New Designs and Coverings

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
in the Department
of
Mathematics and Statistics

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SIMON FRASER UNIVERSITY
August 1997

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0-612-24295-1

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Abstract

In this thesis we study coverings and designs with good covering properties; that is, designs in which the size of the maximum intersection between any two blocks is small. The first chapter contains some basic facts that are used throughout the thesis.

In chapter two we give many new constructions of coverings. We focus on coverings on a reasonably small set of points as they can be directly applied. An interesting "competition" to improve the upper bounds on the sizes of coverings started in 1993 when Nurmela and Östergård used simulated annealing to obtain good coverings for up to 13 points. In 1995, Gordon, Kuperberg and Patashnik obtained good coverings for up to 32 points using somewhat less precise, but faster algorithms and that same year Chang, Etzion and Wei made improvements using combinatorial constructions based on previous results of Etzion. In this chapter we improve many of the bounds in the works of these authors; most of the improvements are accomplished by purely combinatorial arguments, while others are assisted by computer searches.

In chapter three we describe three new families of minimal $(t+1)$-coverings obtained from t-designs. These coverings produce new covering numbers for an infinite

number of parameters. Aside from t-designs (which are also t-coverings) prior to this work only three infinite families of coverings were known; those obtained by Ray-Chaudhuri, and Abraham, Ghosh and Ray-Chaudhuri in 1968, and Todorov in 1984.

In chapter four we prove the existence of 22 new simple 3-designs on 26 and 28 points. The base of the constructions is two designs in each of which the size of the intersection of any two blocks is small. We also use methods introduced in previous work and a new approach based on designs obtained via Driessen's Theorem and its corollaries. The designs on 26 points are obtained from the inversive geometry of order 5. The designs on 28 points are obtained from a code of van Lint and MacWilliams.

Acknowledgements

I would like to thank my supervisor, Dr. Katherine Heinrich, for all her help and advice, and for financial support through NSERC research grants. I would also like to thank Simon Fraser University for financial support. Special thanks to aunt Toni and uncle Jordan for their continuous support of my studies.

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Chapter 1

Some basic facts

In this chapter we establish the notation to be used throughout the thesis and present some basic facts and definitions.

1.1 Design theory

Let $D = \{B_1, B_2, ..., B_b\}$ be a finite family of k-subsets (called **blocks**) of a v-set $X(v)$ $= \{1, 2, ..., v\}$ (with elements called **points**). Then D is a t-(v, k, λ) **design** if every t-subset of $X(v)$ is contained in exactly λ blocks of D. Next, define

$$p = \max_{1 \le i < j \le b} |B_i \cap B_j|.$$

We call p the **maximal intersection number of** D.

Given a point x in $X(v)$, the blocks of D that contain x, after removing x from each of these blocks, form a $(t-1)$-$(v-1, k-1, \lambda)$ design D^x on $X\backslash\{x\}$ called the **derived design of D with respect to x**. The blocks of D that do not contain x form a $(t-1)$-$(v-1, k, \frac{v-k}{k-t+1}\lambda)$ design D_x on $X\backslash\{x\}$ called the **residual design of D with respect to x**. We say also that D^x and D_x form a **matching pair of designs**.

If every t-subset of $X(v)$ is contained in at most λ blocks of D, then D is a t-(v, k, λ) **packing design** (or **packing**). If every t-subset of $X(v)$ is contained in at least λ blocks of D, then D is a t-(v, k, λ) **covering design** (or **covering**). Given a covering, the number of blocks is the **size** of the covering, and the minimum size of a t-(v, k, λ) covering is called the **covering number**, denoted $C_\lambda(v, k, t)$. A covering of size $C_\lambda(v, k, t)$ is called a **minimal covering**. When $\lambda = 1$ we write $C(v, k, t)$ instead of $C_1(v, k, t)$, and we say a (v, k, t) covering instead of a t-$(v, k, 1)$ covering. We also say a (v, k, t) packing instead of a t-$(v, k, 1)$ packing. A **Steiner system** $S(v, k, t)$ is a (v, k, t) covering in which every t-set is covered exactly once.

A general lower bound on $C_\lambda(v, k, t)$ is due to Schönheim [45].

THEOREM 1.1.1

$$C_\lambda(v, k, t) \geq \left\lceil \frac{v}{k} C_\lambda(v-1, k-1, t-1) \right\rceil .$$

Proof. The total number of points involved in all blocks of a t-(v, k, λ) covering design of the minimum possible size $b = C_\lambda(v, k, t)$ is bk. On the other hand, removing a point from all blocks that contain it produces a $(t-1)$-$(v-1, k-1, \lambda)$ covering. Consequently, each point is in at least $C_\lambda(v-1,k-1,t-1)$ blocks. Thus $bk \geq vC(v-1,k-1,t-1)$, so

$$b \geq \left\lceil \frac{v}{k} C_\lambda(v-1,k-1,t-1) \right\rceil .$$

$\square$.

By iterating the inequality of Theorem 1.1.1 we obtain the following.

Corollary 1.1.2

$$C_\lambda(v,k,t) \geq \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \cdots \left\lceil \frac{v-t+1}{k-t+1} \lambda \right\rceil \cdots \right\rceil \right\rceil .$$

A well-known necessary condition for the existence of a t-(v,k,λ) design D is that the λ_q, $1 \leq q \leq t$, defined by

$$\lambda_0 := b = |D|, \qquad \text{and} \qquad \lambda_q = \frac{k-q+1}{v-q+1}\,\lambda_{q-1},$$

be integers. Obviously, $\lambda_t = \lambda$ and the number of the blocks of the design is

$$\lambda_0 = b = \frac{v(v-1)\ldots(v-t+1)}{k(k-1)\ldots(k-t+1)}\,\lambda.$$

Let $D = \{B_1, B_2, \ldots, B_b\}$ be a t-(v,k,λ) design. It is known [34] that $D_s = \{X(v) \setminus B : B \in D\}$ is a t-$\left(v, v-k, \lambda\binom{v-k}{t} \Big/ \binom{k}{t}\right)$ design called the **supplemental design** of D. So we need only be concerned with designs having block-size at most $\left\lfloor \frac{v}{2} \right\rfloor$. The set of all k-subsets of $X(v)$ will be denoted by $X^{(k)}(v)$. (We will use $X^{(k)}$ instead of $X^{(k)}(v)$ whenever the value of v is clear from the context.)

Consider the set $X^{(s)}(v) = X^{(s)}$, where $t+1 \le s \le \left\lfloor \frac{v}{2} \right\rfloor$.

The **intersection numbers of an s-subset S of $X(v)$ with respect to the blocks of a t-(v,k,λ) design D** are defined by

$$n_i = n_i(S) = |\{B : B \in D, |B \cap S| = i\}|, \; i = 0, 1, ..., s.$$

So n_i is the number of blocks of D that intersect S in i points. The **intersection equations for S** are then given by

$$\sum_{i=m}^{s} \binom{i}{m} n_i = \binom{s}{m} \lambda_m \qquad \text{for} \qquad m = 0, 1, ..., \min(s,t) \;\; [52].$$

The last equation is obtained by counting in two ways the pairs (M, B) such that $M \subset S \cap B$, $B \in D$, and $|M| = m$.

The **spectrum of $A \in X^{(s)}$ under D** is the ordered $(m-t)$-tuple

$$Spec_D(A) = (n_{t+1}, n_{t+2}, ..., n_m),$$

where $m = min\{k, s\}$ and n_i, $i = t+1, ..., m$, are intersection numbers of A with respect to the blocks of the design D.

The **spectral set of $X^{(s)}$ under D** is the collection of all possible spectra of the elements of $X^{(s)}$ under D.

The equivalence relation $\Re$ on $X^{(s)}$ is defined by $A_i \Re A_j$ if and only if $Spec_D(A_i) = Spec_D(A_j)$. Therefore $\Re$ partitions $X^{(s)}$ into equivalence classes $X_1^{(s)}, X_2^{(s)}, ..., X_q^{(s)}$ and we write $Spec_D(A) = Spec_D(X_i^{(s)})$ for all $A \in X_i^{(s)} \subset X^{(s)}$. It turns out that some of these classes, or unions of some of these classes, are t'-designs for some t'.

A design without repeated blocks is called a **simple design**. Given two simple designs D_1 and D_2 on X we say that D_1 **is isomorphic to** D_2 if there exists a permutation $\pi : X \to X$ such that $\pi(B) \in D_2$ for every $B \in D_1$. The permutation π is called an **isomorphism from** D_1 **to** D_2. When $D_1 = D_2$, the permutation π is called an **automorphism of** D_1.

The set of all automorphisms of a design D form a group, $Aut(D)$, called the **automorphism group** of D. This group acts as a permutation group on the points and also as a permutation group on the blocks.

A t-(v, k, λ) design on $X(v)$ is said to be **cyclic** if whenever B is a block, $\{x + 1 : x \in B\}$ is also a block, where addition is performed modulo v. In this case Z_v, the cyclic group of order v, is a subgroup of $Aut(D)$.

1.2 Coding theory

Given a vector space $V = V_n(K)$ of dimension $n < \infty$ over the field K, a **code** C is a subset of V. The vectors in the code are called **codewords** or simply **words**. The **(Hamming) distance** between two codewords $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ is the number of places in which they differ; that is,

$$d(\mathbf{x}, \mathbf{y}) = |\{i : 1 \leq i \leq n, x_i \neq y_i\}|.$$

The **(Hamming) weight** of a vector $\mathbf{x} = (x_1, ..., x_n)$ is the number of nonzero coordinates, and is denoted by $wt(\mathbf{x})$; that is, $wt(\mathbf{x}) = d(\mathbf{x}, \mathbf{0})$ where $\mathbf{0}$ is the all zero vector. More generally, $wt(\mathbf{x} - \mathbf{y}) = d(\mathbf{x}, \mathbf{y})$. The **minimal distance** of a code is

$$d = \min\{d(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in C, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}\}.$$

The **support** of a codeword is the set of positions of nonzero coordinates.

A code is **linear** if it is a subspace of V. Given a code C, and a vector $\mathbf{v} \in V$, the set

$$\mathbf{v} + C = \{\mathbf{v} + \mathbf{c} : \mathbf{c} \in C\}$$

is called a **translate of the code** C **by the vector** $\mathbf{v}$. A translate of a code is also a code with the same minimal distance as the original.

Chapter 2

New Upper Bounds on the size of a covering

2.1 Introduction

In this chapter we give many new constructions of coverings. Constructions for specific sets of parameters are typical for this research area and, in fact, there is no general theory behind obtaining good coverings. Computer searches have been made but the existing computer algorithms either produce coverings of poor quality in a reasonable amount of time or coverings of good quality but at a very large cost (CPU time).

An interesting "competition" to improve the upper bounds on the sizes of coverings started in 1993 when Nurmela and Östergård used simulated annealing to obtain good coverings for up to 13 points. In 1995, Gordon, Kuperberg and Patashnik obtained good coverings for up to 32 points using somewhat less precise, but faster algorithms and that same year Chang, Etzion and Wei improved some of the results of Gordon et al. by using combinatorial constructions based on previous results of Etzion. Continuing in the spirit of this competition we improve many of the bounds in the works of these authors; most of the improvements are accomplished by purely combinatorial arguments, while others are assisted by computer searches. Some of the results presented in this chapter have been published in [10].

We focus on coverings on a reasonably small set of points as they can be directly applied; for example, in error-trapping decoding. In this particular case the complexity of the decoding procedure depends on the size of the covering [16], and thus we are interested in finding coverings of the smallest size possible. Further applications are to data compression (see [19] and [27]) and in strategies for selecting lottery tickets (see [38],[40] and [41]).

Let us illustrate the last application. A k/n **lottery** is a game where players initially buy tickets each containing a chosen k-subset of the set $\{1, 2, ..., n\}$ and then a k-subset is drawn randomly from the same n-set. A player gets an s-**win** if the intersection between the k-subset chosen and the k-subset drawn is s. Suppose a player or a group of players (called a syndicate) wants to play a 6/49 (or any $6/n$ for $n \geq 14$) lottery by choosing only tickets with 6-sets from a particular size 14 subset of the n-set. If the syndicate chooses to play with tickets that correspond to the blocks of a $(14,6,4)$ covering then they will get at least one 4-win whenever any 4 of their 14 numbers are drawn. Thus the syndicate will secure a certain guarantee. Since any $(14,6,4)$ covering gives the same guarantee, they should choose the most "economical" covering; that is, the covering with the smallest known number of blocks, which is currently 80, and hence they purchase the fewest number of tickets.

Naturally, one can ask: What is the advantage of playing for such a guaranteed win? If we compare playing with 80 random tickets against 80 tickets forming a $(14,6,4)$ covering we see that the probability of a 6-win ("hitting the jackpot") is the same for each ticket; namely $\binom{49}{6}^{-1}$. However, if any 4 of the numbers drawn are among the 14 numbers chosen by the syndicate, then the 80 tickets of a $(14,6,4)$ covering guarantee at least one 4-win while 80 random tickets (on the same 14 numbers) guarantee nothing! This property of the coverings is attractive to some lottery players and there are many books (see, for example, [32] or [46]) and computer software available describing coverings. In this application, the coverings are often referred to as wheels or lottery systems.

There is an extensive literature on the covering numbers $C(v,k,t)$ and [48], [28] and [36] provide excellent surveys. Techniques for finding good coverings (that is, coverings of size as small as possible for fixed v,k,t) are discussed in [28], [40], [16] and [27].

Table 2.1: Comparison between the old and new bounds

$v\backslash(k,t)$	(5,3)	(6,4)	(7,5)	(5,4)	(6,5)	(6,3)
13			78(88)			
14	43(47)	80(87)	143(154)	232(235)	377(385)	
15	57(60)	118(134)	203(224)	303(313)	609(620)	
16	65(68)	160(178)	321(358)	416(437)	808(840)	
17		188(243)	408(506)	492(558)	1215(1277)	
18		236(258)	596(696)	671(732)	1547(1791)	
19	113(114)	330(352)	707(930)	850(926)	2175(2501)	63(66)
20	138(145)	400(456)	1037(1239)	1095(1165)	2900(3297)	72(75)

This chapter is concerned with new constructions of coverings and hence presents many new upper bounds on the covering number $C(v,k,t)$.

The Table 2.1 shows a comparison between the new and the old bounds on the covering number $C(v,k,t)$ in the same range as given in the tables in the recent CRC Handbook of Combinatorial Designs [48]. The old bounds from [48] are given in parentheses. A complete account on the improvements done in this chapter over the (more extensive) tables in [28] is given in Tables 2.3 and 2.4.

The blank spaces in Tables 2.1, 2.3 and 2.4 correspond to parameters for which we have not found a new bound. In some cases this is due to the fact that the covering number is known. For example, $C(17,5,3) = 68$, because there exists a Steiner system $S(17,5,3)$. The covering number $C(18,5,3) = 94$ follows from the fact that $C(17,4,2) = 26$ [36] and the following sequence of results [36].

Let

$$L(v,k,t) = \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \cdots \left\lceil \frac{v-t+1}{k-t+1} \right\rceil \cdots \right\rceil \right\rceil$$

(note that this is the expression in the right hand side of the inequality of Corollary 1.1.2 for $\lambda = 1$).

Lemma 2.1.1 *If there exists a Steiner system $S(v,k,t)$, then*

$$L(v+1,k,t) = \frac{v+1}{k}L(v,k-1,t-1) + \frac{k-t}{k} = L(v,k,t) + L(v,k-1,t-1).$$

THEOREM 2.1.2 (SCHÖNHEIM) *If there exists a Steiner system $S(v,k,t)$, then $C(v+1,k,t) = L(v+1,k,t)$.*

Corollary 2.1.3 *If there exists a Steiner system $S(v,k,t)$, then*

$$C(v+1,k,t) = C(v,k,t) + C(v,k-1,t-1).$$

There is no general method for finding good coverings although many constructions are based on the study of particular properties of designs and codes. The techniques presented in this chapter might be described as "combining smaller coverings". We also use a "partitioning construction" similar to the one used by Etzion and Van Pul [26] for constructing constant weight codes (also see [13] and [27]).

2.2 Preliminary results

First we discuss some facts and notation that will be used throughout the chapter. We begin with three simple constructions.

Construction 2.2.1 *Given a (v, k, t) covering design D and a point $x \in X(v)$, the blocks of D that contain x form a $(v-1, k-1, t-1)$ covering on $X(v) \setminus \{x\}$.*

A covering of the smallest size among those obtainable from D is produced by choosing x to be a point that occurs in the fewest blocks of D.

Construction 2.2.2 *Given a (v, k, t) covering and a $(v, k-1, t-1)$ covering on the same set $X(v)$, we obtain a $(v+1, k, t)$ covering on $X(v) \cup \{x\}$ by adding the new point x to all the blocks of the $(v, k-1, t-1)$ covering and taking the union of these blocks with those of the (v, k, t) covering.*

The size of the $(v + 1, k, t)$ covering is the sum of the sizes of the initial two coverings. Thus we get the following.

Corollary 2.2.3 $C(v+1, k, t) \leq C(v, k, t) + C(v, k-1, t-1)$.

Construction 2.2.4 (SIDORENKO - TURAN [28]) *Given a (v, k, t) covering, let $x \in X(v)$. Choose two new points x' and x''. If a block B does not contain x, replace it by the two blocks, $B \cup \{x'\}$ and $B \cup \{x''\}$; if B contains x, replace it by the single block*

$(B \setminus \{x\}) \cup \{x', x''\}$. Finally, add a $(v-1, k+1, t+1)$ covering on $X(v) \setminus \{x\}$. The result is a $(v+1, k+1, t+1)$ covering on $(X(v) \setminus \{x\}) \cup \{x', x''\}$.

