the case that there exists an Archimedean valuation w of F which does not belong to V. Let

$\gamma \in M(F)$ be as in Lemma 18.2. It is straightforward task (*cf.* p193 of [10]) to choose a finite set of valuations

$$S = V \cup \{w, v_1, \dots, v_k\}$$

containing the Archimedean valuations, V and at least two other nonArchimedean valuations, and a function

$$f = f_V f_w \prod_{i=1}^{k} f_i \in \mathcal{H}^{\mathrm{met}}(G(F_S), M)$$

such that $I_G(\gamma, f_i) \neq 0$, $I_G(\gamma, f_w) \neq 0$. Then

$$I_M^{\mathcal{M}}(\gamma, f) - I_M^{\mathcal{E}}(\gamma, f)$$

has the expansion of Lemma 16.1. Thanks to Lemma 18.2, we may assume that

$$I_M^{\mathcal{M}}(\gamma, f) - I_M^{\mathcal{E}}(\gamma, f) = 0.$$

A repeated application of Lemma 18.1 to this equation yields

$$\begin{aligned} & (I_M^{\mathcal{M}}(\gamma_V, f_V) - I_M^{\mathcal{E}}(\gamma_V, f_V)) I_G(\gamma, f_w) \prod_{j=1}^{k} I_G(\gamma, f_j) \\ + & \sum_{i=1}^{k} (I_M^{\mathcal{M}}(\gamma, f_i) - I_M^{\mathcal{E}}(\gamma, f_i)) I_G(\gamma, f_V) I_G(\gamma, f_w) \prod_{j \neq i} I_G(\gamma, f_j) \\ = & \ 0. \end{aligned}$$

From this equality and our choice of functions we can readily see that if $I_G(\gamma, f_V) = 0$, then

$$I_M^{\mathcal{M}}(\gamma_V, f_V) - I_M^{\mathcal{E}}(\gamma_V, f_V) = 0$$

as well. As a result, there exists a constant $\varepsilon_M(\gamma)$ such that

$$I_M^{\mathcal{M}}(\gamma_V, f_V) - I_M^{\mathcal{E}}(\gamma_V, f_V) = \varepsilon_M(\gamma) I_G(\gamma, f_V). \tag{60}$$

We may prove (60) in much the same fashion if V consists of a single valuation not contained in $S_{(n)}$.

Since both

$$I_M^{\mathcal{M}}(\cdot, f_V) - I_M^{\mathcal{E}}(\cdot, f_V)$$

and $I_G(\cdot, f_V)$ are smooth functions of $M(F_V) \cap G_{\bullet\text{reg}}$ and the image of the set of F-elliptic elements in $M(F) \cap G_{\bullet\text{reg}}$ is dense in $M(F_V) \cap G_{\bullet\text{reg}}$, we may clearly extend the definition of ε_M to obtain a smooth function on $M(F_V) \cap G_{\bullet\text{reg}}$.□

It is not too hard to see that Theorem 9.1 (i) follows if the function ε_M vanishes. Indeed, if ε_M vanishes then the induction argument begun in §16 is complete. This in turn implies that M may be taken to be M_0 and since

$$\mathcal{H}^{\text{met}}(G(F_S), M_0) = \mathcal{H}^{\text{met}}(G(F_S)),$$

Theorem 9.1 (i) follows after an application of Lemma 9.5. We will show that ε_M vanishes by showing that it vanishes on some subgroups whose product generates M, and then showing that ε_M is a homomorphism. First we show that the values of ε_M do not depend on the Archimedean valuations S_∞.

