Quantum Field-Theoretical Investigations in Nonperturbative QCD and Beyond-The-Standard-Model Electroweak Physics

by

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ABSTRACT

In this thesis, the lowest-order gluon condensate contributions to the QED vertex are calculated. The conclusions appear to be that the presence of a gluon condensate eliminates any possibility of an anomalous magnetic moment.

Single instanton contributions to pseudo-scalar finite energy sum rules are derived both asymptotically and theoretically. The one-resonance finite energy sum rule fit in the scalar channel is explored, in an effort to see if such sum rules support the existence of a very light (σ) scalar resonance.

Cancellation of infrared singularities in two-gluon condensate contributions to finite energy sum rules are demonstrated not to be peculiar to the channels in which they are studied. The explicit cancellation of quark-mass singularities via operator mixing is also demonstrated for the channels in which they naively occur.

The hard photon spectrum in radiative leptonic τ decays $\tau \rightarrow \mu\bar{\nu}_\mu\nu_\tau\gamma$ is analyzed in the presence of possible $\tau\bar{\tau}\gamma$ anomalous couplings. The possibility of anomalous $\tau\bar{\tau}\gamma$ couplings is also examined in the unpolarized differential cross section for the scattering process $e^+e^- \rightarrow \tau^+\tau^-$ at both LEP I and LEP II energies.

To My Mother and Father

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Chapter 1

Introduction

1.1 Overview of the Standard Model

The Standard Model [1] describes the interactions of three generations of quarks and leptons defined by a non-Abelian gauge theory based on the group $SU_C(3) \otimes SU_L(2) \otimes U_Y(1)$. Quantum Chromodynamics (QCD) is the theory of strong interactions based on the colour $SU_C(3)$ group. This group acts on the quarks which are the elementary constituents of matter and the interaction force is mediated by the gluons (g) which are the gauge bosons of the group. The quarks and the gluons are colored fields. One consequence of the non-Abelian nature of the colour symmetry is the existence of self-couplings of the gluons. The "Coupling" (fine structure constant) between quarks and gluons is denoted by α_s which can be ≥ 1. Under some conditions, however, α_s is very small and perturbation theory applies [2]. The $SU_C(3)$ colour symmetry is exact and consequently the gluons are massless.

$SU_L(2) \otimes U_Y(1)$ is the gauge group of the unified weak and electromagnetic interactions, where $SU_L(2)$ is the weak isospin group, acting on left-hand fermions, and

$U_Y(1)$ is the hypercharge group. At low energy ($M < 250 GeV$) the $SU_L(2) \otimes U_Y(1)$ symmetry is spontaneously broken and the residual group is $U_{emg}(1)$ whose generator is a linear combination of the $U_Y(1)$ generator and a generator of $SU_L(2)$: the corresponding gauge boson is the photon (γ) and the associated "coupling" is $\alpha \simeq \frac{1}{137}$. Symmetry breaking implies that the other gauge bosons acquire a mass: they are the heavy $W^{\pm}, Z^0$ bosons discovered at CERN in the mid'80s. The symmetry breaking mechanism is referred to as the Higgs mechanism. The electro-weak theory, based on spontaneously broken $SU_L(2) \otimes U_Y(1)$ gauge invariance, is known as the Glashow-Salam-Weinberg model. At present, the precision tests of the Standard Model have given an extraordinary confirmation of the validity of the Standard Model up to the electroweak energy scale [3], and we have no firm experimental indications for failure of this theory at higher energies. Our belief that the Standard Model is a low-energy approximation of a new and fundamental theory is based only on theoretical, but well-motivated, arguments.

1.2 Two-Point Function Sum Rules in QCD and Non-Perturbative Power Corrections in QCD

There are several types of sum rules which differ in the way that the analyticity properties of two-point functions are exploited and /or in various limits that one can define on sum rule moments. Two of the most common types of sum rules are the Laplace transform sum rules and finite energy sum rules. These sum rules have found extensive success in the determination of hadronic properties. From the phenomenological point of view, one of the most successful examples of the application of finite energy sum rules is the determination of the QCD coupling constant $\alpha_s(m_\tau^2)$

from LEP data on hadronic τ decay [3]. Another recent application uses very general properties of QCD sum rules to set lower bounds [4, 5, 6] on the light quark masses.