Corollary 2.2.5 *If the number of the blocks in a (v, k, t) covering is b, and b_x is the number of blocks in which x occurs, then the size of the new covering obtained by Construction 2.2.4 is $b_x + 2(b - b_x) + s$, where s is the size of the $(v-1, k+1, t+1)$ covering.*

Note that by choosing x to be a point in the largest number of blocks, we minimize the size of the resulting $(v+1, k+1, t+1)$ covering.

A slightly weaker bound follows on averaging the occurrence of a point.

Corollary 2.2.6

$$C(v+1, k+1, t+1) \leq \left\lfloor (2 - \frac{k}{v}) C(v, k, t) \right\rfloor + C(v-1, k+1, t+1).$$

Proof. Using the pigeon-hole argument, there is a point x which is in at least $\left\lceil \frac{k}{v} C(v, k, t) \right\rceil$ blocks of a (v, k, t) covering of size $C(v, k, t)$. Therefore we can assume $b_x = \left\lceil \frac{k}{v} C(v, k, t) \right\rceil$. Then the size of the $(v+1, k+1, t+1)$ covering from Construction 2.2.4 is

$$\begin{aligned} 2b - b_x + s &= 2C(v, k, t) - \left\lceil \tfrac{k}{v} C(v, k, t) \right\rceil + s \\ &= \left\lfloor (2 - \tfrac{k}{v}) C(v, k, t) \right\rfloor + s, \end{aligned}$$

which implies the result. □

We continue with various structures that will be used to produce coverings. A **large set of mutually disjoint Steiner systems** $LS(v,k,t)$ is a partition of $X^{(k)}(v)$ into Steiner systems. Two important results are the following.

THEOREM 2.2.7 *(*TIERLINCK [49]*) An $LS(v,3,2)$ exists if and only if $v \equiv 1$ or 3 $\pmod 6$, $v \geq 9$.*

THEOREM 2.2.8 *(*BARANYAI [4]*) An $LS(v,k,1)$ exists if and only if k divides v.*

We will use a particular instance of this theorem, namely, the existence of an $LS(v,2,1)$ for v even (that is, a 1-factorization of the complete graph K_v, v-even).

In some cases we use a union of (v,k,t) coverings which produce a $(v,k,t+1)$ covering or a union of (v,k,t) coverings and k-sets which together form a $(v,k,t+1)$ covering. The following example is due to Griggs and Rosa [30] and should be read in reference to Theorem 2.2.7.

THEOREM 2.2.9 *(*GRIGGS AND ROSA [30]*)There exist six $(7,3,2)$ coverings of size 7 whose union is a $(7,3,3)$ covering.*

Proof. Each column in the array below contains one of the coverings.

123	123	124	125	126	127
145	146	137	136	135	134
167	157	156	147	147	156
246	247	235	234	237	236
257	256	267	267	245	245
347	345	346	357	346	357
356	367	457	456	567	467

□

The next theorem [2] can be used to extend the coverings of Theorem 2.2.9.

THEOREM 2.2.10 *(ALLTOP [2]) Let D be a t-$(2k+1, k, \lambda)$ design with t even. Then*

$$\{B' : B' = X \backslash B,\ B \in D\} \cup \{B'' : B'' = B \cup \{2k+2\}, B \in D\}$$

is a $(t+1)$-$(2k+2, k+1, \lambda)$ design.

THEOREM 2.2.11 *There exist six $(8,4,3)$ coverings of size 14 whose union is an $(8,4,4)$ covering.*

Proof. The result follows by applying Theorem 2.2.10 to each of the $(7,3,2)$ coverings defined in Theorem 2.2.9. □

Below is a particular case of a result obtained by Etzion [24].

THEOREM 2.2.12 *(*ETZION [24]*) There exist four* $(6,4,3)$ *coverings of size* 6 *whose union is a* $(6,4,4)$ *covering.*

Proof. Each column contains one of the coverings.

1234	1234	1234	1234
1235	1245	1256	1256
1236	1246	1345	1356
1456	1356	1346	1456
2456	2356	2356	2345
3456	3456	2456	2346

□

The following are small minimal coverings that will be used in later proofs.

THEOREM 2.2.13 $C(6,5,3) = 4$, $C(8,5,3) = 8$, $C(9,5,3) = 12$, $C(7,5,4) = 9$.

Proof. Each column contains the corresponding covering.

C(6,5,3)=4	C(8,5,3)=8	C(9,5,3)=12	C(7,5,4)=9
12345	12378	12348	12347
12346	12468	12567	12357
12356	12567	12589	12367
12456	13467	13456	14567
	13458	13479	24567
	23457	13578	34567
	23568	15689	12345
	45678	23459	12346
		23679	12356
		24678	
		34689	
		45789	

□

A t-$(v, \{k_1, k_2, ..., k_n\}, \lambda)$ **design (covering design)** is a pair $(X(v), D)$, where $X(v) = \{1, 2, ..., v\}$ is a set of points and D is a subset of $X^{(k_1)}(v) \cup X^{(k_2)}(v) \cup ... \cup X^{(k_n)}(v)$ with elements called blocks (of size $k_1, k_2, ..., k_n$) so that every t-set of $X(v)$ is contained in exactly (at least) λ blocks.

Let the set X be the disjoint union of the sets X_1 and X_2 of sizes n_1 and n_2, respectively. We define an $[m_1, m_2]$-**set** to be an $(m_1 + m_2)$-subset of X with m_1 of its elements in X_1 and the remaining m_2 elements in X_2.

It is convenient to represent a covering by a $b \times k$ matrix whose rows are the blocks of the covering. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1q} \\ b_{21} & b_{22} & \dots & b_{2q} \\ \dots & \dots & \dots & \dots \\ b_{p1} & b_{p2} & \dots & b_{pq} \end{pmatrix}$$

be a set of m blocks of size n and a set of p blocks of size q, respectively. We use the notation AB to represent the following set of mp blocks:

$$\{\{a_{i1}, a_{i2}, \dots, a_{in}, b_{j1}, b_{j2}, \dots, b_{jq}\} : \ i = 1, 2, ..., m; \ j = 1, 2, ..., p\}.$$

A design is said to be **resolvable** if there exists a partition of its set D of blocks into subsets called **resolution classes** each of which in turn partitions the set $X(v)$. In particular, a Steiner system $S(v, k, t)$ is a (v, k, t) covering, and if it is resolvable, each resolution class is a $(v, k, 1)$ covering.

We use the following variation on the greedy algorithm (see [7]) for finding other partitions; in particular, a partition of the blocks of a design into designs with smaller numbers of blocks.

Input: Design D with blocks arranged in random order and a number p', the maximal intersection number of the design being searched for. (Obviously, we should take $p' < p$, where p is the maximal intersection number of the input design.)

Description: Take the first block of D to be the first block of D', say B_1. Having chosen blocks $B_1, B_2, ..., B_l$ of D', $l \geq 1$, choose the next, B_{l+1}, to be the first block of D that meets the conditions $|B_{l+1} \cap B_s| \leq p'$, $s = 1, 2, ..., l$. When the process ends we have a collection D' of blocks of D. If D' is not a design we select another random ordering of the blocks of D and start again. Continue until a design is found or we run out of time and stop.

Output: D', which is a design.

Suppose we have obtained a design D'. Consider $D \setminus D'$. We can apply the same algorithm to $D \setminus D'$ to (possibly) produce a new design D'' and so on. It is clear that if $D', D'',...$ are designs, obtained consecutively from $D, D \setminus D',...$ by the algorithm and $D^* = D \setminus \{D' \cup D'' \cup ...\}$ is nonempty, then D^* must be a design as well.

THEOREM 2.2.14 *(a) There exists a* $3\text{-}(4^m, 6, \frac{1}{3}(4^m - 4))$ *design* D *with* $p = 3$. *The family of all* 4*-subsets of* $X(4^m)$ *that are not covered by any block of* D *is a Steiner system* $S(4^m, 4, 3)$.

(b) When $m = 2$, *the* $S(16, 4, 3)$ *constructed in (a) can be partitioned into seven* $S(16, 4, 2)$.

The existence of the design D described in Theorem 2.2.14 and the corresponding $S(4^m, 4, 3)$ is discussed in [3]. It is clear that the union of these two designs is a 4-$(4^m, \{6, 4\}, 1)$ design.

The partitioning of an $S(16, 4, 3)$ into seven $S(16, 4, 2)$ is mentioned in [15]. An alternative way of obtaining a 3-$(16, 6, 4)$ design with $p = 3$ is described in [7]. The application of the aforementioned algorithm to the $S(16, 4, 3)$ corresponding to the 3-$(16, 6, 4)$ design with $p = 3$ described in [7] produces the partitioning in (b).

These designs are given below. To avoid listing all blocks of the designs we use the following compressed notation. Suppose the k-subsets of $X(v)$ are arranged in lexicographical order (for example, if $v = 4$ and $k = 3$, then the order is 123,124,134,234). We encode the design according to the following rule: Let the blocks of the design have positions $c_1, c_2, ..., c_b$ in the lexicographical ordering of $X^{(k)}(v)$. Now, form the sequence $\{a_i\}_{i=1}^{b}$ by $a_1 = c_1$, $a_i = c_i - c_{i-1}$, $i = 2, 3, ..., b$. Given the sequence $a_1, a_2, \ldots, a_b$, the n-th block of the design can be recovered as the $(\sum_{i=1}^{n} a_i)$-th k-set from the lexicographical ordering of $X^{(k)}(v)$, where $1 \leq n \leq b$.

D: 28, 58, 69, 108, 65, 31, 84, 144, 56, 55, 36, 65, 151, 13, 107, 116, 1, 37, 151, 3, 85, 65, 14, 82, 135, 13, 38, 97, 154, 33, 114, 53, 32, 61, 121, 72, 174, 11, 89, 45, 17, 58, 101, 115, 22, 84, 27, 113, 30, 99, 119, 66, 2, 128, 138, 17, 39, 20, 57, 173, 12, 61, 97, 5, 139, 30, 131, 106, 22, 92, 38, 59, 23, 93, 70, 22, 201, 72, 103, 60, 42, 77, 135, 47, 113, 1, 42, 15, 184, 81, 49, 100, 53, 55, 10, 138, 19, 147, 7, 112, 9, 9, 133, 18, 188, 29, 39, 24, 135, 34, 129, 63.

A_1 : 60, 32, 217, 43, 103, 56, 68, 37, 77, 253, 88, 53, 88, 45, 97, 37, 55, 68, 150, 133.

A_2 : 75, 89, 22, 63, 129, 100, 82, 116, 73, 107, 109, 31, 137, 181, 25, 145, 42, 95, 150, 27.

A_3 : 26, 106, 79, 180, 34, 95, 77, 110, 58, 114, 44, 27, 177, 36, 234, 58, 138, 15, 39, 79.

A_4 : 25, 117, 119, 36, 139, 67, 125, 53, 47, 116, 47, 213, 51, 28, 67, 141, 153, 70, 166, 13.

A_5 : 9, 215, 43, 34, 67, 196, 86, 8, 140, 69, 29, 87, 77, 55, 118, 223, 110, 77, 92, 11.

A_6 : 48, 88, 82, 66, 47, 126, 163, 189, 1, 106, 104, 58, 45, 143, 10, 82, 136, 66, 101, 52.

A_7 : 79, 45, 67, 81, 141, 78, 49, 130, 111, 81, 67, 111, 152, 113, 22, 106, 149, 96, 13, 4.

Corollary 2.2.15 *There exists a* 2-$(15, 5, 4)$ *design with* $p = 2$ *so that the family of all* 3*-subsets of* $X(15)$ *that are not covered by any block of this design form a resolvable Steiner system* $S(15, 3, 2)$.

Proof. Take the derived designs of all the designs in Theorem 2.2.14 when $m = 2$. □

Observe that taking the union of the blocks of the 2-(15,5,4) and the $S(15, 3, 2)$ of Corollary 2.2.15 results in a 3-(15,{5,3},1) design.

2.3 New Upper Bounds on $C(v, k, t)$

It is known that $C(v, 4, 3) = L(v, 4, 3)$ for all values of v except for $v \equiv 7 \pmod{12}$ [36]. We start our discussion with the covering numbers $C(v, 5, 3)$.

2.3.1 Bounds on $C(v,5,3)$

In this section we improve upper bounds on $C(v,5,3)$ for seven values of v in the range $14 \le v \le 24$. At present, the best upper bound on the size of a $(14,5,3)$ covering is 43 and the corresponding covering was found by the program cover [39]. A compressed description of such a covering is given in the appendix.

THEOREM 2.3.1 $C(15,5,3) \le 57$.

Proof. Let $N_j = (3j-2, 3j-1, 3j)$, $j = 1,2,...,5$. Partition $X(15)$ into two sets, $X_1 = \{1,2,...,6\}$ and $X_2 = \{7,8,...,15\}$. Then

$$I = \begin{pmatrix} 7 & 11 & 15 \\ 8 & 12 & 13 \\ 9 & 10 & 14 \end{pmatrix} \quad J = \begin{pmatrix} 7 & 12 & 14 \\ 8 & 10 & 15 \\ 9 & 11 & 13 \end{pmatrix}, \quad K = \begin{pmatrix} 7 & 10 & 13 \\ 8 & 11 & 14 \\ 9 & 12 & 15 \end{pmatrix} \quad L = \begin{pmatrix} N_3 \\ N_4 \\ N_5 \end{pmatrix}$$

are the resolution classes of a 2-(9,3,1) design on X_2. Let

$$A = \begin{pmatrix} 4 & 13 \\ 5 & 14 \\ 6 & 15 \\ 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 10 \\ 2 & 11 \\ 3 & 12 \\ 7 & 13 \\ 8 & 14 \\ 9 & 15 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \\ 10 & 13 \\ 11 & 14 \\ 12 & 15 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 4 \\ 7 & 13 \\ 8 & 14 \\ 9 & 15 \end{pmatrix} \quad E = \begin{pmatrix} 1 & 6 \\ 2 & 4 \\ 3 & 5 \\ 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{pmatrix}$$

and

$$F = \begin{pmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 5 \end{pmatrix} \quad G = \begin{pmatrix} 1 & 5 \\ 2 & 4 \\ 3 & 6 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{pmatrix}.$$

We claim that the 57 blocks

$$\begin{array}{cc} 1\,2\,3\,A & FI \\ 4\,5\,6\,B & GJ \\ N_3C & HK \\ N_4D & \\ N_5E & \end{array}$$

form a (15,5,3) covering.

To see this observe that the [2,1]-sets in which the 2-component is a subset of N_j, $j = 1, 2$, are covered by the blocks of 1 2 3 A and 4 5 6 B. Further, the blocks of 1 2 3 A and 4 5 6 B contain as subblocks $N_1N_2^T$ and $N_1^TN_2$ which constitute a $(6, 4, 3)$ covering on X_1, and therefore all of the [3,0]-sets are covered.

The [2,1]-sets in which the 2-component is neither a subset of N_1 nor N_2, and the [1,2]-sets are covered by the blocks of

$$\begin{pmatrix} 14 \\ 26 \\ 35 \end{pmatrix} I \quad \begin{pmatrix} 15 \\ 24 \\ 36 \end{pmatrix} J \quad \begin{pmatrix} 16 \\ 25 \\ 34 \end{pmatrix} K \quad N_3 \begin{pmatrix} 14 \\ 25 \\ 36 \end{pmatrix} \quad N_4 \begin{pmatrix} 15 \\ 26 \\ 34 \end{pmatrix} \quad N_5 \begin{pmatrix} 16 \\ 24 \\ 35 \end{pmatrix}$$

Finally, the [0,3]-sets are contained in the blocks

$$N_3C \quad FI$$

$$N_4D \quad GJ$$

$$N_5E \quad HK.$$

□

THEOREM 2.3.2 $C(20,5,3) \leq 138$.

Proof. Partition $X(20)$ into the two sets $X_1 = X(16)$ and $X_2 = \{17,18,19,20\}$. The base of the construction is the unique 3-(17,5,1) design D on $X_1 \cup \{17\}$ [53]. Without loss of generality, assume that 13 14 15 16 17 is a block of D. Let D^i, $i = 14,15,16$ be the derived design of D with respect to i. Let $D(i)$ be the design obtained from D^i by replacing the point 17 by the point i. Simple counting shows there is a collection A of 12 blocks of D such that each of these 12 blocks contains exactly two of the three points 14, 15 and 16. Let $D' = D \setminus A$. Let B be a 1-factor of the complete graph K_{12} on {1,2,...,12} and let $C = (1\ 2\ \ldots\ 16)^T$. We claim that the 138 blocks of

$$D(i)\ (i+4), \quad i = 14,15,16$$

$$B\ 14\ 15\ 16$$

$$C\ 17\ 18\ 19\ 20$$

$$D'$$

form a (20,5,3) covering.

The [2,1]-sets $a\ b\ (i+4)$, where $a, b \in X_1,\ i = 14, 15, 16$, are covered by $D(i)\ (i+4)$. The [2,1]-sets $a\ b$ 17 are covered by D' (the 12 blocks of A do not contain the point 17 as 13 14 15 16 17 is a block of D).

The [1,2]-sets and the [0,3]-sets are covered by C 17 18 19 20.