Lemma 18.4 *Suppose V is a finite set of valuations with the closure property and containing $S_{(n)}$. Let $V_\infty = S_\infty \cap V$. Then ε_M is invariant under $M(F_{V_\infty}) \cap G_{\bullet\text{reg}}$.*

Proof. Suppose first that V_∞ contains at least two Archimedean valuations, v_1 and v_2. Let $V' = V - \{v_1\}$ and let $\gamma_V = \gamma_{V'}\gamma_1 \in M(F_V) \cap G_{\bullet\text{reg}}$, $f_V = f_{V'}f_1 \in \mathcal{H}^{\text{met}}(G(F_V))$ be corresponding decompositions. Then V' has the closure property and $V' \supset S_{(n)}$. Therefore we may combine Lemma 18.1 with Lemma 18.3 to find that

$$\begin{aligned}\varepsilon_M(\gamma_V)I_G(\gamma_V, f_V) &= I_M^{\mathcal{M}}(\gamma_V, f_V) - I_M^{\mathcal{E}}(\gamma_V, f_V) \\ &= \varepsilon_M(\gamma_{V'})I_G(\gamma_{V'}, f_{V'})I_G(\gamma_1, f_1) \\ &= \varepsilon_M(\gamma_{V'})I_G(\gamma_V, f_V).\end{aligned}$$

In other words, $\varepsilon_M(\gamma_V) = \varepsilon_M(\gamma_{V'})$, and so ε_M is independent of γ_1. This argument may be repeated with any other Archimedean valuation in place of v_1, so the lemma follows for this choice of V.

Now if $V \cap S_\infty$ consists of a single Archimedean valuation v_1, then let $V' = V \cup \{v_2\}$, where v_2 is an Archimedean valuation not contained in V. The above argument then yields

$$\varepsilon_M(\gamma_{V'}) = \varepsilon_M(\gamma_V),\ \gamma_{V'} = \gamma_V\gamma_2 \in M(F_{V'}) \cap G_{\bullet\text{reg}}.$$

By the earlier argument, $\varepsilon_M(\gamma_{V'})$ is independent of the factor of $\gamma_{V'}$ corresponding to the valuation v_1. The last equation implies that $\varepsilon_M(\gamma_V)$ must have this independence as well. The Lemma is now complete.□

Lemma 18.5 *Suppose V is as in Lemma 18.3, $f \in \mathcal{H}^{\text{met}}(G(F_V))$, and $\gamma \in M_1(F_V) \cap G_{\bullet\text{reg}}$ where $M_1 \in \mathcal{L}$ and $M_1 \subsetneqq M$. Then $\varepsilon_M(\gamma) = 0$.*

Proof. Lemma 9.2 tells us that

$$I_M^M(\gamma, f) - I_M^{\mathcal{E}}(\gamma, f) = 0.$$

Since ε_M is smooth on $M(F_v) \cap G_{\bullet\text{reg}}$, it is easily seen that we may choose ε_M in Lemma 18.3 so that it vanishes on some open neighbourhood of $\gamma \in M(F_v) \cap G_{\bullet\text{reg}}$.□

Recall decomposition (2),

$$M = \prod_{i=1}^{\ell} M(i),$$

where $M(i) \cong \text{GL}(r_i)$ and $\sum_{i=1}^{\ell} r_i = r$. For the sake of simplicity, we will identify the subgroup $\prod_{i=1}^{\ell} \text{SL}(r_i)$ of $\prod_{i=1}^{\ell} \text{GL}(r_i)$ with its image in M via the above isomorphism.

Lemma 18.6 *Suppose V is as in Lemma 18.4. Then the function ε_M vanishes on $\prod_{i=1}^{\ell} \text{SL}(r_i, F_V)$.*

Proof. Let $f_V \in \mathcal{H}^{\text{met}}(G(F_V))$ and $\gamma_V \in \prod_{i=1}^{\ell} \text{SL}(r_i, F_V) \cap G_{\bullet\text{reg}}$. Fix two nonArchimedean valuations, v_1 and v_2, not contained in V. Choose

$$\gamma_j \in \prod_{i=1}^{\ell} \text{SL}(r_i, F_{v_j}) \cap G_{\bullet\text{reg}}, \; j = 1, 2,$$

such that γ_j is F_{v_j}-elliptic in $M(F_{v_j})$. By strong approximation ([21]) on $\prod_{i=1}^{\ell} \text{SL}(r_i, \mathbb{A})$, there exists $\gamma \in \prod_{i=1}^{\ell} \text{SL}((r_i, F) \cap G_{\bullet\text{reg}}$ satisfying the following properties:

- γ is F-elliptic in $M(F)$.
- The image of γ is $M(F_V)$ is close to γ_V at the nonArchimedean valuations of V.
- The image of γ in $M(F_{v_j})$ is close to γ_j, for $j = 1, 2$.

- The image of γ in $M(F_v)$ belongs to K_v for all nonArchimedean valuations v such that $v \notin V \cup \{v_1, v_2\}$.

Let S_0 be the set of Archimedean valuations of F not contained in V. Choose $f_0 \in \mathcal{H}^{\text{met}}(G(F_{S_0}))$ so that the image of γ in $M(F_{S_0})$ lies in its support. Choose $f_j \in \mathcal{H}^{\text{met}}(G(F_{v_j}))$ supported on small F_{v_j}-elliptic sets of $M(F_{v_j})$, such that $I_G(\gamma_j, f_j) \neq 0$, $j = 1, 2$, and

$$f = f_0 f_1 f_2 f_V \prod_{v \notin V \cup S_0 \cup \{v_1, v_2\}} f_v^0 \in \mathcal{H}^{\text{met}}(G(\mathbf{A}), M).$$

After a possible application of Lemma 18.2, we obtain

$$\begin{aligned} I^{\mathcal{M}}(f) - I(f) &= \operatorname{vol}(M_\gamma(F) \backslash M_\gamma(\mathbf{A})^1)(I_M^{\mathcal{M}}(\gamma, f_S) - I_M^{\mathcal{E}}(\gamma, f_S)) \\ &= 0. \end{aligned}$$

If we substitute the expansion of Lemma 18.3 into this equation, apply Lemma 18.1 repeatedly, use the invariance of Lemma 18.4 and use Lemma 7.2, we end up with

$$\varepsilon_M(\gamma_V) + \varepsilon_M(\gamma_1) + \varepsilon_M(\gamma_2) = 0.$$

Now v_1 and v_2 were chosen to be arbitrary nonArchimedean valuations not contained in V. Therefore the whole argument may be repeated with a different pair of valuations, v_1' and v_2', with the final result,

$$\varepsilon_M(\gamma_V) + \varepsilon_M(\gamma_1') + \varepsilon_M(\gamma_2') = 0,$$

for some

$$\gamma_j' \in \prod_{i=1}^{\ell} \operatorname{SL}(r_i, F_{v_j'}) \cap G_{*\text{reg}}, \ j = 1, 2.$$

This implies that $\varepsilon_M(\gamma_V)$ is constant for all $\gamma_V \in \prod_{i=1}^{\ell} \operatorname{SL}(r_i, F_V) \cap G_{*\text{reg}}$. Since $\varepsilon_M(\gamma_V)$ vanishes for γ_V belonging to a proper Levi subgroup of M, this constant must be zero.□

Observe that $M = M_0 \prod_{i=1}^{\ell} \operatorname{SL}(r_i)$ and that ε_M vanishes on both M_0 and $\prod_{i=1}^{\ell} \operatorname{SL}(r_i)$. As stated before, the following lemma completes the proof of Theorem 9.1 (i).