The physical vacuum of QCD is not the vacuum state which one uses in perturbation theory. Physical effects like spontaneous chiral symmetry breaking and /or confinement do not appear in an order-by-order perturbative treatment of QCD. Unfortunately, most physical processes involve both low energy (large-distance) and high energy (short-distance) aspects. Since it is difficult to separate the short-distance from the long-distance effects, perturbative QCD is modified by non-perturbative effects at short-distance. The non-perturbative effects in two-point functions evaluated at large Q^2 values appear as inverse power corrections in Q^2. They can be systematically evaluated by utilizing Wilson's operator product expansion (OPE) [7] in the physical vacuum. The power corrections then appear as the product of Wilson coefficients, which are calculable perturbatively, with universal non-zero vacuum expectation values of gauge invariant operator, the so called QCD-vacuum condensates which, although excluded by definition in purely perturbative field theory, can have non-zero values in the physical vacuum. Typical examples are the lowest dimension quark condensate $\langle\bar{q}q\rangle$ and gluon condensate $\langle\alpha_s G^2\rangle$.

1.3 Beyond the Standard Model

Despite all these successes, there are theoretical motivations for going beyond the Standard Model. First of all, in the electro-weak symmetry breaking sector, the Higgs mechanism, which is invoked by the Standard Model to generate the Z^0 and $W^{\pm}$ mass, predicts the existence of a new scalar particle, still to be discovered. From the theoretical point of view, if the Standard Model is embedded within a large gauge theory based upon a unifying symmetry, the Higgs mechanism suffers from the so-called hierarchy [8] problem: radiative corrections would tend to equilibrate the vastly different electroweak (m_Z) and unification mass scales. Furthermore, the complexity of the fermionic and gauge structures makes the Standard Model look like an improbable fundamental theory. It provides no explanation for the duplication of particle families, the elementary particle quantum numbers (colour, electroweak isospin, hypercharge), and it contains many free parameters (e.g. the three gauge coupling constants, the nine fermion masses and the four Cabibbo-Kabayashi-Maskawa mixing parameters): these correspond to important physical quantities, but can not be computed in the context of the Standard Model. Simplifying the Standard Model structure and predicting its free parameters are therefore basic tasks of a successful theory. In fact, there exist many theories beyond the Standard Model such as Grand Unified Theories [9], Supersymmetry [10], Supersymmetric GUT [11], Technicolour [12] etc. Among many other possible ways to explore new physics beyond the Standard Model is to study the anomalous coupling of heavy flavor fermions to the conventional SM gauge bosons, i.e., Z, W, γ and g. With the joint effort of experiments and theory, we are likely to unravel the mystery of the fundamental principles of particle interactions lying beyond the Standard Model.

1.4 Content of the Thesis

The main objective of this thesis is to explore *non*-perturbative topics in QCD as well as perturbative *non*-Standard Model contributions to the electro-weak interactions that may be manifest in the radiative leptonic τ decay $\tau \to \mu\bar{\nu}_\mu\nu_\tau\gamma$ and in the $e^+e^- \to \tau^+\tau^-$ scattering process.

In Chapter 2, gluon condensate $\langle\alpha_s G^2\rangle$ contributions to the quark's electromagnetic F_2 form factor are calculated. We find that the gluon condensate does not appear to contribute to the anomalous magnetic moment of quarks, once the renormalization procedure for the electromagnetic vertex is suitably redefined to account for divergent order-unity condensate contributions. In Appendix A, we have also shown that the self-energy contributions to the vertex involving $\langle\alpha_s G^2\rangle$ do not contribute to the anomalous magnetic moment of quarks.

In Chapter 3, we derive the direct single-instanton contributions to finite-energy sum rules $F_{0,1}$ in the pseudo-scalar meson channel. This contribution is very important for understanding why isosinglet and isovector mesons in both the scalar and pseudo-scalar channels differ in mass.