It remains to show that all of the [3,0]-sets are covered. Consider a partition of X_1 into two sets: $X' = \{1, 2, ..., 13\}$ and $X'' = \{14, 15, 16\}$. In what follows, the $[a, b]$ notation is applied on $X' \cup X''$. The [0,3]-sets and the [1,2]-sets are covered by B 14 15 16.

The collection D' covers all of the triples of X_1 with the exception of those contained in the blocks of A. Let $xyzij$ be a block of A. Then $x, y, z \in X'$ and $i, j \in X''$. But then $xyzi$ is a block of $D(j)$, and $xyzj$ is a block of $D(i)$. Therefore, the [3,0]-sets and the [2,1]-sets are covered by the blocks of D' and $D(i)\ (i+4),\ i \in X''$. □

Several good coverings with block size 5 can be obtained using the designs given in Corollary 2.2.15.

THEOREM 2.3.3 $C(23, 5, 3) \leq 190$.

Proof. Partition $X(23)$ into the two sets $X_1 = X(15)$ and $X_2 = \{16, 17, ..., 23\}$. Let D be the 2-(15,5,4) design on X_1 from Corollary 2.2.15. Let $A_1, A_2, ..., A_7$ be the resolution classes of the 2-(15,3,1) design formed by the 3-sets that are not covered by D. Let $B_1, B_2, ..., B_7$ be the 1-factors of a 1-factorization of the complete graph K_8

on X_2. Let C be an (8,5,3) covering of size 8 on X_2. We claim that the 190 blocks of

$$D$$

$$A_iB_i, \quad i = 1, 2, ..., 7$$

$$C$$

form a (23,5,3) covering.

Since these blocks contain as subblocks the blocks of a 3-(15,{5,3},1) design on X_1 (see Corollary 2.2.15 and the comment following it) all the [3,0]-sets are covered.

The [2,1]-sets and the [1,2]-sets are covered by the blocks of A_iB_i, $i = 1, 2, ..., 7$.

The [0,3]-sets are covered by the blocks of C. □

THEOREM 2.3.4 $C(24, 5, 3) \leq 234$.

Proof. Using the notation from the previous theorem, let

$$B_i' = \begin{pmatrix} & B_i & \\ i+15 & & 24 \end{pmatrix} \quad \text{for} \quad i = 1, 2, ..., 6 \quad \text{and} \quad B_7' = \begin{pmatrix} & B_7 & \\ 22 & & 24 \\ 23 & & 24 \end{pmatrix}.$$

Let C' be a $(9, 5, 3)$ covering of size 12 on $X_2 \cup \{24\}$ (Theorem 2.2.13). Then the 234 blocks of

$$D$$

$$A_iB_i', \quad i = 1, 2, ..., 7$$

$$C'$$

form a (23,5,3) covering. □

THEOREM 2.3.5 $C(21,5,3) \leq 151$.

Proof. Repeat the proof of Theorem 2.3.3, but take instead $X_2 = \{16, 17, ..., 21\}$ and let B_i, $i = 1, 2, ..., 5$ be the 1-factors of a 1-factorization of the complete graph K_6 on X_2. Let $B_6 = B_7 = B_1$ (note that B_6 and B_7 could also be any two 1-factors of K_6 on X_2). Take C to be a (6,5,3) covering of size 4 on X_2 (Theorem 2.2.13). □

THEOREM 2.3.6 $C(19,5,3) \leq 113$.

Proof. Repeat the proof of Theorem 2.3.3, but take $X_2 = \{16, 17, 18, 19\}$ and B_i, $i = 1, 2, 3$ to be the 1-factors of the 1-factorization of K_4 on X_2. Let $B_4 = B_5 = B_6 = B_7 = B_1$. Take C to be the block 1 16 17 18 19. □

The results of this section are summarized as follows. (The value in parentheses indicates the best previously known upper bound on $C(v,5,3)$.)

$C(14,5,3) \leq 43(47)$, $C(15,5,3) \leq 57(60)$, $C(19,5,3) \leq 113(114)$, $C(20,5,3) \leq 138(145)$, $C(21,5,3) \leq 151(171)$, $C(23,5,3) \leq 190(227)$, $C(24,5,3) \leq 234(260)$.

2.3.2 Bounds on $C(v,5,4)$

In this section we improve the upper bounds on $C(v,5,4)$ for $v = 14, 15, 16, 29$ and give some general upper bounds on $C(v,5,4)$. In particular, we observe that some of the upper bounds on $C(v,5,4)$ found by Etzion et al. [27] can be improved or generalized.

THEOREM 2.3.7 $C(15,5,4) \leq 303$.

Proof. Partition $X(15)$ into the two sets $X_1 = \{1,2,...,7\}$ and $X_2 = \{8,9,...,15\}$. Let $A_1, A_2, ..., A_6$ be the (7,3,2) coverings on X_1 from Theorem 2.2.9, and $A_7 = A_1$. Let $B_1, B_2, ..., B_7$ be the 1-factors of a 1-factorization of the complete graph K_8 on X_2. The blocks of A_iB_i, $i = 1,2,...,7$ cover each of the [3,1] and [2,2]-sets of $X(15)$. Let $C_1, C_2, ..., C_6$ be the (8,4,3) coverings on X_2 from Theorem 2.2.11, and $C_7 = C_1$. Then the blocks of iC_i, $i = 1,2,...,7$, cover each of the [1,3] and [0,4]-sets of $X(15)$. Add a (7,5,4) covering on X_1 to cover the [4,0]-sets of $X(15)$. Thus we obtain a (15,5,4) covering. Since $C(7,5,4) = 9$ we get $C(15,5,4) \leq (7)(7)(4) + (7)(14) + 9 = 303$. □

We should mention that, in what follows, whenever we use an upper bound for an application of the constructions 2.2.1, 2.2.2 and 2.2.4, it is either one obtained in the present thesis, or it is from [28] or [27]. When it is from [28] or [27] this will be indicated.

The following is a generalization of a result in [27].

THEOREM 2.3.8 *If* $v \equiv 1$ *or* $3 \pmod 6$, $v \geq 9$, *then*

$$C(2v-1,5,4) \leq C(v,5,4) + C(v-1,5,4) + vC(v-1,4,3) + \frac{v-1}{2}\binom{v}{3}.$$

Proof. Partition $X(2v-1)$ into the two sets $X_1 = \{1,2,...,v\}$ and $X_2 = \{v+1, v+2, ..., 2v-1\}$. According to Theorem 2.2.7 there exists an $LS(v,3,2)$. Let

$A_1, A_2, ..., A_{v-2}$ be the designs of the system on X_1. Let $B_1, B_2, ..., B_{v-2}$ be the 1-factors of a 1-factorization of the complete graph K_{v-1} on X_2. Then the blocks of A_iB_i, $i = 1, 2, ..., v-2$, cover exactly once each of the [3,1] and [2,2]-sets of $X(2v-1)$. The number of blocks of A_iB_i, $i = 1, 2, ..., v-2$, is

$$(v-2)\frac{v(v-1)}{3.2}\frac{v-1}{2} = \frac{v-1}{2}\binom{v}{3}.$$

The union of the blocks of A_iB_i, $i = 1, 2, ..., v-2$, with a $(v, 5, 4)$ covering on X_1, $(v-1, 5, 4)$ covering on X_2, and iC, $i = 1, 2, ..., v$, where C is a $(v-1, 4, 3)$ covering on X_2, produces a $(2v-1, 5, 4)$ covering on $X(2v-1)$. Thus the upper bound is obtained. ∎

For example, in [28], it is shown that $C(29, 5, 4) \leq 5427$, whereas Theorem 2.3.8 produces the bound $C(29, 5, 4) \leq 5085$, which is a significant improvement. The bounds needed for the calculation are $C(14, 5, 4) \leq 232$ (see the Appendix), and $C(14, 4, 3) = 91$ (because there exists a Steiner system $S(14, 4, 3)$). As described in the next theorem slightly better bounds can be obtained under certain conditions.

THEOREM 2.3.9 *Let $v \equiv 1$ or $3 \pmod 6$, $v \geq 9$. If there exists a family of at most v $(v-1, 4, 3)$ coverings so that their union is a $(v-1, 4, 4)$ covering, then*

$$C(2v-1, 5, 4) \leq C(v, 5, 4) + vC(v-1, 4, 3) + \frac{v-1}{2}\binom{v}{3}.$$

Proof. We basically follow the proof of the previous theorem, defining X_1, X_2, A_i and B_i as was done there. Let $C_1, C_2, ..., C_m$ be the $(v-1,4,3)$ covering on X_2, $m \leq v$, and let $C_{m+1} = C_{m+2} = ... = C_v = C_1$. Now for the covering, use the blocks of a $(v,5,4)$ covering on X_1, the blocks A_iB_i, $1 \leq i \leq v-2$ and the blocks iC_i, $i = 1,2,...,v$,. □

For example, Theorem 2.3.9 yields $C(17,5,4) \leq 492$ (determined earlier in [27]), which is the best known bound. The best bounds $C(14,5,4) \leq 232$ and $C(16,5,4) \leq 416$ were established by the program cover [39]. A compressed description of these coverings is given in the appendix.

Aside from the general results in this section, we have made the following improvements:

$C(14,5,4) \leq 232(235)$, $C(15,5,4) \leq 303(313)$, $C(16,5,4) \leq 416(437)$, $C(29,5,4) \leq 5085(5427)$,

where the value in parentheses indicates the previously best known upper bound on $C(v,5,4)$ for the particular value of v.

2.3.3 Bounds on $C(v,6,4)$

In this section we improve the bounds on $C(v,6,4)$ for certain values of v in the range $14 \leq v \leq 24$.

THEOREM 2.3.10 $C(14,6,4) \leq 80$.

Proof. Partition $X(14)$ into the two sets $X_1 = X(10)$ and $X_2 = \{11, 12, 13, 14\}$. Let C_i, $i = 1, 2, 3$, be the 1-factors of a 1-factorization of the complete graph K_4 on X_2. Let D' be a 2-(10,4,2) design on X_1 as given below (we write 0 instead of 10):

D'

1489	1235	2678
1670	1280	3460
2369	1347	3789
2457	1569	4568
3580	2490	5790.

Let D''' be the supplemental design of D'. The design D''' has the following interesting properties. (Properties (2) and (3) may be tediously verified.)

1) No block of D''' covers a block of D' as any two blocks of D' have at least one point in common.

2) There are exactly 15 4-sets of X_1 each of which is covered by precisely three blocks of D'''. These 15 sets form a 2-(10,4,2) design, denoted D'', such that $D' \cup D''$ is a 3-(10,4,1) design.

3) The blocks of D''' cover exactly once any 4-set of X_1 except those of D''.

D''

1368	1246	2560
1450	1279	3459
2370	1390	3567
2589	1578	4780
4679	2348	6890

Let $B = \begin{pmatrix} 12 \\ 34 \\ 56 \\ 78 \\ 90 \end{pmatrix}$. There are exactly 5 blocks that do not contain any of the pairs of B in each of the designs D' and D''. These blocks are given in the first column of blocks for each of the designs D' and D''. Let A_1 be the collection of these 10 blocks. Let $A_2 = D' \setminus A_1$ and $A_3 = D'' \setminus A_1$. We claim that the 80 blocks of

$$A_iC_i,\ i = 1, 2, 3$$

$$B\ 11\ 12\ 13\ 14$$

$$D'''$$

form a (14,6,4) covering.

Now, the [0,4]-sets and the [1,3]-sets are covered by the blocks of B 11 12 13 14.

The [4,0]-sets are covered because the union of D' and D''' covers all of the 4-sets of X_1.

The [3,1]-sets are covered by the blocks of A_iC_i, $i = 1,2,3$, since $D' \cup D''$ is a 3-(10,4,1) design.

To prove that all of the [2,2]-sets are covered, we note that each pair from X_1 (with the exception of the pairs of B) is in a collection of blocks with each of the 1-factors C_i, $i = 1,2,3$. Therefore, the [2,2]-sets are covered by either A_iC_i, $i = 1,2,3$, or B 11 12 13 14, which completes the proof. □

Now we introduce a particular construction for some well-known designs. The constructions leads to a design that proves to be useful in in the building of a $(15,6,4)$ covering.

Lemma 2.3.11 *Arrange the points of $X(9)$ in a 3×3 matrix $A = (a_{ij})$ with the entries of row i designated by r_i, $i = 1,2,3$ and column j by c_j, $j = 1,2,3$. Consider two sets on $X(9)$:*

$$M_{ij} = (r_i \cup c_j) \setminus \{a_{ij}\} \text{ and } N_{ij} = X(9) \setminus (r_i \cup c_j).$$

Then:

(a) The collection of blocks M_{ij} and N_{ij}, $1 \leq i < j \leq 3$, is a 2-$(9,4,3)$ design, denoted $D(A)$;

(b) The union of $D'(A) = \{M_{ij} \cup \{a_{ij}\} |\ i,j \in \{1,2,3\}\}$ and $D''(A) = \{N_{ij} \cup \{a_{ij}\} |\ i,j \in \{1,2,3\}\}$, is a 2-$(9,5,5)$ design; and

(c) The triples given by the three rows, three columns and six diagonals of A form a 2-$(9,3,1)$ *design.*

Proof. The verification of all three parts (a), (b) and (c) is straightforward. $\square$

Lemma 2.3.12 *The* 2-$(9,3,1)$ *design and the* 2-$(9,4,3)$ *design from Lemma* 2.3.11 *form a matching pair of designs. Thus we obtain a* 3-$(10,4,1)$ *design.*

THEOREM 2.3.13 $C(15,6,4) \leq 120$.

Proof. Partition $X(15)$ into the two sets $X_1 = \{1,2,...,9\}$ and $X_2 = \{a,b,c,d,e,f\}$. Let

$$S = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 5 & 9 \\ 2 & 6 & 7 \\ 3 & 4 & 8 \end{pmatrix} \quad V = \begin{pmatrix} 1 & 6 & 8 \\ 2 & 4 & 9 \\ 3 & 5 & 7 \end{pmatrix}$$

be the resolution classes of a 2-$(9,3,1)$ design on X_1. Let

$$S' = \begin{pmatrix} a & c & f \\ a & d & e \\ b & c & e \\ b & d & f \end{pmatrix} \quad T' = \begin{pmatrix} a & c & e \\ a & d & f \\ b & c & f \\ b & d & e \end{pmatrix} \quad U' = \begin{pmatrix} a & b & e \\ a & b & f \\ a & c & d \\ b & c & d \\ c & e & f \\ d & e & f \end{pmatrix} \quad V' = \begin{pmatrix} a & b & c \\ a & b & d \\ a & e & f \\ b & e & f \\ c & d & e \\ c & d & f \end{pmatrix}$$

be a partition of $X_2^{(3)}$. Note that U' and V' are $(6,3,2)$ coverings, while S' and T' are "almost coverings", each leaving only the three pairs ab, cd, and ef uncovered. Let

$$W = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 3 & 7 & 9 \\ 2 & 3 & 4 & 6 \\ 2 & 4 & 7 & 8 \\ 5 & 6 & 8 & 9 \end{pmatrix}$$

noting that the blocks of W cover all pairs contained in the blocks of $S \cup T$. Using the notation from Lemma 2.3.11, we claim that the 120 blocks of

$$\begin{array}{lll} SS' & D(U)ab & Wef \\ TT' & D(V)cd & abcdef \\ UU' & D'(T)e & \\ VV' & D''(T)f & \end{array}$$

form a (15,6,4) covering.

The [0,4]-sets are covered by the block $abcdef$.

The [1,3]-sets are covered by the union of the blocks of SS', TT', UU', and VV'. The same blocks cover the [2,2]-sets with the exception of the [2,2]-sets $xypq$, where x, y is a pair in the blocks of $S \cup T$, and pq is either ab, cd or ef. The remaining [2,2]-sets are covered as follows. The [2,2]-sets $xyab$, where $x, y \in X_1$, are covered by $D(U)ab$. The [2,2]-sets $xycd$ are covered by $D(V)cd$. The [2,2]-sets $xyef$ are covered by the blocks of Wef.

The [3,1]-sets $xyzp$, where xyz is a block of the 2-(9,3,1) design on X_1, are covered by the union of the blocks of SS', TT', UU', and VV'. The triples of X_1, that are not blocks of the 2-(9,3,1) design, are covered by each of the designs $D(U)$ and $D(V)$ (cf. Lemma 2.3.12), as well as by each of the collections $D'(T)$ and $D''(T)$. Therefore, the remaining [3,1]-sets are covered by the union of the blocks of $D(U)ab$, $D(V)cd$, $D'(T)e$ and $D''(T)f$.

It is easy to check (although tedious) that the designs $D(U)$ and $D(V)$ are disjoint. Moreover, the union $D(U) \cup D(V) \cup D'(T) \cup D''(T)$ is a 4-(9,{4,5},1) design on X_1. Therefore, the [4,0]-sets are covered by the union of the blocks of $D(U)ab$, $D(V)cd$, $D'(T)e$ and $D''(T)f$, which completes the proof. □

Remark 2.3.14 *The covering described in Theorem* 2.3.13 *was further transformed and reduced by local search to* 118 *blocks. The blocks of a* (15,6,4) *covering of size* 118 *are given in the appendix.*

The next constructions are based, in one way or another, on the two designs described in Theorem 2.2.14 (a) when $m = 2$. As mentioned there, the union of these two designs is a 4-(16,{6,4},1) design.