Lemma 18.7 *Let V be a finite set of valuations with the closure property and containing $S_{(n)}$. Suppose $f_V \in \mathcal{H}^{\text{met}}(G(F_V)$ and ε_M is as in Lemma 18.3. Then ε_M is a homomorphism from $M(F_V)$ to $\mathbf{C}$.*

Proof. Let $\gamma_1, \gamma_2 \in M(F_V) \cap G_{\bullet\mathrm{reg}}$. By weak approximation, we may choose $\gamma \in M(F) \cap G_{\bullet\mathrm{reg}}$ to be an F-elliptic element in $M(F)$ such that the image of γ in $M(F_V)$ is close to γ_1. Embed γ_1 into $M(\mathbf{A})$ in the obvious way and let $\delta_1 = \gamma\gamma_1^{-1} \in M(\mathbf{A})$. Then the factors $\delta_{1,v}$ of δ_1 at the valuations $v \in V$ are close to the identity. Let V_1 denote the finite set of valuations $\{v_1, \ldots, v_k\}$ at which the factors of δ_{1,w_j} of δ_1 do not lie in K_{v_j}, $1 \leq j \leq k$. By construction, V and V_1 are disjoint sets. We may use an argument similar to that of Lemma 18.6 to conclude that

$$0 = \varepsilon_M(\gamma) = \varepsilon_M(\delta_1\gamma_1) = \sum_{v \in V_1} \varepsilon_M(\delta_{1,v}) + \varepsilon_M(\gamma_1).$$

We repeat this argument for γ_2 to obtain a finite set of valuations V_2, disjoint from V, and an element $\delta_2 \in M(\mathbf{A})$ such that

$$\sum_{v \in V_2} \varepsilon_M(\delta_{2,v}) + \varepsilon_M(\gamma_2) = 0.$$

Weak approximation allows us to assume that V_2 is disjoint from V_1 as well.

Once again we use an argument similar to the one in Lemma 18.6 on the product

$$\delta_1\gamma_1\delta_2\gamma_2 = \gamma_1\gamma_2\delta_1\delta_2 \in M(F) \cap G_{\bullet\mathrm{reg}},$$

to arrive at

$$0 = \varepsilon_M(\gamma_1\gamma_2) + \sum_{v \in V_1} \varepsilon_M(\delta_{1,v}) + \sum_{v \in V_2} \varepsilon_M(\delta_{2,v}) = \varepsilon_M(\gamma_1\gamma_2) - \varepsilon_M(\gamma_1) - \varepsilon_M(\gamma_2).$$

Otherwise stated, ε_M is a homomorphism.□

All that remains to be done now is to prove the rest of Theorem 9.1 and Theorem 12.1.

Proof of Theorem 9.1 (ii). We wish to show that

$$a^{\tilde{M}}(S, \gamma^*) = a^M(S, \gamma), \ \gamma \in M(F).$$

Suppose first that $\gamma \in M(F)$ has Jordan decomposition $\gamma = \sigma u$, where the semisimple element σ is not in $A_G(F)$ if $M = G$. Then

$$\dim_F(M_\sigma(F)) < \dim_F(G(F)),$$

so we may apply the induction hypothesis of §9 to decompositions (22) and (23) and the lemma follows.

On the other hand, if $M = G$ and $\sigma \in A_G(F)$ then

$$a^{\tilde{G}}(S, \gamma^*) = a^{\tilde{G}}(S, u^*)$$

and

$$a^G(S, \gamma) = a^G(S, u),$$

by (22) and (23) respectively. It follows from Theorem 9.1 (i) and Lemma 16.1 (where we may now take $M = M_0$) that

$$\sum_{\sigma \in A_G(F)} \sum_{u \in (\mathcal{U}_G(F))} (a(S, u^*) - a(S, u)) I_G(\sigma u, f) = 0,$$

for any $f \in \mathcal{H}^{\text{met}}(G(\mathbf{A}))$. We may choose $f \in \mathcal{H}^{\text{met}}(G(\mathbf{A}))$ above so that for a fixed element $u_1 \in \mathcal{U}_G(F)$, we have

$$I_G(\sigma u, f) = \begin{cases} 1, & \text{if } \sigma = 1 \text{ and } u = u_1 \\ 0, & \text{otherwise} \end{cases}$$

(§3.3 [31]). This clearly implies that $a(S, u_1^*) = a(S, u_1)$.□

Theorem 12.1 (i) follows from Theorem 9.1 (i) and §13. This leaves us with a single proof to be completed.