In chapter 4, a one-resonance finite energy sum rule fit is constructed in the scalar channel in an effort to see if such sum rules support the existence of a very light (σ) scalar resonance.

In Chapter 5, we demonstrate explicitly the cancellation of infrared singularities of two-gluon condensate contributions to finite energy sum rules in pseudo-scalar, scalar, vector and transverse component of the axial-vector channels. The explicit cancellation of quark-mass singularities via operator mixing is also demonstrated for the channels in which such singularities naively occur.

In Chapter 6, electroweak physics beyond the Standard Model is investigated. we calculate the hard photon spectrum in radiative leptonic τ decays $\tau \rightarrow \mu\bar{\nu}_\mu\nu_\tau\gamma$ in the presence of possible $\tau\bar{\tau}\gamma$ anomalous couplings. We analyze the unpolarized differential cross section in the scattering process $e^+e^- \rightarrow \tau^+\tau^-$ with the possible anomalous couplings at LEP 200 energies.

Finally Chapter 7 presents a summary and concluding remarks as well as some possible directions for future research.

Chapter 2

Gluon Condensate $\langle G^2 \rangle$ Contributions to the Quark's Anomalous Magnetic Moment

2.1 The Purely-Perturbative Electromagnetic Vertex Correction: A Methodological Review

The purely-perturbative three-point Green's function G_μ as shown in Fig. 2.1 containing the truncated fermion-antifermion-photon vertex Green's function $-ieQ\Gamma_\sigma(p_2, p_1)$ is expressed in terms of the vacuum expectation values of time-ordered products of Heisenberg fields $\Psi, \bar{\Psi}$ and A_μ^h as follows:

$$\begin{aligned} G_\mu(p_2, p_1) &= \left[\frac{-i}{(p_2-p_1)^2}\left(g_{\mu\sigma} - (1-a)\frac{(p_2-p_1)_\mu(p_2-p_1)_\sigma}{(p_2-p_1)^2}\right)\right] \\ &\quad \left[\frac{i}{\not{p}_2 - m}\right]\{-ieQ\Gamma^\sigma(p_2, p_1)\}\left[\frac{i}{\not{p}_1 - m}\right] \\ &= \int d^4x' \int d^4y' < 0|T\Psi(x')A_\mu^h(0)\bar{\Psi}(y')|0>_{Heis} e^{ip_2\cdot x'}e^{-ip_1\cdot y'} \end{aligned} \quad (2.1)$$

eQ is the electromagnetic fermion charge and a is the gauge parameter. The one-loop correction to this vertex is obtained via a Wick-Dyson expansion of the time-ordered product of Heisenberg fields [13].

The Heisenberg-field vacuum expectation value in Equation (2.1) can be written

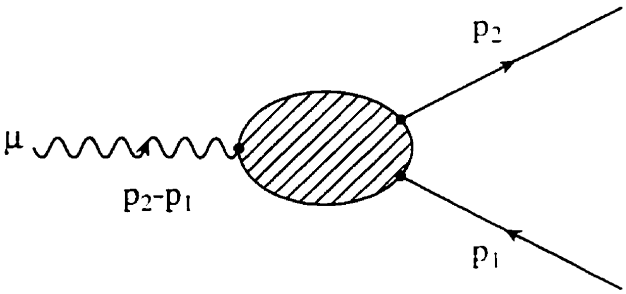

Figure 2.1: The fermion-antifermion-photon three point Green's function

in terms of interaction-picture fields ψ, $\bar{\psi}$ and A_μ as

$$\begin{aligned} < 0|T\Psi(x')A^h_\mu(0)\bar{\Psi}(y')|0 >_{Heis} &= < 0|T\psi(x') exp\left[-ieQ\int d^4w\, \bar{\psi}(w)\gamma^\tau\psi(w)A_\tau(w)\right] \\ & \quad A_\mu(0)\bar{\psi}(y')|0 >. \end{aligned} \tag{2.2}$$

with a one-loop(1L) correction given by

$$\begin{aligned} < \cdots >^{1L}_{Heis} &= i(eQ)^3 < 0|T\psi(x')\left[\int d^4x\, \bar{\psi}(x)\gamma^\tau\psi(x)A_\tau(x)\right] \\ & \quad \left[\int d^4y\, \bar{\psi}(y)\gamma^\sigma\psi(y)A_\sigma(y)\right]\left[\int d^4z\, \bar{\psi}(z)\gamma^\rho\psi(z)A_\rho(z)\right] \\ & \quad \bar{\psi}(y')A_\mu(0)|0 > \end{aligned} \tag{2.3}$$