THEOREM 2.3.15 $C(16, 6, 4) \leq 160$.

Proof. The 140 4-sets of $X(16)$ that are not covered by the 112 blocks of the 3-(16,6,4) design D with $p = 3$ (Theorem 2.2.14, part (b)) can be covered by 48 6-sets of $X(16)$. To find these 6-sets we used a greedy algorithm described in [28], but acting on the 140 4-sets arranged randomly rather than on the entire family $X^{(4)}(16)$. Below is the compressed description (defined in Chapter 1) of the 48 blocks that should be added to the design D from Theorem 2.2.14, part (b), in order to obtain a (16,6,4) covering:

177, 41, 117, 182, 94, 213, 183, 189, 233, 160, 65, 110, 64, 370, 230, 169, 105, 290, 91, 193, 66, 223, 64, 272, 188, 109, 210, 152, 156, 28, 488, 126, 89, 65, 171, 488, 230, 89, 60, 121, 78, 460, 30, 289, 30, 292, 23, 85. □

THEOREM 2.3.16 $C(4^m+2,6,4) \leq \binom{4^m}{3}\frac{4^m+11}{60}$.

Proof. Let D be the 3-$(4^m, 6, \frac{1}{3}(4^m-4))$ design with $p=3$ from Theorem 2.2.14 and D^* be the corresponding $S(4^m,4,3)$. It is easy to see that the blocks of

$$D$$

$$D^* \ (4^m+1)\ (4^m+2)$$

form a $(4^m+2,6,4)$ covering of size the sum of sizes of D and D^*, which produces the desired bound. □

THEOREM 2.3.17 $C(17,6,4) \leq 188$.

Proof. Let D be the unique 3-$(17,5,1)$ design [53] on $X(17)$. Let D^{17} and D_{17} be the derived and residual designs of D with respect to the point 17. Consider the set

$$D' = \{B' \in X^{(6)}(16) : |B' \cap B| \leq 3 \ \ \forall B \in D_{17}\}.$$

Table 2.2: Spectral sets

Spectral set of $X^{(6)}(16)$ under D_{17}	Number of 6-sets in the class	Spectral set of $X^{(4)}(16)$ under D'	Number of 4-sets in the class
(10,0,1)	48	(0,6)	20
(8,0,1)	480	(8,2)	120
(6,3,0)	640	(14,1)	960
(10,2,0)	2400	(12,1)	480
(9,2,0)	1920	(16,0)	240
(8,2,0)	240		
(13,1,0)	960		
(12,1,0)	240		
(11,1,0)	960		
(16,0,0)	120		

This is an appropriate place to illustrate how the spectral sets can be used for finding designs and studying covering properties of designs. First we find by computer the spectral set of $X^{(6)}(16)$ under D_{17} (Table 2.2). The blocks of D' correspond to the spectrum $(16,0,0)$.

As was mentioned in Chapter 1, some of the classes or union of these classes can be designs. In particular, D' is a 2-$(16,6,15)$ design with $p = 4$. (In fact, all ten classes are 2-designs, and some unions are 3-designs [7].) Now we compute the spectral set of $X^{(4)}(16)$ under D' (Table 2.2). It is clear that no block of D' covers a quadruple contained in a block of D_{17}. The number of such quadruples is $48\binom{5}{4} = 240$ (because D_{17} has $p = 2$). On the other hand, from the spectral set of $X^{(4)}(16)$ under D' we see that there are exactly 240 quadruples that are not covered by any block of D'. Therefore the blocks of D' cover all quadruples of $X(16)$ except for those covered by the blocks of D_{17}. Consequently, the blocks of $D' \cup D_{17}$ cover all elements of $X^{(4)}(16)$. The blocks of $D^{17} \cup D_{17}$ cover all elements of $X^{(3)}(16)$.

Now, adding the point 17 to each block of $D^{17} \cup D_{17}$, we get 20 blocks of size 5 and 48 blocks of size 6 whose union covers all quadruples on $X(17)$ that contain the point 17. Therefore, the blocks of

$$\begin{array}{l} D^{17}\ 17 \\ D_{17}\ 17 \\ D' \end{array}$$

form a 4-$(17, \{6,5\}, 1)$ covering design in which 168 blocks have size 6 and the remaining 20 – size 5. We now arbitrarily add points to the blocks of size 5 to increase their size. This completes the proof. □

Corollary 2.3.18 $C(16,5,3) \leq 65$.

Proof. Count the number of occurrences of each point in the preceding construction. The point 17 is in 68 blocks and each of the remaining points is in 65 blocks. The result follows by applying Construction 2.2.1. □

The bound $C(18,6,4) \leq 252$ from Theorem 2.3.16 can be further improved in the smallest non-trivial case $m = 2$, where we had $C(18,6,4) \leq 252$.

THEOREM 2.3.19 $C(18,6,4) \leq 236$.

Proof. Using the notation from Theorem 2.3.17 we claim that the blocks of

$$D^{17}\ 17\ 18$$

$$D_{17}\ 18$$

$$D_{17}\ 17$$

$$D'$$

form a $(18,6,4)$ covering of size 236 on $X(18)$.

The design described above "extends" the design of Theorem 2.3.17, and therefore it covers all the elements of $X^{(4)}(17)$. The quadruples $a\ b\ 17\ 18$, where $a, b \in X(16)$, are covered by the blocks of D^{17} 17 18 because D^{17} is a 2-$(16,4,1)$ design. The quadruples $a\ b\ c\ 18$, where $a, b, c \in X(16)$, are covered by the blocks of D^{17} 17 18 and D_{17} 18 since $D^{17}\ 17 \ \cup D_{17}$ is a 3-$(17,5,1)$ design on $X(17)$. This completes the proof. □

Corollary 2.3.20 $C(19,6,4) \leq 330$.

Proof. The result follows by applying Construction 2.2.2 to the covering obtained in Theorem 2.3.19 and a (18,5,3) covering with 94 blocks (discussed in Section 2.1) □

THEOREM 2.3.21 $C(24,6,4) \leq 784$.

Proof. Partition $X(24)$ into the two sets $X_1 = X(16)$ and $X_2 = \{17, 18, ..., 24\}$. Let D be the 3-(16,6,4) design on X_1 (described in Theorem 2.2.14), and D^* be the corresponding $S(16,4,3)$. Let $A_1, A_2, ..., A_7$ be the partition of D^* into seven $S(16,4,2)$'s.

Let $B_1, B_2, ..., B_7$ be the 1-factors of a 1-factorization of the complete graph K_8 on X_2. Let F_i, $i = 1, 2, ..., 6$, be the (8,4,3) coverings on X_2 from Theorem 2.2.11 whose union is an $(8, 4, 4)$ covering, and let $F_7 = F_8 = F_1$. We claim that the 784 blocks of

$$\begin{array}{ll} D & \\ A_i B_i, & i = 1, 2, ..., 7 \\ i\,(i+8)\;F_i, & i = 1, 2, ..., 8 \end{array}$$

form a (24,6,4) covering.

All the [4,0]-sets are covered because the set of blocks contains a 4-(16,{6,4},1) design.

The [3,1]-sets and the [2,2]-sets are covered by the blocks of $A_i B_i$, $i = 1, 2, ..., 7$.

The [1,3]-sets and the [0,4]-sets are covered by the blocks of $i(i+8)F_i$, $i = 1, 2, ..., 8$.

□

THEOREM 2.3.22 $C(22, 6, 4) \leq 580$.

Proof. Repeat the proof of Theorem 2.3.21, but instead take $X_2 = \{17, ..., 22\}$ and $B_1, B_2, ..., B_5$ to be the 1-factors of a 1-factorization of the complete graph K_6 on X_2. Let $B_6 = B_7 = B_5$. Take F_i, $i = 1, 2, 3, 4$, to be the four (6,4,3) coverings of size 6 on X_2 from Theorem 2.2.12, and let $F_i = F_1$, $i = 5, 6, 7, 8$. □

THEOREM 2.3.23 $C(20, 6, 4) \leq 400$.

Proof. This is again similar to the proof of Theorem 2.3.21. Take $X_2 = \{17, 18, ..., 20\}$ and let B_i, $i = 1, 2, 3$, be the 1-factors of the 1-factorization of the complete graph K_4 on X_2. Let $B_i = B_1$, $i = 4, 5, 6, 7$. Let F be the 4-set 17 18 19 20. Then the 400 blocks of

$$D$$

$$A_i B_i, \qquad i = 1, 2, ..., 7$$

$$i\ (i+8)\ F, \quad i = 1, 2, ..., 8$$

form a (20,6,4) covering. □

Summarizing the results of this section, we have made the following improvements: $C(16,5,3) \leq 65(68)$, $C(14,6,4) \leq 80(87)$, $C(15,6,4) \leq 118(134)$, $C(16,6,4) \leq 160(178)$, $C(17,6,4) \leq 188(243)$, $C(18,6,4) \leq 236(258)$, $C(19,6,4) \leq 330(352)$, $C(20,6,4) \leq 400(456)$, $C(22,6,4) \leq 580(721)$, $C(24,6,4) \leq 784(1035)$,

where the old bounds are given in parentheses.

2.3.4 Bounds on $C(v,k,t)$, $t \geq 5$

In this section we improve the bounds on the covering number $C(v,7,5)$ for some values of v in the range $13 \leq v \leq 25$. The results lead to improvements on the covering number $C(v,k+2,k)$ for $k = 6,7,8$ (see Table 2.4).

The best currently known bound on the size of a (14,6,5) covering is 377. The corresponding covering was found by the program **cover** described in [39]. A compressed description of a covering that attains this bound is given in the appendix.

THEOREM 2.3.24 $C(13,7,5) \leq 78$.

Proof. Start with a projective plane of order 3; that is, with a 2-$(13,4,1)$ design D, generated by the block $[1,2,4,10] \pmod{13}$. The set of blocks

$$\{B_i \cup B_j \ : \ B_i, B_j \in D, \ 1 \leq i < j \leq 13\}$$

is a 2-$(13,7,21)$ cyclic design whose blocks cover all elements of $X^{(5)}(13)$ and is therefore a $(13,7,5)$ covering design. The representatives of the orbits of the blocks under the cyclic group of order 13 are

$$\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 10 & 11 \\
1 & 2 & 3 & 4 & 6 & 10 & 12 \\
1 & 2 & 3 & 4 & 7 & 8 & 10 \\
1 & 2 & 3 & 5 & 6 & 8 & 11 \\
1 & 2 & 3 & 5 & 7 & 11 & 12 \\
1 & 2 & 4 & 5 & 9 & 10 & 12.
\end{array}$$

□

Corollary 2.3.25 $C(14,8,6) \leq 165$.

Proof. Morley [37] has proved $C(12,8,6) \leq 51$. The result follows by applying Construction 2.2.4 to the covering of Theorem 2.3.24. □

Several upper bounds on the covering numbers $C(v,7,5)$ are based on the following.

THEOREM 2.3.26 $C(18,8,6) \leq 918$.

Proof. We use a computer to verify most of the steps of the proof (cf. Theorem 2.3.17.) Let $X = X(17)$ and D' be the unique 3-(17,5,1) design [53]. The size of the spectra set of $X^{(7)}$ under D' is 4. The equivalence class D^* corresponding to the spectrum (0,1) has 408 blocks and is a 4-(17,7,6) design with $p = 5$. Similarly, the size of the spectra set of $X^{(8)}$ under D' is 6. The equivalence class D_* corresponding to the spectrum (10,0) has 510 blocks and is a 4-(17,8,15) design with $p = 6$.

The two designs D^* and D_* prove to be a matching pair of designs, thus producing a 5-(18,8,6) design D with $p = 6$ where $D^{18} = D^*$ and $D_{18} = D_*$. Further computer investigation shows that the design D is also an (18,8,6) covering. The compressed descriptions of representatives of the orbits under the cyclic group of order 17 are:

D^{18}: 19, 71, 87, 5, 64, 36, 19, 123, 8, 46, 27, 19, 15, 113, 19, 18, 21, 71, 26, 33, 71, 111, 100, 13, and

D_{18}: 30, 7, 6, 83, 27, 97, 35, 101, 59, 3, 41, 91, 38, 40, 9, 71, 65, 79, 28, 4, 72, 60, 85, 23, 24, 35, 108, 30, 4, 22. □

Corollary 2.3.27 $C(17,7,5) \leq 408$.

Proof. To the $(18,8,6)$ covering in Theorem 2.3.26 apply Construction 2.2.1 and delete the point 18. □

We next study some properties of the Steiner systems $S(24,8,5)$ and $S(23,7,4)$

and use them to construct a $(19, 7, 5)$ covering. It is known that these two Steiner systems are unique [5]. The proof of the following lemma is also given in [5], p. 207.

Lemma 2.3.28 *Let A and C be blocks of the Steiner system $S(24, 8, 5)$ where $|A \cap C| = 4$. Then $(A \cup C) \setminus (A \cap C)$ is also a block.*

Lemma 2.3.29 *Let B be a block of the Steiner system $D = S(23, 7, 4)$. Removing each point of B from all blocks that contain it gives a collection of blocks of sizes 4 and 6 forming a Steiner system $S(16, 4, 3)$ and a 3-$(16, 6, 4)$ design.*

Proof. Without loss of generality, let D be defined on the set $X(23)$ and $B =$ 1 2 3 4 5 6 7. Noting that $n_4 = n_5 = n_6 = 0$ and $n_7 = 1$ the intersection equations for B with respect to D become

$$n_0 + n_1 + n_2 + n_3 = 252$$

$$n_1 + 2n_2 + 3n_3 = 532$$

$$n_2 + 3n_3 = 420$$

$$n_3 = 140$$

which yield $n_1 = 112$, $n_3 = 140$ and $n_0 = n_2 = 0$. So, on deleting the points of B, we get 112 blocks of size 6 and 140 blocks of size 4 on $X = X(23) \setminus X(7)$. Let us denote the collection of blocks of size 6 by S'.

We first prove that S' is a 3-$(16, 6, 4)$ design. Since the initial design D has $p = 3$, any two blocks of S' meet in at most three points. Let T be an arbitrary 3-subset of

X. Then T is covered by at most 4 blocks of S' (otherwise p would be more than 3). Therefore, the maximum number of triples that can be covered by the blocks of S' is $4\binom{16}{3} = 2240$. However, the number of triples covered by the blocks of S' is exactly $112\binom{6}{3} = 2240$. This shows that each triple of X is covered by exactly 4 blocks of S'. That is, S' is a 3-$(16, 6, 4)$ design.

Now, let us denote the collection of blocks of size 4 by Q. Consider again the 3-subset T. There are $16 - 3 = 13$ different quadruples of $X^{(4)}$ containing T. The intersection of a block of Q and a block of S' has size at most 3 because the initial design D has $p = 3$. Since T is covered by exactly 4 blocks of S', there are exactly $4 \cdot 3 = 12$ quadruples covering T that are subsets of blocks of S'. Therefore, T is contained in $13 - 12 = 1$ quadruple that is not a subset of a block of S'. This quadruple must therefore be a block of Q. Thus any triple $T \in X^{(4)}$ is contained in exactly one block of Q, which completes the proof. □

THEOREM 2.3.30 $C(19, 7, 5) \leq 707$.

Proof. It is known [21] that an $S(24, 8, 5)$ can be obtained by a variation on the greedy algorithm as follows. Take 1 2 3 4 5 6 7 8 to be the first block. Then the block B_i, $2 \leq i \leq 759$, is the first block in the lexicographical ordering of $X^{(8)}(24)$ such that $|B_i \cap B_j| \leq 4 \; \forall j \in X(i-1)$.

We will need some information on the block intersections of the Steiner system $S(24, 8, 5)$. Let U with $|U| = u < t$, be a subset of a block B of a Steiner system

$D = S(v,k,t)$. It is shown in [5], pp. 199-200, that the number of blocks $Y \in D$ with $Y \cap B = U$ depends only on $|U|$; that is, it is independent of both the choice of B and U. Hence we may set

$$m_u = m(U,B) = |\{Y \in D \ : \ Y \cap B = U\}|.$$

By counting, we get the recurrence

$$m_u = \lambda_u - 1 - \sum_{i=u+1}^{t-1} m_i \binom{k-u}{i-u}.$$

(The numbers λ_i, $i = 0, 1, ..., t$, were defined in Chapter 1. Also, by definition, $\sum_i^j = 0$ if $i > j$.) For any block of the Steiner system $S(24,8,5)$ we obtain

$$m_0 + 8m_1 + 28m_2 + 56m_3 + 70m_4 = 758$$

$$m_1 + 7m_2 + 21m_3 + 35m_4 = 252$$

$$m_2 + 6m_3 + 15m_4 = 76$$

$$m_3 + 5m_4 = 20$$

$$m_4 = 4$$

which gives $m_4 = 4$, $m_3 = 0$, $m_2 = 16$, $m_1 = 0$, and $m_0 = 30$. This shows in particular that there are exactly 16 blocks having two fixed points in common with B_1.

Consider the intersection equations for B_1. Since $n_5 = n_6 = n_7 = 0$ and $n_8 = 1$

we get

$$n_0 + n_1 + n_2 + n_3 + n_4 = 758$$

$$n_1 + 2n_2 + 3n_3 + 4n_4 = 2016$$

$$n_2 + 3n_3 + 6n_4 = 2128$$

$$n_3 + 4n_4 = 1120$$

$$n_4 = 280$$

which yield $n_1 = n_3 = 0$, $n_0 = 30$, $n_2 = 448$, and $n_4 = 280$.