Proof of Theorem 12.1(ii). By the induction hypothesis of §12, we need only show that

$$a_{\text{disc}}^{\mathcal{M}}(\pi) = a_{\text{disc}}(\pi), \; \pi \in \Pi^{\text{met}}(G(\mathbf{A})^1).$$

Let ν_1 be the infinitesimal character of the Archimedean factor of some fixed representation π in $\Pi^{\text{met}}(G(\mathbf{A})^1)$, and let K_1 be a compact open subgroup of $\prod_{v \notin S_\infty} K_v$ such that π is bi-K_1-invariant. Let $\Pi^{\text{met}}_{\nu_1, K_1}(G(\mathbf{A})^1)$ be the set of bi-K_1-invariant representations in $\Pi^{\text{met}}(G(\mathbf{A})^1)$ with infinitesimal character ν_1. In the process of proving Proposition 17.1 (*cf.* (16.6) [10]), one obtains

$$\sum_{\pi \in \Pi^{\text{met}}_{\nu_1, K_1}(G(\mathbf{A})^1)} (a_{\text{disc}}^{\mathcal{M}}(\pi) - a_{\text{disc}}(\pi)) \text{tr}(f^1) = 0,$$

for any $f \in \mathcal{H}^{\text{met}}(G(\mathbb{A})^1)$ which is bi-K_1-invariant. This sum is finite by Lemma 4.2 [7], and the linear forms,

$$f \mapsto \operatorname{tr}\pi(f^1), \;\; \pi \in \Pi^{\text{met}}_{\nu_1, K_1}(G(\mathbb{A})^1),$$

on the space of bi-K_1-invariant functions in $\mathcal{H}^{\text{met}}(G(\mathbb{A}))$ are linearly independent. The result follows.□

19 Appendix. Tensor Products of Metaplectic Representations

In §26.2 [15] a method of induction from parabolic subgroups of $\tilde{G}(F_v)$ is delineated. Recall decomposition (2),

$$M \cong \prod_{i=1}^{\ell} M(i).$$

This method of induction relates tensor products of representations of $\tilde{M}(i)$ to representations of $\tilde{M}$. We describe this relation and prove all of the claims made in §26.2 [15] concerning it. It seems that the claims are not true in general ([30]). The assumptions made on n and r in §2 remain in force.

Let $(\cdot,\cdot)_{F_v} : F_v^\times \times F_v^\times \to \mu_n$ be the nth Hilbert symbol on F_v and let B be a maximal subgroup of $F_v^\times$ with respect to the property that $(x, x')_{F_v} = 1$ for all $x, x' \in B$. Let $1 \leq i \leq \ell$ and set

$$\tilde{M}^B(i)(F_v) = \{\tilde{\gamma} \in M(i)(F_v) : \det(\mathrm{p}(\tilde{\gamma})) \in B\}.$$

It is a simple matter to check that $\tilde{M}^B(i)$ is a normal subgroup of $\tilde{M}(i)$ of finite index. Let $\tilde{\rho}_i$ be a genuine irreducible $\widetilde{A^n}\tilde{M}(i)$-module whose restriction to $\mathrm{s}_0(A^n)$ the central character $\tilde{\omega}$ of §3. Let $\tilde{\rho}_i''$ be the restriction of $\tilde{\rho}_i$ to $\widetilde{A^n}\tilde{M}(i)$. The $\widetilde{A^n}\tilde{M}(i)$-module $\tilde{\rho}_i''$ is a sum of conjugates of some irreducible $\widetilde{A^n}\tilde{M}(i)$-module $\tilde{\rho}_i'$. Otherwise stated,

$$\tilde{\rho}_i'' = \sum_{\gamma} \tilde{\rho}_i'^{\gamma},$$

where the sum runs over representatives γ of certain cosets in $\tilde{M}(i)/\tilde{M}^B(i)$.