Putting Equation (2.3) back into Equation (2.1), the one-loop correction to the Green's function $G_\mu(p_2, p_1)$ is found to be [Fig. 2.2]

$$\begin{aligned} \Delta G_\mu(p_2, p_1) &= (-ieQ)^3\int d^4x'\int d^4y'\, e^{ip_2\cdot x'}e^{-ip_1\cdot y'} \\ & \quad \cdot\left\{\int d^4x\int d^4y\int d^4z < 0|T\psi(x')\bar{\psi}(x)|0 > \gamma^\tau\right. \\ & \quad \cdot < 0|T\psi(x)\bar{\psi}(y)|0 > \gamma^\sigma < 0|T\psi(y)\bar{\psi}(z)|0 > \gamma^\rho \\ & \quad \cdot < 0|T\psi(z)\bar{\psi}(y')|0 >< 0|TA_\mu(0)A_\sigma(y)|0 > \\ & \quad \left.\cdot < 0|TA_\tau(x)A_\rho(z)|0 >\right\} \end{aligned} \tag{2.4}$$

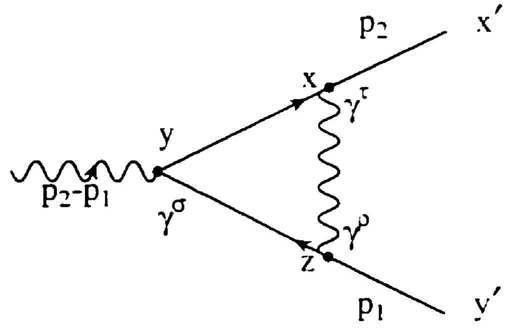

Figure 2.2: The one loop purely perturbative contribution to the vertex function in configuration space

where the term in curly brackets is just the fully-contracted third-order term in the Wick-Dyson expansion of (2.2). Equation (2.4) can be evaluated by explicit use of the configuration-space fermion and photon propagators

$$< 0|T\psi(x)\bar{\psi}(y)|0 >= i\int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(x-y)} \frac{\not q - m}{q^2 - m^2} \tag{2.5}$$

$$< 0|TA_\tau(x)A_\rho(z)|0 >= -i\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-z)} \frac{g_{\tau\rho}}{k^2} \tag{2.6}$$

The gauge-dependent longitudinal term from (2.6) does not contribute to the vertex function. Upon substitution of configuration-space propagators (2.5) and (2.6) into (2.4), one gets the following results:

$$\begin{aligned}\Delta G_\mu(p_2,p_1) &= (-ieQ)^3 \int d^4x' \int d^4y'\, e^{ip_2\cdot x'} e^{-ip_1\cdot y'} \\ &\cdot \int d^4x \int d^4y \int d^4z \left[i\int \frac{d^4q_1}{(2\pi)^4} e^{-iq_1\cdot(x'-x)} \frac{(\not q_1 + m)}{q_1^2 - m^2} \right] \gamma^\tau \\ &\cdot \left[i\int \frac{d^4q_2}{(2\pi)^4} e^{-iq_2\cdot(x-y)} \frac{(\not q_2 + m)}{q_2^2 - m^2} \right] \gamma^\sigma \\ &\cdot \left[i\int \frac{d^4q_3}{(2\pi)^4} e^{-iq_3\cdot(y-z)} \frac{(\not q_3 + m)}{q_3^2 - m^2} \right] \gamma^\rho\end{aligned}$$

$$
\cdot \left[i \int \frac{d^4 q_4}{(2\pi)^4} e^{-iq_4\cdot(z-y')} \frac{(\not{q}_4 + m)}{q_4^2 - m^2} \right]
\left[-ig_{\mu\sigma} \int \frac{d^4 k_1}{(2\pi)^4} \frac{e^{-ik_1\cdot(0-y)}}{k_1^2} \right]
\cdot \left[-ig_{\tau\rho} \int \frac{d^4 k_2}{(2\pi)^4} \frac{e^{-ik_2\cdot(x-z)}}{k_2^2} \right] \tag{2.7}
$$