Removing each point of B_1 from all blocks that contain it we obtain a collection of blocks of sizes 4,6, and 8 on $X = X(24) \setminus X(8)$.

We claim that Lemmas 2.3.28 and 2.3.29 imply that the 280 blocks of size 4 form two identical $S(16,4,3)$. More explicitly, if a is a point of $X(8)$, then by Lemma 2.3.29,

$$Q = \{B \setminus \{i,j,k,a\} \;:\; B \in D,\ i,j,k,a \in B,\ i,j,k \in X(8) \setminus \{a\},\ i < j < k\}$$

is an $S(16,4,3)$. Applying Lemma 2.3.28, it is easy to check that

$$P = \{B \setminus \{i,j,k,l\} \;:\; B \in D,\ i,j,k,l \in B,\ i,j,k,l \in X(8) \setminus \{a\},\ i < j < k < l\}$$

has the same blocks as Q.

There are 448 blocks of size 6. We will denote this collection by S. There is a collection E of 30 blocks of size 8. The union $S \cup E$ is a 5-$(16, \{8,6\}, 1)$ design on $X = X(24) \setminus X(8)$ because the initial design D is a 5-design on $X(24)$ and $\binom{16}{5} = 448\binom{6}{5} + 30\binom{8}{5}$. Let $S_i \subset S$, $i \in X(8)$ be the collection of 6-tuples obtained from the blocks containing the point i in the initial $S(24,8,5)$, that is,

$$S_i = \{B \setminus \{i,j\} \ : \ B \in D, \ i,j \in B, \ j \in X(8) \setminus \{i\}\}.$$

The maximal intersection number of the initial design D is 4 which implies that the maximal intersection number of S does not exceed 4. Now, $S_i \subset S$, and S_i is obtained from blocks of the initial design D containing the common point i. Therefore, any two blocks of S_i intersect in at most 3 points.

We claim that the same is valid for any pair U, V, where U is a block of Q and V is a block of S. For suppose $|U \cap V| = 4$. Then, in fact, $U \subset V$. Now, U and V originate from different blocks, say A and C, of the initial design D where $U = A \setminus B_1$ and $V = C \setminus B_1$, $|A \cap B_1| = 4$ and $|C \cap B_1| = 2$. Since $|U \cap V| = |U| = 4$ we have $|A \cap C| \geq 4$. On the other hand, $|A \cap C| \leq 4$ because the maximal intersection number of an $S(24,8,5)$ is 4. Thus $|A \cap C| = 4$ and, in fact, $A \cap C = U$. By Lemma 2.3.28, $(A \cup C) \setminus (A \cap C)$ is a block of D. But this is impossible as it has precisely 6 points in common with B_1.

Thus any two blocks of S_i have at most three points in common, and the size of the intersection of a block of Q with a block of S_i is at most 3. On the other hand,

$$\begin{aligned} |S_i| &= |\{B \setminus \{i,j\} \ : \ B \in D, \ i,j \in B, \ j \in X(8) \setminus \{i\}\}| \\ &= m_2(7) = 16(7) = 112. \end{aligned}$$

Therefore, the blocks of S_i cover $112\binom{6}{4} = 1680$ different quadruples on X. The blocks of Q constitute an additional 140 distinct quadruples on X. Consequently, the blocks of $S_i \cup Q$ cover any element of of $X^{(4)}$ exactly once ($1680 + 140 = \binom{16}{4}$); that is, $S_i \cup Q$ is a 4-$(16, \{6,4\}, 1)$ design (cf. Theorem 2.2.14).

Let x_1, x_2, x_3 be three new points. Now, it is easy to check that the blocks of

$$Q\, x_1\, x_2\, x_3$$

$$S_{i+1}\ x_i,\ \ i = 1,2,3$$

$$S \setminus \textstyle\bigcup_{i=2}^{4} S_i$$

$$E$$

form a 5-$(19, \{8,7,6\}, 1)$ covering design, where $|S \setminus \bigcup_{i=2}^{4} S_i| = 160$ blocks have size 6, $140 + 112(3) = 476$ have size 7, and 30 have size 8. (The equality $|S \setminus \bigcup_{i=2}^{4} S_i| = 160$ follows by inclusion-exclusion from $|S| = 448$, $|S_2| = |S_3| = |S_4| = 112$, $|S_i \cap S_j| = 16$, $2 \le i < j \le 4$, and $|S_2 \cap S_3 \cap S_4| = 0$.) In order to obtain a $(19,7,5)$ covering of size 707 it suffices to cover the 5-tuples contained in the 160 blocks of size 6 and the 30 blocks of size 8 by 231 blocks of size 7. The last step has been completed by computer. A compressed description of the 231 blocks of size 7 is given below:

42, 37, 40, 38, 81, 26, 34, 1, 4, 85, 14, 14, 9, 22, 48, 76, 5, 18, 36, 42, 17, 27, 124, 90, 73, 138, 12, 45, 12, 95, 8, 130, 32, 56, 38, 109, 32, 61, 15, 11, 155, 75, 29, 13, 41, 31, 60, 3, 107, 16, 93, 27, 5, 69, 18, 35, 53, 136, 7, 52, 1, 26, 134, 31, 24, 70, 18, 5, 39, 5, 52, 60, 111, 8, 102, 3, 7, 77, 65, 1, 19, 9, 175, 63, 71, 76, 157, 51, 86, 29, 48, 15, 27, 141, 13, 13, 39, 23, 135, 174, 39, 38, 7, 25, 67, 15, 10, 18, 45, 9, 144, 39, 8, 32, 8, 26, 49, 33, 43, 50, 67, 38, 178, 81, 4, 26, 61, 33, 29, 19, 42, 48, 93, 36, 4, 110, 62, 41, 12, 98, 123, 44, 29, 226, 24, 56, 37, 98, 13, 126,

42, 15, 35, 42, 48, 43, 41, 13, 153, 35, 53, 22, 7, 11, 41, 35, 7, 55, 64, 38, 64, 29, 39, 26, 48, 19, 20, 77, 40, 44, 2, 31, 135, 10, 93, 32, 9, 44, 43, 3, 73, 48, 12, 3, 43, 3, 36, 5, 9, 5, 103, 1, 26, 138, 150, 28, 34, 7, 94, 23, 15, 121, 160, 14, 101, 51, 142, 9, 20, 31, 26, 47, 1, 107, 109, 23, 18, 335, 66, 2, 2. □

Corollary 2.3.31 $C(20,7,5) \leq 1037$.

Proof. The result follows from Construction 2.2.2, Corollary 2.3.20 and Theorem 2.3.30. □

Corollary 2.3.32 $C(21,7,5) \leq 1359$.

Proof. There is a point, for example 20, in the (20,6,4) covering constructed in Theorem 2.3.23 which is in 148 blocks. The result follows from Construction 2.2.4 and Theorem 2.3.30. □

The application of Construction 2.2.4 for obtaining coverings of small size does not necessarily require the initial covering to be of the smallest known size. In other words, we can start with initial covering designs of sizes s_1 and s_2, where $s_1 < s_2$, to obtain covering designs of sizes s'_1 and s'_2, where $s'_1 > s'_2$, provided the second design had a point which was in many blocks. The next lemma and theorem illustrate this fact.

Lemma 2.3.33 *There exists a* $(21,6,4)$ *covering of size* 565 *with a point occurring in* 293 *blocks.*

Proof. Let D, A_i, B_i, $i = 1, 2, ..., 7$ be the same as in Theorem 2.3.22 except that the pair 21 22 is contained in B_1. Replace the point 22 with a point from $X_2 \setminus \{21, 22\}$ in B_1, and change 22 to 21 in the remaining 1-factors B_i, $i = 2, 3, ..., 7$. A (5,4,3) covering F of size 4 on $X_2 \setminus \{22\}$ such that the point 21 is in all 4 blocks is easily constructed. Let C be the block a 17 18 19 20 21, where $a \in X_1$. It is easy to check that the blocks of

$$\begin{array}{ll} D & \\ A_i B_i, & i = 1, 2, ..., 7 \\ i\,(i+8)\;F, & i = 1, 2, ..., 8 \\ C & \end{array}$$

form a (21,6,4) covering of size 565 such that the point 21 is in $260 + 32 + 1 = 293$ blocks. □

Although this result is useful in the next theorem, it does not yield the best upper bound on $C(21, 6, 4)$. Belić (personal communication) has obtained $C(21, 6, 4) \leq 502$. He has also found $C(23, 6, 4) \leq 723$.

THEOREM 2.3.34 $C(22, 7, 5) \leq 1874$.

Proof. To obtain the covering apply construction 2.2.4 and Corollaries 2.3.33 and 2.3.31.

□

THEOREM 2.3.35 $C(23, 7, 5) \leq 2342$.

Proof. First we prove that $C(23,7,5) \leq 2347$. There is a point, say 22, in the (22,6,4) covering constructed in Theorem 2.3.22 which is in $140 + 32 = 172$ blocks. The proof now follows from Construction 2.2.4.

The covering described above was further reduced by local computer search to 2342 blocks. Thus $C(23,7,5) \leq 2342$. □

So far, we have proved the following new upper bounds:

$C(13,7,5) \leq 78(88)$, $C(14,8,6) \leq 165(179)$, $C(18,8,6) \leq 918(1240)$, $C(17,7,5) \leq 408(506)$, $C(19,7,5) \leq 707(930)$, $C(20,7,5) \leq 1037(1239)$, $C(21,7,5) \leq 1359(1617)$, $C(22,7,5) \leq 1874(2088)$, $C(23,7,5) \leq 2342(2647)$,

where the value in parentheses indicates the best previously known upper bound.

Some other new bounds on $C(v,k,t)$ are obtained by local search or by direct applications of Constructions 2.2.2 and 2.2.4 or Corollary 2.2.6 (see Tables 2.3 and 2.4).

In the tables the superscript a indicates the covering is given in the appendix; e indicates the result is obtained in [27]; b indicates the result is found by R. Belić and o that it is found by F. Oats (personal communication). The coverings of Belić and Oats can be found in the La Jolla Covering Repository, a website maintained by Dan Gordon at http://sdcc12.ucsd.edu/~xm3dg/cover.html

The coverings of the last two authors are found via computer local search. The entries with superscripts $c2$ and $c3$ are obtained by Constructions 2.2.2 and 2.2.4.

Table 2.3: New Upper Bounds on $C(v, k, t)$ (A)

$v \backslash (k,t)$	(5,3)	(5,4)	(6,4)	(6,5)	(7,5)
13					78
14	43[a]	232[a]	80	377[a]	143[a]
15	57	303	118[a]	609[c2]	203[c3]
16	65	416[a]	160	808[e]	321[c2]
17		492[e]	188	1215[b]	408
18		671[e]	236	1547[b]	596[c2]
19	113	850[e]	330	2175[b]	707
20	138	1095[o]	400	2900[e]	1037
21	151	1251[e]	502[b]	3995[c2]	1359
22	179[o]	1573[e]	580	4692[b]	1874
23	190		723[b]	6197[b]	2342
24	234		784		3065[c2]
25			1018[c2]		3714[c3]
29		5085		*23711*	
30		6035[c2]		28796[c2]	

Table 2.4: New Upper Bounds on $C(v, k, t)$ (B)

$v \backslash (k,t)$	(6,3)	(8,6)	(9,7)
14		161[b]	
15		304[c2]	291[b]
16		443[b]	595[c2]
17		718[b]	937[b]
18		918	1586[b]
19	63[a]	1507[b]	2335[b]
20	72[a]	2042[c3]	3841[b]
21		3079[c2]	5584[c3]
22		4251[c3]	8663[c2]
23		6125[c2]	12456[c3]
24		8124[c3]	18581[c2]
25		11189[c2]	*25770*
26			36959[c2]

The entries in italics are old bounds from [28] needed to calculate bounds obtained by Construction 2.2.2. The (v, k, t) coverings with less than 6000 blocks obtained by Construction 2.2.4 are explicitly constructed after finding the point which is in the largest number of blocks in an appropriate $(v-1, k-1, t-1)$ covering. For coverings with more than 6000 blocks the upper bounds on $C(v, k, t)$ are calculated by Corollary 2.2.6. The remaining entries are explained in the text.

2.4 Comparative and asymptotic results

Let $L'(v, k, t)$ denote the best possible lower bound which can be obtained by successive application of Theorem 1.1.1 and call it the Schönheim bound. Given a (v, k, t) covering C^* of size b, define the **deviation** δ by

$$\delta = \delta(C^*) = \frac{b}{L'(v, k, t)}.$$

It seems natural to use this definition to estimate how good a covering is: the closer the deviation to 1, the better the covering. However, given that there are cases where the Schönheim bound cannot be attained, for example, $C(8, 4, 3) = 14$ (there exists a Steiner system $S(8, 4, 3)$) and by Theorem 1.1.1, we obtain $C(9, 5, 4) \geq 26 = L'(9, 5, 4)$, whereas it is known that $C(9, 5, 4) = 30$ [14], it may be that δ is far from 1 and yet we do have the best covering. There are also cases where the Schönheim bound is the size of the best covering; for example $C(10, 6, 5) \geq \lceil \frac{10}{6} C(9, 5, 4) \rceil = 50$,

Table 2.5: Bounds on $C(v, 6, 4)$

v	Lower bound	New upper bound	Previously known upper bound [28]	$\frac{\textit{New upper bound}}{\textit{Lower bound}}$
14	75	80	87	1.07
15	93	118	134	1.27
16	144	160	178	1.11
17	173	188	243	1.09
18	205	236	258	1.15
19	298	330	352	1.11
20	344	400	456	1.16
21	434	502	594	1.16
22	539	580	721	1.08
23	625	723	871	1.16
24	720	784	1035	1.09
25	921	1018	1170	1.11

which is, in fact, the exact value of $C(10, 6, 5)$ [14].

In Table 2.5 we give a comparison between the bounds on the covering number $C(v, k, t)$ for the third column of Table 2.3.

The next lemma gives a general upper bound on the covering number $C(v, k, k-1)$.

Lemma 2.4.1 $C(v, k, k-1) \leq \binom{v}{k-1} - \frac{k-1}{v}\binom{v}{k}$.

Proof. First we form $(v, k, k-1)$ packings defined by

$$P_j = \{\{a_1, a_2, ..., a_k\} \ : \ a_i \in X(v), \ \sum_{i=1}^{k} a_i \equiv j \pmod{k}\}, \ j = 1, 2, ..., v \ [29].$$

It is clear that

$$\bigcup_{j=1}^{v} P_j = X^k(v), \ \ P_i \cap P_j = \emptyset, \ \ i \neq j$$

and so $\sum_{j=1}^{v} |P_j| = \binom{v}{k}$. Therefore, at least one of these packings has size at least $\frac{1}{v}\binom{v}{k}$. The blocks of the packing cover at least $\frac{k}{v}\binom{v}{k}$ $(k-1)$-subsets of $X(v)$. At most $\left(\binom{v}{k-1} - \frac{k}{v}\binom{v}{k}\right)$ $(k-1)$-subsets remain and in the worst case can be covered by $\left(\binom{v}{k-1} - \frac{k}{v}\binom{v}{k}\right)$ k-sets. Thus we obtain

$$C(v,k,k-1) \leq \frac{1}{v}\binom{v}{k} + \binom{v}{k-1} - \frac{k}{v}\binom{v}{k} = \binom{v}{k-1} - \frac{k-1}{v}\binom{v}{k}.$$

□

This lemma does not produce good bounds for small values of v, but it implies a general bound which asymptotically produces coverings of deviation 1. More precisely, we have the following.

THEOREM 2.4.2 *Let k be fixed. Then*

$$\lim_{v\to\infty} \frac{\binom{v}{k-1} - \frac{k-1}{v}\binom{v}{k}}{C(v,k,k-1)} = 1.$$

Proof.

$$\begin{aligned} 1 &\leq \frac{\binom{v}{k-1} - \frac{k-1}{v}\binom{v}{k}}{C(v,k,k-1)} \\ &\leq \frac{\binom{v}{k-1} - \frac{k-1}{v}\binom{v}{k}}{\left\lceil \frac{v}{k}\left\lceil \frac{v-1}{k-1} \cdots \left\lceil \frac{v-k+2}{2}\right\rceil \cdots \right\rceil\right\rceil} \\ &\leq \frac{\binom{v}{k-1} - \frac{k-1}{v}\binom{v}{k}}{\frac{v}{k}\frac{v-1}{k-1}\cdots\frac{v-k+2}{2}} \\ &= \frac{a_1 v^{k-1} + a_2 v^{k-2} + \ldots}{b_1 v^{k-1} + b_2 v^{k-2} + \ldots}, \end{aligned}$$

where

$$\frac{a_1}{b_1} = \frac{\frac{1}{(k-1)!} - \frac{k-1}{k!}}{\frac{1}{k!}} = 1 ,$$

and the result follows. □

Lemma 2.4.1 and Theorem 2.4.2 provide an easy constructive proof of a particular instance of the famous Erdős-Hanani conjecture [23] that for fixed t and k, $t < k$,

$$\lim_{v\to\infty} \frac{C(v,k,t)}{\frac{\binom{v}{t}}{\binom{k}{t}}} = 1$$

Our result provides a direct proof for the case $t = k-1$. The Erdős-Hanani conjecture was proved in 1985 by V. Rödl [44], and shortly after that, Spencer [47] simplified the proof. Both proofs make use of probabilistic methods.