Lemma 19.1 *If γ is as above then $\tilde{\rho}'_i$ is not equivalent to $\tilde{\rho}'^{\gamma}_i$ unless $M(i) = \mathrm{GL}(1)$ or $\gamma \in \tilde{M}^B(i)$.*

Proof. If $M(i) = \mathrm{GL}(1)$ then $\tilde{M}(i) = F_v^{\times} \times \mu_n$. In particular $\tilde{M}(i)$ is abelian and $\tilde{\rho}'_i = \tilde{\rho}'^{\gamma}_i$. Suppose that $r_i \geq 2$. By using the Iwasawa decomposition, it is easy to see that representatives of $\tilde{M}(i)/\tilde{M}^B(i)$ may be taken to be diagonal matrices. Let

$$\gamma = \begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_{r_i} \end{pmatrix}$$

be such a representative. Suppose that $\tilde{\rho}'^{\gamma}_i$ is equivalent to $\tilde{\rho}'_i$. More precisely, suppose that there exists a linear isomorphism T such that

$$T \circ \tilde{\rho}'^{\gamma}_i(\tilde{\gamma}) = \tilde{\rho}'_i(\tilde{\gamma}) \circ T,$$

for all $\tilde{\gamma} \in \widetilde{A^n}\tilde{M}^B(i)$. Let $x \in B$ and choose $\tilde{\gamma} \in \tilde{M}^B(i)$ such that

$$\mathbf{p}(\tilde{\gamma}) = \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix}.$$

It may be verified by following 0.1.1 [19] that $\tilde{\gamma}$ is in the centre of $\widetilde{A^n}\tilde{M}^B(i)$. Thus, we have,

$$\begin{aligned} \tilde{\rho}'_i(\tilde{\gamma}) &= T \circ \tilde{\rho}'^{\gamma}_i(\tilde{\gamma}) \circ T^{-1} \\ &= T \circ \tilde{\rho}'_i(\gamma^{-1}\tilde{\gamma}\gamma) \circ T^{-1} \\ &= ((\det(\gamma), \det(\mathbf{p}(\tilde{\gamma})))_{F_v} / \prod_{j=1}^{r_i} (\gamma_i, x)_{F_v}) T \circ \tilde{\rho}'_i(\tilde{\gamma}) \circ T^{-1}, \end{aligned}$$

by Proposition 0.1.5 [19] and the multiplicativity of the Hilbert symbol. By Schur's lemma $\tilde{\rho}'_i(\tilde{\gamma})$ is a nonzero scalar operator. Consequently this last equality may be rewritten as

$$(\det(\gamma), x)^{r_i - 1} = 1.$$

Since $\gcd(n, r_i - 1) = 1$ (§1), we have that $(\det(\gamma), x) = 1$. The element $x \in B$ was chosen arbitrarily so this means that $\gamma \in \tilde{M}^B(i)$.□

Continuing with the discussion on tensor products, we set $\tilde{\rho}'$ to be the $\widetilde{A^n}\tilde{M}^B$-module, $\otimes_{i=1}^{\ell}\tilde{\rho}'_i$. This module is irreducible and, from arguments similar to those in the proof of Lemma 19.1, we find that it is inequivalent to any of its conjugates by elements in $\tilde{M} - \widetilde{A^n}\tilde{M}^B$. Mackey's criterion then yields that the $\tilde{M}$-module, $\tilde{\rho}$, induced from $\tilde{\rho}'_i$ is irreducible. This process may be reversed without difficulty. Thus every irreducible $\tilde{M}$-module corresponds to a unique set of $\widetilde{A^n}\tilde{M}(i)$-modules.

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