Performing the integrals over x' and y' before doing any other integrals, we find that:

$$
\int d^4 x' e^{ip_2\cdot x'} e^{-iq_1\cdot x'} \int d^4 y' e^{-ip_1\cdot y'} e^{iq_4\cdot y'} = (2\pi)^8 \delta^4(q_1 - p_2)\delta^4(q_4 - p_1) \tag{2.8}
$$

If one then integrates over q_1 and q_4 variables, one finds that $\Delta G_\mu(p_2, p_1)$ can be rewritten as follows:

$$
\begin{aligned}
\Delta G_\mu(p_2, p_1) \quad = \quad & \left[i \frac{(\not{p}_2 - m)}{p_2^2 - m^2} \right] \\
& \cdot (-ieQ)^3 \int \frac{d^4 q_2}{(2\pi)^4} \int \frac{d^4 q_3}{(2\pi)^4} \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} (i)^2 \\
& \cdot \gamma^\tau \frac{(\not{q}_2 + m)}{q_2^2 - m^2} \gamma^\sigma \frac{(\not{q}_3 + m)}{q_3^2 - m^2} \gamma_\tau \left(\frac{-ig_{\mu\sigma}}{k_1^2} \right) \left(\frac{-i}{k_2^2} \right) \\
& \cdot \int d^4 x \int d^4 y \int d^4 z \, e^{i(p_2 - q_2 - k_2)\cdot x} e^{i(q_2 - q_3 + k_1)\cdot y} \\
& \cdot e^{i(q_3 - p_1 - k_2)\cdot z} \left[i \frac{(\not{p}_1 + m)}{p_1^2 - m^2} \right]
\end{aligned} \tag{2.9}
$$

Integrating over x, y and z variables in (2.9), we obtain the product of three δ-functions:

$$
\begin{aligned}
& (2\pi)^{12} \delta^4(p_2 - q_2 - k_2)\delta^4(q_2 - q_3 + k_1)\delta^4(q_3 - p_1 + k_2) \\
= \; & (2\pi)^{12} \delta^4(k_1 - (p_1 - p_2))\delta^4(q_2 - (p_2 - k_2))\delta^4(q_3 - (p_1 - k_2))
\end{aligned} \tag{2.10}
$$

Substituting this result into (2.9) and integrating over k_1, q_2 and q_3 we find that

$$
\begin{aligned}
\Delta G_\mu(p_2, p_1) \quad = \quad & \left[\frac{i(\not{p}_2 + m)}{p_2^2 - m^2} \right] \left[\frac{-ig_{\mu\sigma}}{(p_2 - p_1)^2} \right] (-ieQ) \\
& \left\{ \left[\frac{-i(eQ)^2}{(2\pi)^4} \int \frac{d^4 k_2}{k_2^2} \gamma^\tau \frac{(\not{p}_2 - \not{k}_2 + m)}{(p_2 - k_2)^2 - m^2} \right. \right. \\
& \left. \left. \gamma^\sigma \frac{(\not{p}_1 - \not{k}_2 + m)}{(p_1 - k_2)^2 - m^2} \gamma_\tau \right] \right\} \left[\frac{i(\not{p}_1 + m)}{p_1^2 - m^2} \right]
\end{aligned} \tag{2.11}
$$

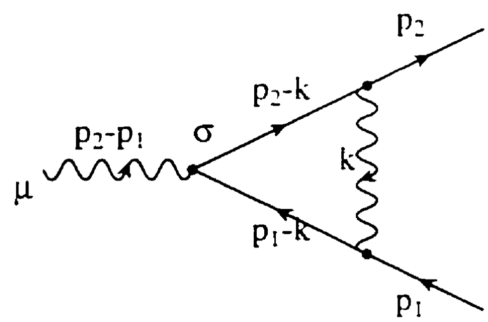

Figure 2.3: The one loop purely perturbative contribution to the vertex function in momentum space.