2.5 Other generalizations

After studying the paper of Zaitsev et al. [54] we noticed that it contains (although not explicitly stated) the following result which is an extension of Theorem 2.2.14.

THEOREM 2.5.1 *There exists a* 3-$(4^m, 6, \frac{1}{3}(4^m-4))$ *design* D *with* $p = 3$ *so that the family of all* 4*-subsets of* $X(4^m)$ *not covered by any block of* D *is a Steiner system* $S(4^m, 4, 3)$. *This Steiner system can be partitioned into* $(2^{2m-1}-1)$ $S(4^m, 4, 2)$*'s.*

Note that the extension of Theorem 2.2.14 is the decomposition of the $S(4^m, 4, 3)$ into $S(4^m, 4, 2)$'s for every $m \geq 2$, a result of Zaitsev et al. [55]. This result is based on the remarkable proof of Preparata [42] that the binary Hamming code decomposes into translates of the Preparata code. The design D is obtained from the codewords of weight 6 of the Preparata code. The partition of the Steiner system $S(4^m, 4, 3)$ (formed by the codewords of weight 4 in the Hamming code) into $(2^{2m-1} - 1)$ $S(4^m, 4, 2)$'s is described in [55]. Taking the derived designs of the designs given in Theorem 2.5.1 we obtain the following.

Corollary 2.5.2 *There exists a 2-$(4^m - 1, 5, \frac{1}{3}(4^m - 4))$ design D with $p = 2$. The family of all 3-subsets of $X(4^m - 1)$ that are not covered by any block of D form a resolvable Steiner system $S(4^m - 1, 3, 2)$.*

The result of Zaitsev et al. leads to the following general upper bound.

THEOREM 2.5.3 *Let $0 \le d \le 2^{2m-2} - 3$. Then*

$$\begin{aligned} & C(3(2^{2m-1}) - 2d, 6, 4) \\ \le\ & \binom{4^m}{3}\left[\tfrac{1}{15}(4^{m-1} - 1) + (4^{m-2} - \tfrac{d}{4})\right] + 2^{2m-1}C(2^{2m-1} - 2d, 4, 3) \\ & + C(2^{2m-1} - 2d, 6, 4). \end{aligned}$$

Proof. Partition $X(3(2^{2m-1}) - 2d)$ into two sets, $X_1 = \{1, 2, ..., 4^m\}$ and $X_2 = \{4^m + 1, 4^m + 2, ..., 3(2^{2m-1}) - 2d\}$. Let D be the 3-$(4^m, 6, \frac{1}{3}(4^m - 4))$ design on X_1 from Theorem 2.5.1, and D^* be the corresponding $S(4^m, 4, 3)$. Let $A_1, A_2, ..., A_{2^{2m-1}-1}$

be the partition of D^* into $(2^{2m-1}-1)$ $S(4^m,4,2)$'s. Let $B_1, B_2, ..., B_{2^{2m-1}-1-2d}$ be the 1-factors of a 1-factorization of the complete graph $K_{2^{2m-1}-2d}$ on X_2. Let $B_{2^{2m-1}-2d} = B_{2^{2m-1}-2d+1} = ... = B_{2^{2m-1}-1} = B_1$. Let E be a 1-factor of the complete graph K_{4^m} on X_1 and F a $(2^{2m-1}-2d,4,3)$ covering of size $C(2^{2m-1}-2d,4,3)$ on X_2. Let H be a $(2^{2m-1}-2d,6,4)$ covering of size $C(2^{2m-1}-2d,6,4)$ on X_2. We claim that the blocks of

$$D$$

$$A_iB_i, \quad i = 1,2,...,2^{2m-1}-1$$

$$E\ F$$

$$H$$

form a $(3(2^{2m-1})-2d,6,4)$ covering.

All of the [4,0]-sets are covered because the blocks contain as subblocks the blocks of the 4-$(4^m,\{6,4\},1)$ design formed by the union of the design D and the Steiner system $S(4^m,4,3)$.

The [3,1]-sets and [2,2]-sets are covered by the blocks of A_iB_i, $i = 1,2,...,2^{2m-1}-1$.

The [1,3]-sets are covered by the blocks of EF.

The [0,4]-sets are covered by the blocks of H.

Finally, it is easy to check that the number of blocks of this covering is exactly the right hand side of the inequality of the theorem, which completes the proof. ◫

A slightly better bound can be obtained under the condition given in the next theorem which generalizes Theorems 2.3.21, 2.3.22, and 2.3.23.

THEOREM 2.5.4 *Let* $0 \leq d \leq 2^{2m-2} - 2$. *If there exist* 2^{2m-1} $(2^{2m-1} - 2d, 4, 3)$ *coverings each of size* $C(2^{2m-1} - 2d, 4, 3)$ *whose union is a* $(2^{2m-1} - 2d, 4, 4)$ *covering, then*

$$\begin{aligned} & C(3(2^{2m-1}) - 2d, 6, 4) \\ \leq\ & \binom{4^m}{3}\left[\tfrac{1}{15}(4^{m-1} - 1) + (4^{m-2} - \tfrac{d}{4})\right] + 2^{2m-1}C(2^{2m-1} - 2d, 4, 3). \end{aligned}$$

Proof. We basically follow the proof of the preceding theorem. The difference is in the covering of the [1,3] and [0,4]-sets. Let F_i, $i = 1, 2, ..., 2^{2m-1}$, be the $(2^{2m-1} - 2d, 4, 3)$ coverings on X_2 whose union is a $(2^{2m-1} - 2d, 4, 4)$ covering. Instead of using the union of the blocks of EF and H we use the blocks of $E\ F_i$, $i = 1, 2, ..., 2^{2m-1}$, to cover the [1,3] and [0,4]-sets. □

For example, if $m = 2$ and $d = 1$, Theorem 2.5.3 gives $C(22, 6, 4) \leq 581$ while Theorem 2.5.4 gives $C(22, 6, 4) \leq 580$ which is the best known bound (both use Theorem 2.2.12). Corollary 2.5.2 leads to the following generalization.

THEOREM 2.5.5 *Let* $0 \leq d \leq 2^{2m-2} - 3$. *Then*

$$\begin{aligned} & C(3(2^{2m-1}) - 2d - 1, 5, 3) \\ \leq\ & \binom{4^m - 1}{2}\left[\tfrac{1}{15}(2^{2m-1} - 2) + \tfrac{1}{3}(4^{m-1} - d)\right] + C(2^{2m-1} - 2d, 5, 3). \end{aligned}$$

Proof. Partition $X(3(2^{2m-1})-2d-1)$ into the two sets $X_1=\{1,2,...,4^m-1\}$ and $X_2=\{4^m,4^m+1,...,3(2^{2m-1})-2d-1\}$. Let D be the 2-$(4^m-1,5,\frac{1}{3}(4^m-4))$ design on X_1 from Corollary 2.5.2. Let $A_1,A_2,...,A_{2^{2m-1}-1}$ be the resolution classes of the Steiner system $S(4^m-1,3,2)$. Let $B_1,B_2,...,B_{2^{2m-1}-1-2d}$ be the 1-factors of a 1-factorization of the complete graph $K_{2^{2m-1}-2d}$ on X_2 and $B_{2^{2m-1}-2d}=B_{2^{2m-1}-2d+1}=...=B_{2^{2m-1}}=B_1$. Let C be a $(2^{2m-1}-2d,5,3)$ covering of minimum size on X_2. We claim that the blocks of

$$D$$

$$A_iB_i,\quad i=1,2,...,2^{2m-1}-1$$

$$C$$

form a $(3(2^{2m-1})-2d-1,5,3)$ covering.

Since the blocks contain as subblocks the blocks of the 3-$(4^m-1,\{5,3\},1)$ design formed by the union of the design D and the Steiner system $S(4^m-1,3,2)$, all of the [3,0]-sets are covered.

The [2,1]-sets and [1,2]-sets are covered by the blocks of A_iB_i, $i=1,2,...,2^{2m-1}-1$.

The [0,3]-sets are covered by the blocks of C.

Again, the number of blocks of the constructed covering is exactly the right hand of the desired inequality, which completes the proof. □

Chapter 3

Three infinite families of minimal coverings

3.1 Introduction

In this chapter we prove that certain t-designs are minimal $(t+1)$-coverings, thus finding some new covering numbers. The results of this chapter have been published in [11].

It is clear that any t-(v,k,λ) design is a minimal t-(v,k,λ) covering. Aside from this, little is known about the covering number $C_\lambda(v,k,t)$, $\lambda \geq 2$.

There are some general results on $C_\lambda(v,k,2)$ [36] but for $t \geq 3$, only a few sporadic values are known. Two general results which have been obtained from finite geometries are:

$$C\left(\frac{q^{t+1}-1}{q-1},\frac{q^t-1}{q-1},t\right)=\frac{q^{t+1}-1}{q-1}\ \text{[43] and }\ C\left(q^t,q^{t-1},t\right)=q\frac{q^t-1}{q-1}\ \text{[1]},$$

where q is a prime power. The corresponding coverings are 2- and 3-designs. In fact, the only other known infinite family producing coverings with $t \geq 3$ is

$$C\left(\sum_{i=0}^{t} a_i q^{t-i},\sum_{i=0}^{t-1} a_i q^{t-i-1},t\right)=\frac{q^{t+1}-1}{q-1}\ \text{for any integers }\ a_0\geq a_1\geq\ldots\geq a_t\geq 1;$$

a generalization of the result in [43] and found by Todorov [50], [51]. In a recent paper, Chee and Ling [18] determined several covering numbers $C_\lambda(v,k,3)$ and in particular showed that $C_{3\lambda}(15,6,3) = 70\lambda$, for $\lambda = 1,2,3$. The corresponding covering designs also arise from results in [7] and are, in fact, 2-$(15,6,10\lambda)$ designs, $\lambda = 1,2,3$. These results suggest that there might be other designs that are minimal coverings and the results of a search for such designs are presented in this chapter. We find three new infinite families of minimal coverings with $t \geq 3$.

3.2 Minimal 4-Coverings

THEOREM 3.2.1 *Let $v \equiv 2$ or $4 \pmod 6$, $v > 8$, and $m = \frac{1}{24}(v-4)(v^2-15v+62)-1$. Then $C_m(v,v-4,4) = \frac{1}{24}v(v-1)(v-2) = C(v,4,3)$.*

Proof. There exists a 3-$(v,4,1)$ design D for any $v \equiv 2$ or $4 \pmod 6$ [31]. The number of blocks of D is $\frac{1}{24}v(v-1)(v-2) = C(v,4,3)$. The supplemental design $\overline{D}$ of

D is a 3-$(v, v-4, \frac{1}{4}\binom{v-4}{3})$ design with $\lambda_0 = \frac{1}{24}v(v-1)(v-2)$, $\lambda_1 = \frac{1}{24}(v-1)(v-2)(v-4)$, $\lambda_2 = \frac{1}{24}(v-2)(v-4)(v-5)$, $\lambda_3 = \frac{1}{24}(v-4)(v-5)(v-6)$.

Consider an arbitrary 4-subset S of $X(v)$. The intersection equations of S with respect to $\overline{D}$ are

$$n_0 + n_1 + n_2 + n_3 + n_4 = \lambda_0$$

$$n_1 + 2n_2 + 3n_3 + 4n_4 = 4\lambda_1$$

$$n_2 + 3n_3 + 6n_4 = 6\lambda_2$$

$$n_3 + 4n_4 = 4\lambda_3.$$

An essential step of the proof is to show that this system has only two distinct solutions in non-negative integers n_0, n_1, n_2, n_3 and n_4. To prove this, we find bounds on n_4. From the last three equations we obtain

$$n_1 + 4n_4 = 4\lambda_1 - 12\lambda_2 + 12\lambda_3.$$

Since $n_1 \geq 0$ we get

$$n_4 \leq \lambda_1 - 3\lambda_2 + 3\lambda_3 = \frac{1}{24}(v-4)(v^2 - 15v + 62).$$

On the other hand, solving the system for n_0, n_1, n_2 and n_3, we have

$$n_3 = 4\lambda_3 - 4n_4$$

$$n_2 = 6\lambda_2 - 12\lambda_3 + 6n_4$$

$$n_1 = 4\lambda_1 - 12\lambda_2 + 12\lambda_3 - 4n_4$$

$$n_0 = \lambda_0 - 4\lambda_1 + 6\lambda_2 - 4\lambda_3 + n_4.$$

A lower bound on n_4 now follows from $n_0 \geq 0$:

$$n_4 \geq 4\lambda_3 - 6\lambda_2 + 4\lambda_1 - \lambda_0 = \frac{1}{24}(v-4)(v^2 - 15v + 62) - 1.$$

Thus n_4 can take only two values, either m or $m+1$, where $m = \frac{1}{24}(v-4)(v^2 - 15v + 62) - 1$. It is easy to check that both values produce a non-negative integral solution to the intersection equations for S. So, we proved that each 4-subset of $X(v)$ is contained in either m or $m+1$ blocks of $\overline{D}$; that is, $\overline{D}$ is a 4-$(v, v-4, m)$ covering. To prove that $\overline{D}$ is a minimal covering, we show that the number λ_0 of blocks of $\overline{D}$ meets the Schönheim lower bound (Corollary 1.1.2). Thus we claim

$$\left\lceil \frac{v}{v-4} \left\lceil \frac{v-1}{v-5} \left\lceil \frac{v-2}{v-6} \left\lceil \frac{v-3}{v-7} m \right\rceil \right\rceil \right\rceil \right\rceil = \lambda_0.$$

Let $v = 6k+4$, for $k \geq 1$. Then $m = \frac{1}{2}(18k^3 - 21k^2 + 9k - 2)$. We obtain

$$\frac{6k+1}{6k-3} m = \frac{1}{2}(18k^3 - 9k^2 + k) - \frac{1}{6k-3} = \frac{1}{2}k(6k-1)(3k-1) - \frac{1}{6k-3},$$

$$\left\lceil \frac{6k+1}{6k-3} m \right\rceil = \frac{1}{2}k(6k-1)(3k-1) = \frac{1}{24}6k(6k-1)(6k-2),$$

and the claim follows after simple arithmetic.

If $v = 6k+2$, for $k \geq 2$, then $m = \frac{1}{2}(18k^3 - 39k^2 + 29k - 8)$. We have

$$\frac{6k-1}{6k-5} m = \frac{1}{2}(18k^3 - 27k^2 + 13k - 2) - \frac{1}{6k-5} = \frac{1}{2}(3k-1)(2k-1)(3k-2) - \frac{1}{6k-5},$$

$$\left\lceil \frac{6k-1}{6k-5} m \right\rceil = \frac{1}{2}(3k-1)(2k-1)(3k-2) = \frac{1}{24}(6k-2)(6k-3)(6k-4),$$

and the claim follows analogously. $\square$.

The existence of many minimal covering designs without repeated blocks follows from this result. The following result illustrate this.

Corollary 3.2.2 *Suppose there exist l disjoint 3-$(v,4,1)$ designs, where $v > 8$ and $l \leq v-8$, and let $m = \frac{1}{24}(v-4)(v^2-15v+62)-1$. Then there exists a 4-$(v, v-4, ms)$ minimal covering without repeated blocks for $1 \leq s \leq l$.*

Proof. Let D be the 3-$(v,4,s)$ design formed by s disjoint 3-$(v,4,1)$ designs. Then $\overline{D}$ is the desired covering. The Schönheim lower bound argument works for $1 \leq s \leq v-8$. □.

For example, there exist five non-intersecting 3-$(10,4,1)$ designs [33]. Therefore, there exists a 4-$(10,6,2s)$ minimal covering without repeated blocks for $s = 1,2$. There exist at least four non-intersecting 3-$(14,4,1)$ designs [25]. Therefore, there exists a 4-$(14,10,19s)$ minimal covering without repeated blocks for $s = 1,2,3,4$. Similar extensions are possible for many other sets of parameters. For a survey of the results on finding non-intersecting 3-$(v,4,1)$ designs we refer the reader to [25].

THEOREM 3.2.3 *Let $q \geq 4$ be a prime power and $\lambda = q(q-1)(q-3)+2$. Then $C_\lambda(q^2+1, q^2-q, 4) = q(q^2+1)$.*

Proof. The proof is similar to the proof of the preceding theorem. Let $\overline{D}$ be the supplemental design of a 3-$(q^2+1, q+1, 1)$ design D (for example, the inversive geometry of order q). The design $\overline{D}$ is a 3-$(q^2+1, q^2-q, (q-2)(q^2-q-1))$ design with $\lambda_0 = q(q^2+1)$, $\lambda_1 = q^2(q-1)$, $\lambda_2 = (q-1)(q^2-q-1)$ and $\lambda_3 = (q-2)(q^2-q-1)$. Let S be a 4-subset of $X(q^2+1)$ and n_i, $i = 0,1,2,3,4$, be the intersection numbers of S with respect to $\overline{D}$. As in Theorem 3.2.1,

$$n_4 \le \lambda_1 - 3\lambda_2 + 3\lambda_3 = q(q-1)(q-3) + 3,$$

$$n_4 \ge 4\lambda_3 - 6\lambda_2 + 4\lambda_1 - \lambda_0 = q(q-1)(q-3) + 2,$$

and both bounds produce a solution in non-negative integers to the intersection equations for S. Thus any 4-subset of $X(q^2+1)$ is covered by either λ or $\lambda+1$ blocks of $\overline{D}$. So, $\overline{D}$ is a 4-(q^2+1, q^2-q, λ) covering. Furthermore,

$$\frac{q^2-2}{q^2-q-3}\lambda = q^3 - 3q^2 + q + 2 - \frac{q-2}{q^2-q-3} = (q-2)(q^2-q-1) - \frac{q-2}{q^2-q-3},$$

$$\left\lceil \frac{q^2-2}{q^2-q-3}\lambda \right\rceil = (q-2)(q^2-q-1),$$

and the equality

$$\left\lceil \frac{q^2+1}{q^2-q} \left\lceil \frac{q^2}{q^2-q-1} \left\lceil \frac{q^2-1}{q^2-q-2} \left\lceil \frac{q^2-2}{q^2-q-3}\lambda \right\rceil \right\rceil \right\rceil \right\rceil = q(q^2+1)$$

follows directly. Therefore, the number of blocks of $\overline{D}$ meets the Schönheim bound, which completes the proof. $\square$.