Equation (2.11) shows the same factorization of the external legs as in the intermediate line of (2.1). As is evident from the δ-function integrals (2.8) and (2.10), this factorization is a direct consequence of the explicit translation invariance of (2.5) and (2.6) which guarantees the translational invariance of all vacuum expection values in (2.4). We equate the curly-bracketed terms in both equations to obtain the usual perturbative expression for the one-loop correction to the vertex function, as shown in Fig. 2.3 :

$$\Delta\Gamma^{\sigma}(p_2, p_1) = \frac{-i(eQ)^2}{(2\pi)^4} \int \frac{d^4k}{k^2} \gamma^{\tau} \frac{1}{\gamma \cdot (p_2 - k) - m} \gamma^{\sigma} \frac{1}{\gamma \cdot (p_1 - k) - m} \gamma_{\tau} \qquad (2.12)$$

The unrenormalized one-loop vertex correction $\Lambda^{\sigma}(p_2, p_1)$ can be written as

$$\Gamma^{\sigma} = \gamma^{\sigma} + \Lambda^{\sigma} + \cdots \qquad (2.13)$$

The unrenormalized vertex correction $\bar{u}(p_2)\Lambda^{\sigma} u(p_1)$ is defined as

$$\Lambda^{\sigma}(p_2, p_1) \equiv (eQ)^2 \left[\mathcal{R}(q^2)\gamma^{\sigma} + \frac{2\mathcal{S}(q^2)}{m}(p_1^{\sigma} + p_2^{\sigma}) \right] \qquad (2.14)$$

where $q^\sigma \equiv p_2^\sigma - p_1^\sigma$. This unrenormalized vertex can be expressed as follows in terms of the renormalized vertex form factors $F_1(q^2)$ and $\mathcal{K}F_2(q^2)$ via applying the Gordon decomposition of the current $\bar{u}(p_2)\gamma^\mu u(p_1)$

$$\bar{u}(p_2)\gamma^\mu u(p_1) = \bar{u}(p_2)\left[\frac{(p_2+p_1)^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p_2-p_1)_\nu}{2m}\right]u(p_1) \tag{2.15}$$

and substituting (2.14) into (2.13), we find that

$$\begin{aligned}\bar{u}(p_2)\Gamma^\mu(p_2,p_1)u(p_1) &= \bar{u}(p_2)[\gamma^\mu + \Lambda^\mu]u(p_1) \\ &= \bar{u}(p_2)\left[\left(1 + e^2Q^2[\mathcal{R}(q^2) + 4\mathcal{S}(q^2)]\right)\gamma^\mu \right.\\ &\quad \left. -2e^2Q^2\mathcal{S}(q^2)i\sigma^{\mu\nu}q_\nu/m\right]u(p_1) \\ &\equiv Z\bar{u}(p_2)\left[F_1(q^2)\gamma^\mu + i\sigma^{\mu\nu}\mathcal{K}F_2(q^2)/2m\right]u(p_1)\end{aligned} \tag{2.16}$$

The rescaling in the final line of (2.16) is accomplished through the renormalization condition that $F_1(0) = 1$, in which case the (divergent) constant Z is given to order-e^2 by

$$Z = 1 + e^2Q^2\left[\mathcal{R}(0) + 4\mathcal{S}(0)\right] \tag{2.17}$$

To the leading order in e^2, one then finds that

$$\begin{aligned}F_1(q^2) &= \frac{1 + e^2Q^2\left[\mathcal{R}(q^2) + 4\mathcal{S}(q^2)\right]}{1 + e^2Q^2\left[\mathcal{R}(0) + 4\mathcal{S}(0)\right]} \\ &= 1 + e^2Q^2\left[(\mathcal{R}'(0) + 4\mathcal{S}'(0))q^2 + \mathcal{O}(q^4)\right] + \mathcal{O}(e^4)\end{aligned} \tag{2.18}$$

$$\begin{aligned}\mathcal{K}F_2(q^2) &= \frac{-4e^2Q^2\mathcal{S}(q^2)}{1 + e^2Q^2\left[\mathcal{R}(0) + 4\mathcal{S}(0)\right]} \\ &= -4e^2Q^2\mathcal{S}(q^2) + \mathcal{O}(e^4)\end{aligned} \tag{2.19}$$