Similarly, we obtain the following generalization of Theorem 3.2.3.

Corollary 3.2.4 *Suppose there exist l disjoint 3-$(q^2+1, q+1, 1)$ designs, where $l \le q$, $q \ge 4$ is a prime power, and let $\lambda = q(q-1)(q-3) + 2$. Then there exist a 4-$(q^2+1, q^2-q, \lambda s)$ minimal covering without repeated blocks for $1 \le s \le l$.*

Proof. Let D be the 3-$(q^2+1, q+1, s)$ design formed by s disjoint 3-$(q^2+1, q+1, 1)$ designs. Then $\overline{D}$ is the desired covering because the Schönheim lower-bound argument works for $\frac{s(q-2)}{q^2-q-3} < 1$; that is, for $s \leq q$. □.

For example, there are at least three disjoint 3-$(17, 5, 1)$ designs [17]. Therefore, there exists a minimal 4-$(17, 12, 14s)$ covering without repeated blocks for $s = 1, 2, 3$.

3.3 Minimal 3-coverings

The next theorem provides the third of the infinite families referred to in the introduction.

THEOREM 3.3.1 *Let* $n = 4^k - 1$, $k \geq 2$. *Then*

$$C_{\frac{1}{3}n-2}(n, 6, 3) = \frac{1}{360}n(n-1)(n-3)(n-5).$$

Proof. Let D be the 3-$(n+1, 6, \frac{1}{3}n - 1)$ design (with maximal intersection number $p = 3$) obtained from the Preparata code [15], p. 185-193 and described in Theorem 2.5.1. Let D^x be the derived design of D with respect to a point x; which is a 2-$(n, 5, \frac{1}{3}n - 1)$ design with $p = 2$.

For completeness we need a direct proof to part of Corollary 2.5.2 which we now present. We prove that the 3-subsets of $X(n)$ that are not covered by any block of

D^x form a 2-$(n,3,1)$ design D^*. A 2-subset T of $X(n)$ is contained in $(n-2)$ of the 3-subsets of $X(n)$. On the other hand, T is contained in $3(\frac{1}{3}n-1)=n-3$ of the 3-subsets that are subsets of blocks of D^x. Since D^x has $p=2$, all these subsets are distinct. Therefore, T is contained in exactly $(n-2)-(n-3)=1$ 3-subset of $X(n)$ that is not covered by any block of D^x.

Now, consider the residual design D_x of D with respect to the same point x. It is a 2-$(n,6,\frac{1}{12}(n-3)(n-5))$ design with $p=3$. Any 3-subset of $X(n)$ that is not a block of D^* is contained in exactly $(\frac{1}{3}n-2)$ blocks of D_x and any block of D^* is contained in exactly $(\frac{1}{3}n-1)$ blocks of D_x. Thus D_x is a 3-$(n,6,\frac{1}{3}n-2)$ covering. To prove that D_x is a minimal covering we note that

$$\left\lceil \frac{n-2}{4}\left(\frac{1}{3}n-2\right)\right\rceil = \left\lceil \frac{(n-3)(n-5)}{12}-\frac{1}{4}\right\rceil = \frac{(n-3)(n-5)}{12}$$

(because $(n-3)(n-5)$ is divisible by 12). Further, $\frac{n(n-1)(n-3)(n-5)}{2^3.3^2.5}$ is an integer, as one of $(n-1)$, $(n-3)$ and $(n-5)$ is divisible by 5 and $n=4^k-1$ implies $n\equiv 3 \pmod 6$. Then the equality

$$\frac{n(n-1)}{6(5)}\frac{1}{12}(n-3)(n-5) = \left\lceil \frac{n}{6}\left\lceil \frac{n-1}{5}\left\lceil \frac{n-2}{4}\left(\frac{1}{3}n-2\right)\right\rceil\right\rceil\right\rceil$$

is immediate. This completes the proof that D_x meets the Schönheim bound. □.

Chapter 4

New Simple 3-Designs on 26 and 28 Points

4.1 Introduction

The most recent tables of the known simple designs have been published in [34]. In this chapter we prove the existence of 22 new simple 3-designs on 26 and 28 points. The base of the constructions are two designs in which the maximum size of the intersection of any two blocks is small. The work of this chapter has been published in [12].

The following theorem, proved in [22], and its corollaries given in [9], [8], can be used to obtain new designs from designs with sufficiently small maximal intersection number p.

THEOREM 4.1.1 *(*DRIESSEN'S THEOREM [22]*) If D is a t-(v,k,λ) design with a maximal intersection number $p \leq k-m-l-1$, for fixed integers $m, l \geq 0$, then*

$$D_{m,l} = \{(B \cup L) \setminus M : B \in D, M \subseteq B, |M| = m, L \subseteq (X \setminus B), |L| = l\}$$

is a

$$t-\left(v, k+l-m, \lambda\binom{v-k}{l}\binom{k+l-m}{t}\binom{k}{m}\Big/\binom{k}{t}\right)$$

design.

Corollary 4.1.2 *Designs obtained by Driessen's Theorem for pairs m_1, l_1 and m_2, l_2, where $m_1 \neq m_2$, but $m_1 - l_1 = m_2 - l_2$, have the same block size, and are simple and disjoint.*

Corollary 4.1.3 *If the designs produced by Driessen's Theorem are simple and non-trivial, then $m \leq k-t-1$, and the initial design is not trivial.*

4.2 New $3-(26,k,\lambda)$ designs

Let D be the unique 3-(26,6,1) design. We will prove the existence of a 3-(26,8,14) design D' with $p = 5$. We use the method illustrated in Theorem 2.3.17. The size of the spectral set of $X^{(8)}$ under D is 13. One of the equivalence classes is D' with

$|D'| = 650$ and $Spec_D(D') = (12, 0, 0)$. It is a 3-(26,8,14) design with $p = 5$. Thus we have the following.

THEOREM 4.2.1 *The 8-subsets of $X(26)$ that intersect each block of the unique $3-(26,6,1)$ design D in at most 4 points and exactly 12 of the blocks of D in 4 points, form a $3-(26,8,14)$ design with $p = 5$.*

The application of Driessen's theorem and Corollaries to D and D' produces amongst others the following designs:

$D_{1,1} : 3-(26,6,120)$	$D'_{2,0} : 3-(26,6,140)$
$D_{0,1} : 3-(26,7,35)$	$D'_{1,0} : 3-(26,7,70)$
$D_{1,2} : 3-(26,7,1995)$	$D'_{1,1} : 3-(26,8,2016)$
$D_{0,2} : 3-(26,8,532)$	$D'_{0,0} \cup D'_{1,1} : 3-(26,8,2030)$
$D_{0,3} : 3-(26,9,4788)$	$D'_{0,1} : 3-(26,9,378)$
	$D'_{0,2} : 3-(26,10,4590)$.

The designs from the first column are mentioned in [9]; whereas the designs from the second column are new (with the exception of $D'_{2,0}$).

THEOREM 4.2.2 *The following are sets of pairwise disjoint designs: D, $D_{1,1}$ and $D'_{2,0}$; $D_{0,1}$, $D_{1,2}$ and $D'_{1,0}$; $D_{0,2}$, D' and $D'_{1,1}$; $D_{0,3}$ and $D'_{0,1}$.*

Proof. We investigate the intersections of the designs with the initial design D. In what follows, we essentially use the condition of Driessen's theorem.

Consider the designs D, $D_{1,1}$ and $D'_{2,0}$. The maximal intersection number of D is 2. Each block B of $D_{1,1}$ has an intersection of size 5 with some block B^* of D and an intersection of size less than 5 with each of the remaining blocks of D (because B is obtained from B^* by removing a point and adding a new point from the set $X(26) \setminus B^*$). Any block of $D'_{2,0}$ is a 6-subset of a block of D' and any block of D' has at most 4 points in common with a block of D. Therefore, any block of $D'_{2,0}$ has at most 4 points in common with a block of D. On the other hand, each block of $D_{1,1}$ has 5 points in common with a block of D and consequently, the designs D, $D_{1,1}$ and $D'_{2,0}$ are pairwise disjoint.

Consider the designs $D_{0,1}$, $D_{1,2}$ and $D'_{1,0}$. Any block of $D_{0,1}$ contains a block of D. Any block of $D_{1,2}$ has an intersection of size 5 with a block of D and an intersection of size less than 5 with each of the remaining blocks of D. So, it has at most 5 points in common with a block of $D_{0,1}$. The blocks of $D'_{1,0}$ are the 7-subsets of the blocks of D'. Therefore, any block of $D'_{1,0}$ has intersection of size at most 4 with each of the blocks of D and thus at most 5 with the blocks of $D_{0,1}$ and 6 with the blocks of $D_{1,2}$. Consequently, the designs $D_{0,1}$, $D_{1,2}$ and $D'_{1,0}$ are pairwise disjoint.

Consider $D_{0,2}$, D' and $D'_{1,1}$. Any block of $D_{0,2}$ contains a block of D. Any block of D' has at most 4 points in common with each of the blocks of D, so blocks of D' and $D_{0,2}$ have at most 6 points in common. Any block of $D'_{1,1}$ is obtained by removing a point from a block of D' and adding a point from the supplement of the same block.

Therefore, a block of $D'_{1,1}$ cannot have more than 5 points in common with a block of D and hence 7 with a block of $D_{0,2}$. The designs $D' = D'_{0,0}$ and $D'_{1,1}$ are disjoint by Corollary 4.1.2. Consequently, the three designs $D_{0,2}$, D' and $D'_{1,1}$ are pairwise disjoint.

Finally, consider $D_{0,3}$ and $D'_{0,1}$. Any block of $D_{0,3}$ contains a block of D. Any block of $D'_{0,1}$ contains a block of D' and one more element. Consequently, a block of $D'_{0,1}$ cannot have more than 5 points in common with a block of D and hence at most 8 points with a block of $D_{0,3}$. Therefore, the designs $D_{0,3}$ and $D'_{0,1}$ are disjoint. □.

The observations made so far lead to the following result. (See Table 4.1 for the proofs.)

Corollary 4.2.3 *There exist designs with the following parameters:*

$$3-(26,6,m) \quad for \quad m = 141, 260, 261;$$

$$3-(26,7,35m) \quad for \quad m = 2, 3, 59, 60;$$

$$3-(26,8,7m) \quad for \quad m = 2, 78, 288, 290, 364, 368;$$

$$3-(26,9,21m) \quad for \quad m = 18, 246;$$

$$3-(26,10,3m) \quad for \quad m = 1530.$$

Note that the designs are given in the form t-$(v, k, \lambda_{min}m)$, where λ_{min} is the minimum value of λ for which a t-(v, k, λ) design could exist (cf. [34]).

4.3 New $3-(28,k,\lambda)$ designs

Van Lint and MacWilliams [35] have constructed a 3-$(28,9,28)$ design D'' from the subsets of coordinate places holding codewords of weight 9 in a linear code of length 28 over $GF(4)$. The code has minimal distance 9 (equal to the minimal weight of a codeword). We prove that the design D'' has $p \leq 6$. There are three non-zero elements in $GF(4)$ and the code is linear, so there are three codewords for each support. Since the minimal distance of the code is 9, any other word of the code must be at a distance at least 9 from each of these three. If $p > 6$, consider three codewords, c_1, c_2 and c_3 with the same support. There must be a codeword c' which has at least 7 non-zero elements in the support positions. At least three of these elements must be the same as the corresponding elements in one of the three codewords, say c_1. This gives two codewords, c_1 and c', at distance at most 8, which is a contradiction. Thus $p \leq 6$ for the 3-$(28,9,28)$ design D'' obtained in [35].

The application of Driessen's theorem and corollaries, as shown in Table 4.1, now proves the following.

THEOREM 4.3.1 *There exist designs with the parameters* $3-(28,7,420)$*;* $3-(28,8,168)$*;* $3-(28,9,28m)$*,* $m = 171, 172$*;* $3-(28,10,760)$ *and* $3-(28,11,9405)$.

The results of Corollary 4.2.3 and Theorem 4.3.1 are summarized in Table 4.1. Of particular interest are the two designs with parameters 3-(26,8,14) and 3-(26,9,378)

Table 4.1: New designs

New design		Construction
Parameters	m	
3-(26,6,m)	141	$D \cup D'_{2,0}$
	260	$D_{1,1} \cup D'_{2,0}$
	261	$D \cup D_{1,1} \cup D'_{2,0}$
3-(26,7,35m)	2	$D'_{1,0}$
	3	$D_{0,1} \cup D'_{1,0}$
	59	$D_{1,2} \cup D'_{1,0}$
	60	$D_{0,1} \cup D_{1,2} \cup D'_{1,0}$
3-(26,8,7m)	2	D'
	78	$D' \cup D_{0,2}$
	288	$D'_{1,1}$
	290	$D' \cup D'_{1,1}$
	364	$D_{0,2} \cup D'_{1,1}$
	366	$D' \cup D_{0,2} \cup D'_{1,1}$
3-(26,9,21m)	18	$D'_{0,1}$
	246	$D_{0,3} \cup D'_{0,1}$
3-(26,10,3m)	1530	$D'_{0,2}$
3-(28,7,5m)	84	$D''_{2,0}$
3-(28,8,42m)	4	$D''_{1,0}$
3-(28,9,28m)	171	$D''_{1,1}$
	172	$D''_{0,0} \cup D''_{1,1}$
3-(28,10,20m)	38	$D''_{0,1}$
3-(28,11,495m)	19	$D''_{0,2}$

as they have *the smallest known* λ when the other three parameters are fixed.

Appendix

$C(14,5,3) \leq 43$

42, 26, 41, 111, 21, 29, 92, 15, 5, 37, 11, 9, 134, 35, 56, 24, 23, 13, 7, 13, 93, 102, 15, 62, 60, 26, 13, 64, 133, 56, 18, 29, 50, 30, 48, 48, 103, 54, 38, 53, 94, 25, 16.

$C(14,5,4) \leq 232$

6, 8, 6, 20, 7, 4, 3, 3, 9, 21, 7, 4, 7, 1, 3, 20, 6, 19, 3, 14, 3, 6, 2, 6, 3, 9, 23, 13, 5, 3, 9, 20, 5, 5, 2, 2, 22, 11, 1, 3, 7, 8, 24, 9, 9, 11, 15, 9, 2, 6, 12, 12, 6, 4, 6, 8, 5, 7, 11, 2, 10, 7, 31, 9, 5, 5, 11, 2, 7, 9, 2, 11, 15, 8, 4, 11, 3, 22, 5, 9, 10, 8, 14, 2, 12, 1, 14, 6, 12, 12, 3, 14, 2, 14, 3, 14, 1, 2, 12, 4, 10, 8, 7, 16, 14, 6, 12, 2, 9, 6, 8, 9, 8, 3, 3, 2, 25, 16, 10, 19, 5, 13, 8, 6, 8, 13, 4, 3, 11, 7, 21, 2, 10, 18, 10, 6, 12, 6, 4, 5, 5, 16, 2, 2, 6, 8, 21, 16, 3, 11, 3, 12, 15, 1, 1, 3, 21, 2, 3, 17, 27, 1, 1, 2, 15, 2, 1, 9, 9, 4, 8, 25, 5, 2, 10, 12, 2, 9, 16, 9, 8, 6, 11, 9, 6, 8, 26, 5, 16, 5, 1, 10, 13, 5, 11, 7, 1, 3, 21, 14, 1, 6, 4, 11, 2, 19, 14, 5, 9, 2, 11, 5, 4, 2, 12, 4, 13, 5, 17, 27, 1, 16, 1, 9, 2, 21, 4, 14, 3, 3, 11, 11.

$C(16,5,4) \leq 416$

10, 13, 9, 4, 11, 6, 11, 16, 3, 8, 3, 15, 28, 4, 9, 16, 11, 17, 12, 14, 9, 19, 5, 14, 13, 11, 7, 6, 13, 6, 10, 2, 23, 1, 15, 5, 11, 11, 5, 14, 12, 13, 17, 3, 7, 15, 11, 14, 6, 6, 12, 19, 8, 5, 11, 20, 1, 22, 19, 1, 8, 3, 14, 10, 15, 1, 3, 18, 28, 4, 7, 11, 4, 27, 14, 6, 6, 13, 10, 4, 12, 13, 31, 8, 1, 10, 2, 22, 5, 2, 7, 10, 4, 16, 9, 8, 6, 5, 21, 21, 3, 13, 2, 19, 3, 2, 7, 25, 1, 4, 20, 7, 28, 8, 4, 6, 8, 15, 21, 19, 7, 8, 14, 11, 3, 5, 9, 4, 12, 13, 9, 11, 5, 16, 1, 23, 5, 19, 19, 7, 8, 13, 12, 1, 15, 18, 13, 2, 4, 10, 3, 2, 25, 25, 11, 8, 14, 13, 3, 6, 5, 24, 12, 25, 7, 6, 6, 8, 10, 9, 8, 20, 13, 14, 1, 4, 7, 24, 1, 2, 14, 15, 10, 3, 9, 5, 32, 10, 17, 6, 8, 8, 7, 11, 6, 4, 4, 19, 6, 8, 1, 18, 5, 14, 16, 10, 3, 24, 28, 5, 1, 17, 10, 12, 21, 3, 2, 15, 5, 2, 6, 14, 6, 3, 28, 9, 13, 5, 19, 7, 15, 7, 17, 14, 8, 2, 10, 14, 5, 13, 29, 3, 3, 6, 2, 8, 10, 34, 6, 2, 3, 8, 18, 3, 14, 17, 5, 1, 7, 9, 7, 1, 23, 21, 3, 7, 16, 4, 26, 11, 2, 1, 17, 4, 8, 22, 11, 12, 11, 1, 20, 6, 22, 9, 5, 11, 11, 7, 9, 33, 9, 15, 10, 5, 3, 4, 16, 12, 11, 3, 17, 14, 34, 5, 7, 3, 27, 3, 1, 1, 1, 21, 25, 10, 5, 10, 5, 4, 8, 1, 17, 33, 7, 6, 10, 8, 13, 14, 4, 5, 7, 1, 7, 14, 5, 4, 21, 4, 52, 3, 9, 9, 2, 19, 8, 2, 26, 9, 20, 12, 3, 14, 8, 24, 20, 1, 7, 8, 16, 5, 12, 8, 3, 19, 2, 9, 3, 32, 13, 4, 10, 13, 16, 2, 11, 8, 22, 3, 2, 12, 25, 6, 7, 7, 5, 9, 24, 2, 16, 5, 18, 11, 6, 5, 12, 14, 3, 14, 7, 1, 24, 10, 8, 9, 4, 22, 9, 5, 2, 22, 15, 8, 5, 9, 7, 7.

$C(19, 6, 3) \leq 63$

359, 238, 57, 301, 15, 17, 1669, 181, 806, 22, 9, 2426, 173, 64, 19, 213, 518, 26, 19, 879, 571, 737, 26, 19, 35, 1924, 538, 1030, 64, 19, 1044, 103, 97, 98, 22, 9, 847, 718, 64, 19, 203, 1369, 15, 17, 1004, 745, 22, 9, 1678, 26, 19, 455, 1594, 15, 17, 444, 1308, 508, 187, 90, 1329, 19, 64.

$C(20, 6, 3) \leq 72$

264, 487, 366, 948, 791, 69, 372, 557, 244, 952, 359, 227, 344, 1233, 112, 172, 693, 1237, 447, 356, 279, 197, 71, 947, 1501, 81, 457, 970, 191, 287, 814, 297, 327, 803, 188, 498, 1324, 453, 521, 554, 454, 820, 351, 1226, 632, 1408, 52, 277, 494, 325, 640, 217, 514, 540, 1167, 213, 363, 440, 656, 689, 803, 122, 209, 266, 1129, 173, 325, 281, 663, 386, 542, 392.

$C(15,6,4) \leq 118$

14, 19, 50, 44, 36, 97, 64, 35, 27, 56, 52, 73, 8, 14, 24, 44, 5, 56, 50, 3, 94, 28, 25, 20, 38, 96, 40, 12, 15, 6, 22, 82, 71, 91, 28, 41, 4, 4, 49, 23, 51, 7, 12, 6, 7, 73, 48, 9, 42, 7, 140, 78, 41, 8, 58, 8, 9, 18, 36, 78, 133, 24, 32, 22, 39, 85, 40, 15, 78, 26, 13, 50, 5, 11, 55, 81, 30, 51, 36, 52, 14, 60, 93, 30, 108, 41, 32, 57, 27, 58, 35, 31, 34, 51, 44, 49, 8, 18, 40, 101, 55, 74, 23, 38, 86, 13, 45, 47, 14, 79, 57, 34, 52, 21, 23, 53, 153, 31.

$C(14,6,5) \leq 377$

8, 9, 1, 15, 3, 10, 16, 9, 5, 15, 1, 3, 7, 13, 2, 4, 7, 6, 26, 4, 1, 2, 13, 3, 6, 4, 13, 3, 6, 10, 11, 10, 13, 4, 3, 9, 4, 4, 17, 7, 4, 3, 4, 3, 5, 22, 4, 10, 6, 3, 8, 8, 18, 9, 2, 19, 14, 7, 6, 5, 5, 11, 4, 14, 4, 8, 9, 8, 5, 9, 4, 5, 19, 8, 1, 2, 5, 14, 10, 6, 5, 16, 3, 5, 11, 2, 19, 2, 5, 1, 6, 8, 7, 12, 13, 9, 4, 14, 6, 7, 15, 7, 4, 4, 2, 7, 4, 15, 8, 11, 5, 14, 10, 8, 8, 6, 4, 9, 6, 3, 13, 1, 13, 15, 2, 12, 9, 5, 10, 8, 8, 6, 13, 3, 3, 15, 5, 17, 1, 13, 8, 7, 8, 8, 1, 14, 2, 11, 10, 13, 14, 2, 7, 2, 12, 6, 9, 3, 13, 2, 5, 8, 16, 11, 3, 9, 7, 3, 11, 6, 9, 3, 5, 18, 10, 6, 1, 25, 1, 4, 18, 7, 2, 9, 5, 4, 5, 9, 3, 7, 9, 16, 2, 4, 7, 6, 9, 15, 3, 9, 5, 4, 23, 7, 7, 9, 3, 4, 16, 3, 2, 11, 9, 4, 6, 2, 15, 14, 3, 18, 8, 6, 9, 5, 10, 5, 14, 4, 11, 3, 14, 3, 3, 12, 15, 3, 5, 3, 5, 5, 28, 6, 9, 14, 4, 8, 12, 3, 9, 8, 6, 1, 7, 9, 16, 12, 5, 7, 7, 2, 9, 11, 6, 4, 2, 16, 2, 6, 6, 12, 8, 11, 6, 16, 7, 4, 18, 12, 6, 8, 10, 8, 5, 7, 1, 7, 14, 14, 4, 8, 10, 3, 6, 17, 1, 4, 2, 21, 1, 10, 19, 3, 8, 17, 5, 4, 6, 6, 14, 9, 8, 2, 15, 7, 9, 8, 9, 4, 2, 7, 12, 10, 8, 10, 10, 4, 8, 7, 7, 16, 6, 10, 15, 4, 3, 2, 21, 8, 4, 12, 9, 1, 18, 5, 5, 9, 7, 17, 6, 8, 7, 5, 18, 7, 2, 9, 4, 4, 9, 13, 2, 2, 8, 8, 7, 5, 12, 10, 5, 8, 8, 11, 19, 3, 12, 10, 4.

$C(14, 7, 5) \leq 143$

26, 3, 24, 5, 24, 17, 49, 2, 45, 10, 40, 47, 20, 9, 13, 7, 144, 2, 19, 34, 22, 57, 5, 13, 9, 42, 7, 15, 3, 41, 43, 5, 141, 2, 2, 1, 21, 33, 40, 37, 9, 13, 5, 46, 3, 15, 7, 98, 16, 24, 3, 53, 2, 3, 8, 5, 52, 1, 5, 38, 15, 21, 4, 40, 11, 5, 62, 51, 22, 1, 22, 118, 34, 19, 2, 53, 7, 6, 9, 3, 2, 53, 3, 8, 5, 1, 52, 5, 8, 6, 85, 15, 22, 9, 13, 5, 46, 3, 15, 7, 38, 24, 31, 19, 2, 17, 146, 7, 41, 18, 23, 5, 13, 9, 42, 7, 15, 3, 41, 36, 19, 17, 2, 19, 146, 5, 5, 9, 35, 6, 132, 14, 3, 32, 30, 10, 2, 3, 17, 6, 1, 5, 16.

Bibliography

[1] **C.T. Abraham, S.P. Ghosh, and D.K. Ray-Chaudhuri**, File Organization Schemes Based on Finite Geometries, *Information and Control*, 12(1968), 143-163.

[2] **W. O. Alltop**, Extending t-Designs, *Journal of Combinatorial Theory (A)*, 18 (1975), 177-186.

[3] **A.H. Baartmans, I.D. Bluskov, V.D. Tonchev**, The Preparata Codes and a Class of 4-Designs, *Journal of Combinatorial Designs*, 2(1994), 167-170.

[4] **Z. Baranyai**, On the Factorization of the Complete Uniform Hypergraph, *Colloquia Mathematica Society Janos Bolyai*, 10(1975), 91-108.

[5] **Th. Beth, D. Jungnickel, H. Lenz**, *Design Theory*, Cambridge University Press, 1986.

[6] **I.D. Bluskov**, Designs with Maximally Different Blocks, *Utilitas Mathematica*, 48 (1995), 193-201.

[7] **I.D. Bluskov**, Designs with Maximally Different Blocks and $v = 15, 16$, *Utilitas Mathematica*, 50(1996), 203-213.

[8] **I.D. Bluskov**, *Designs with Maximally Different Blocks and New Designs*, M.Sc. Thesis, Department of Mathematics and Statistics, University of Victoria, Canada, 1995.

[9] **I.D. Bluskov**, New Designs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, 23(1997), 212-220.

[10] **I.D. Bluskov and H. Hämäläinen**, New Upper Bounds on the Minimum Size of Covering Designs, *Journal of Combinatorial Designs* (to appear).

[11] **I.D. Bluskov**, Some t-Designs are Minimal $(t+1)$-Coverings, *Discrete Mathematics* (to appear).

[12] **I.D. Bluskov**, New Simple 3-Designs on 26 and 28 Points, *Ars Combinatoria* (to appear).

[13] **A. E. Brouwer, J. B. Shearer, N. J. A. Sloane and W. D. Smith**, A New Table of Constant Weight Codes, *IEEE Transactions on Information Theory*, 36(1990), 1334-1380.

[14] **D. de Caen, D.L. Kreher, W.H. Mills and S.P. Radziszowski**, On the Covering of t-Sets with $(t+1)$-Sets: $C(9,5,4)$ and $C(10,6,5)$, *Journal of Combinatorial Mathematics and Combinatorial Computing*, 7(1990), 53-90.

[15] **P.J. Cameron, J.H. Van Lint**, *Designs, Graphs, Codes and their Links*, Cambridge University Press, New York, 1991.

[16] **A.H. Chan and R.A. Games**, (n,k,t)-Covering Systems and Error-trapping Decoding, *IEEE Transactions on Information Theory*, 27(1981), 643-646.

[17] **Y.M. Chee**, Some Simple Three-Designs of Small Order, *Journal of Combinatorial Mathematics and Combinatorial Computing*, 16(1994), 163-164.

[18] **Y.M. Chee and S. Ling**, The Union Construction for 3-Designs (submitted for publication).

[19] **G.D. Cohen, M.G. Karpovsky, H.F. Mattson and J.R. Schatz**, Covering Radius-Survey and Recent Results, *IEEE Transactions on Information Theory*, 31(1985), 328-343.

[20] **C.J. Colbourn**, Winning the Lottery. In C.J. Colbourn and J.H. Dinitz, editors, *The CRC Handbook of Combinatorial Designs*, CRC Press, 1996, 578-584.

[21] **J.H. Conway, N.J.A. Sloane**, Lexicographical Codes: Error-Correcting Codes from Game Theory, *IEEE Transactions on Information Theory*, 32(1986), 337-348.

[22] **L.H.M.E. Driessen**, *t-Designs, $t \geq 3$*, Doctoral Dissertation, Department of Mathematics, Eindhoven University of Technology, Holland, 1978.

[23] **P. Erdõs and H. Hanani**, On a Limit Theorem in Combinatorial Analysis, *Publ. Math. Debrecen*, 10(1963), 10-13.

[24] **T. Etzion**, Large Sets of Coverings, *Journal of Combinatorial Designs*, 2(1994), 359-374.

[25] **T. Etzion and A. Hartman**, Towards a Large Set of Steiner Quadruple Systems, *SIAM Journal of Discrete Mathematics*, 4(1991), 182-195.

[26] **T. Etzion and C. L. M. Van Pul**, New Lower Bounds for Constant Weight Codes, *IEEE Transactions on Information Theory*, 35(1989), 1324-1329.

[27] **T. Etzion, V. Wei and Z. Zhang**, Bounds on the Sizes of Constant Weight Covering Codes, *Designs, Codes and Cryptography*, 5(1995), 217-239.

[28] **D.M. Gordon, G. Kuperberg and O. Patashnik**, New Constructions for Covering Designs, *Journal of Combinatorial Design*, 3(1995), 269-284.

[29] **R. L. Graham, N. J. A. Sloane**, Lower Bounds for Constant Weight Codes, *IEEE Transactions on Information Theory*, vol. IT-26(1980), 37-43.

[30] **T.S. Griggs and A. Rosa**, Large Sets of Steiner Triple Systems, in *Surveys in Combinatorics*, editor P. Rowlinson, London Mathematical Society, 1995, 25-39.

[31] **H. Hanani**, On Quadruple Systems, *Canadian Journal of Mathematics*, 12(1960), 145-157.

[32] **G. Howard**, *Lotto-How to Wheel a Fortune*, Smart Luck Publishers Inc., 1990, New York.

[33] **E.S. Kramer and D.M. Mesner**, Intersection Among Steiner Systems, *Journal of Combinatorial Theory (A)*, 16(1974), 273-285.

[34] **D.L. Kreher**, t-Designs, $t \geq 3$. In C.J. Colbourn and J.H. Dinitz, editors, *The CRC Handbook of Combinatorial Designs*, CRC Press, 1996, 47-66.

[35] **J.H. Van Lint and F.J. MacWilliams**, Generalized Quadratic Residue Codes, *IEEE Transactions on Information Theory*, 24(1978), 730-737.

[36] **W.H. Mills and R.C. Mullin**, Coverings and Packings, in J.H. Dinitz and D.R. Stinson, editors, *Contemporary Design Theory: A Collection of Surveys*, Wiley, 1992, 371-399.

[37] **M. Morley**, The Complements of Some Covering Designs and Questions on the Overall Significance, *Australasian Journal of Combinatorics*, 4(1991), 285-289.

[38] **M. Morley and G.H.J van Rees**, Lottery Schemes and Covers, *Utilitas Mathematica*, 37(1990), 159-166.

[39] **K.J. Nurmela and P.R.J. Östergård**, Constructing Covering Designs by Simulated Annealing, Helsinki University of Technology, Digital Systems Laboratory, Series B: Technical Reports, No. 10, January 1993, ISSN 0783-540X, ISBN 951-22-1382-6.

[40] **K.J. Nurmela and P.R.J. Östergård**, Upper Bounds for Covering Designs by Simulated Annealing, *Congressus Numerantium* 96(1993), 93-111.

[41] **W. Oberschelp**, Lotto-Garantiesysteme und Blokplane, *Math.-Phys. Semesterberichte* XIX(1972), 55-67.

[42] **F.P. Preparata**, Weight and Distance Structure of Nordstrom-Robinson Quadratic Code, *Information and Control*, 12(1968), 466-473.

[43] **D.K. Ray-Chaudhuri**, Combinatorial Retrieval Information Systems for Files, *SIAM Journal on Applied Mathematics*, 16(1968), 973-992.

[44] **V. Rődl**, On a Packing and Covering Problem, *European Journal of Combinatorics* 5(1985), 69-78.

[45] **J. Schönheim**, On Coverings, *Pacific Journal of Mathematics*, 14(1964), 1405-1411.

[46] **R. Serotic**, *Lotto-How To Play And Win*, LSI Publishers, 1986, San Francisco, California, USA.

[47] **J. Spencer**, Asymptotically Good Coverings, *Pacific Journal of Mathematics*, 118(1985), 575-586.

[48] **D. Stinson**, Coverings, In C.J. Colbourn and J.H. Dinitz, editors, *The CRC Handbook of Combinatorial Designs*, CRC Press, 1996, 260-265.

[49] **L. Tierlinck**, A Completion of Lu's Determination of the Spectrum for Large Sets of Disjoint Steiner Triple Systems, *Journal of Combinatorial Theory (A)*, 57(1991), 302-305.

[50] **D.T. Todorov**, A Method of Constructing Coverings, *Mathematicheskie Zametki*, 35(1984), 869-876.

[51] **D.T. Todorov**, Some Coverings Derived from Finite Planes, *Colloquia Mathematica Society Janos Bolyai*, 37(1984), 697-710.

[52] **V.D. Tonchev**, *Combinatorial Configurations*, Wiley, New York, 1988.

[53] **E. Witt**, Uber Steinersche Systeme, *Abhandlungen aus dem Mathematischen Seminar der Universitat Hamburg*, 12(1938), 265-275.

[54] **G.V. Zaitsev, V.A. Zinovjev, and N.V. Semakov**, Interrelation of Preparata and Hamming Codes and Extension of Hamming Codes to New Double-Error-Correcting Codes, *Proceedings of the Second International Symposium on Information Theory*, Tsahkadsor, Armenia, USSR, 1971 (Akadémiai Kiadó, Budapest, 1973), 257-263.

[55] **G.V. Zaitsev, V.A. Zinovjev, and N.V. Semakov**, Uniformly Packed Codes, *Problemy Peredatshi Informatsii*, 7(1971), 38-50.

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