The $q^2 \to 0$ limit of Equation (2.19) gives the $\mathcal{O}(\alpha)$ anomalous magnetic moment of QED [14] via explicit evaluation of the coefficient $\mathcal{S}(q^2)$ of $(p_1^\sigma + p_2^\sigma)$, as defined by (2.14), within the vertex correction $\Delta\Gamma^\sigma$ (2.12). The eventual result first obtained by

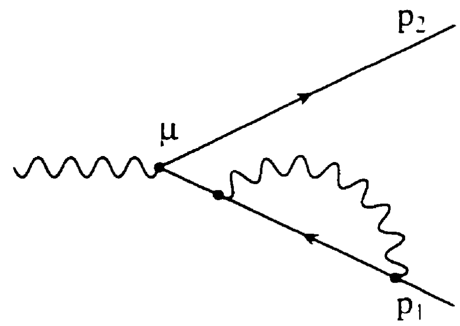

Figure 2.4: An example of fermion self-energy corrections to the QED vertex function Γ^σ.

Schwinger [15] $\mathcal{K}F_2(0) = e^2Q^2/8\pi^2 = \alpha Q^2/2\pi$ is finite and does not entail any further regularizations. The Feynman integrals for $\mathcal{S}(q^2)$ arising from (2.14) are all finite.

To summarize, the anomalous magnetic moment of a Dirac fermion in QED is obtained *entirely* from the coefficient $\mathcal{S}(q^2)$ in the one-loop vertex correction Γ^σ [Fig. 2.3]. Self-energy, vacuum polarization and bremsstrahlung corrections to the bare vertex Fig. 2.4, 2.5 and 2.6 are known not to contribute to $\mathcal{S}(q^2)$, but only to the coefficient $\mathcal{R}(q^2)$ in (2.14). Although these additional contributions are divergent, they are necessarily accompanied by a factor of e^2 in the denominator of (2.19), and consequently do not enter the $\mathcal{O}(e^2)$ determination of the anomalous magnetic moment $\mathcal{K}F_2(0)$. Such divergences are $\mathcal{O}(e^4)$ contributions to $\mathcal{K}F_2(0)$, as the numerator of (2.19) is $\mathcal{O}(e^2)$.

It is interesting to extend this calculation to the case of fundamental fermions coupling to QCD - i.e. quarks. As long as the external boson line remains a photon, as opposed to a gluon with tri-gluon couplings, the only QCD modification to the

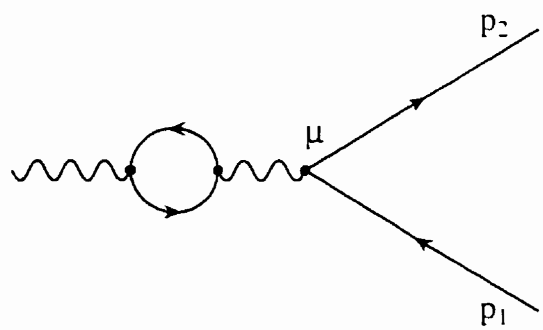

Figure 2.5: An example of photon self-energy corrections to the QED vertex function Γ^σ.

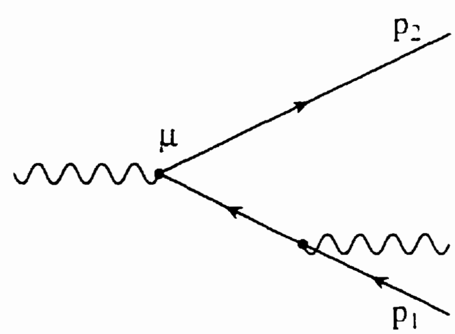

Figure 2.6: An example of bremmstralung corrections to the QED vertex function Γ^σ.

calculation of $S(q^2)$ presented above is the replacement of the momentum-k photon internal-line in Fig. 2.3 with gluons, with appropriate colour factors $\lambda^a_{ij}/2$ at each terminus of the gluon line G^a. The net effect in Equation (2.11) is to replace $(eQ)^2$ with $(eQ)^2 + (4/3)g_s^2$, where g_s is the $SU(3)_c$ group-theoretical factor $(\lambda^a_{ij}\lambda^a_{ji})/4$. Noting that $\alpha_s \gg \alpha$ at any soft momentum scale, we see that the anomalous magnetic moment of a quark will naively be obtained by replacing the QED couplings αQ^2 with the QCD coupling $(4/3)\alpha_s$, thereby leading to the following purely-perturbative QCD contribution to the quark's anomalous magnetic moment:

$$\mathcal{K}F_2(0) = 2\alpha_s(0)/3\pi \tag{2.20}$$

An immediate problem arises when one attempts to discuss α_s at very soft momenta. Because of asymptotic freedom [2], α_s is known to increase as momentum decreases until perturbative QCD is no longer valid. For (2.20) to have any meaning at all, the strong coupling has to be assigned a not-purely-large phenomenological value appropriate for infrared region. Mattingly and Stevenson [16] have argued that α_s should freeze out to a value somewhat less than unity (0.82) at soft momentum scales. Baboukhadia, Elias and Scadron [17] have obtained a similar estimate (0.72) by exploring the linkage between linear-sigma-model hadronic phenomenology and low energy QCD. It has also been argued [18] that values of α_s near unity ($\alpha_s = 3/\pi$) induce the chiral symmetry breaking responsible for a transition from a gauge theory of quarks to a chiral-Lagrangian theory of mesons. Finally, a number of authors have argued for the existence of a non-zero infrared fixed point within the full QCD β-function [19, 20], although contrary arguments based upon Padé summation methods have also been advanced [21].

In any case, we see that there exists motivation for a value for α_s between 0.72 and 1.0 in the infrared region, suggestive [via (2.20)] of an anomalous quark magnetic moment in range $0.14 < \mathcal{K}F_2(0) < 0.20$ arising entirely from purely perturbative single-gluon exchange. The presence of such a large anomalous magnetic moment has significant consequences for baryon magnetic moment estimates in the constituent quark model. Such estimates, which predict successfully the magnetic moments of nucleons and other baryons are consistent with constituent quark masses $m_u = m_d = 363MeV$ obtained from baryon spectroscopy [22]. However, a 15-20% increase in the quark's magnetic moment, as suggested by (2.20), can be accommodated only if the quark mass ($m_{apparent}$) used in naive baryon magnetic moment estimates is 15-20% smaller than the constituent quark mass (m):

$$\frac{e^2}{2m}[1 + \mathcal{K}F_2(0)] = \frac{e^2}{2m_{apparent}} \qquad (2.21)$$

$$m_{apparent} = \frac{m}{1 + \mathcal{K}F_2(0)} \qquad (2.22)$$

These admittedly naive arguments would suggest that the u and d quark masses characterizing magnetic moment calculations ($\sim 360MeV$) necessarily correspond to constituent quark masses in excess of $410MeV$.

There is no hadron-spectroscopic motivation for such large constituent quark masses. Consequently, the result (2.20) is in apparent contradiction to the approximate equality of constituent quark masses (as defined by spectroscopy) with the quark masses characterizing baryon magnetic moments.

Thus, it is reasonable to ask if there exists any further mechanism within QCD, specifically QCD's non-perturbative sector, that could account for the near-equality of spectroscopic-constituent and magneton quark masses - i.e., the absence of an appreciable anomalous magnetic moment.

The very fact that we are concerned with constituent quark masses at all, as opposed to the very much lighter quark masses of the QCD Lagrangian, is indicative of a need to incorporate QCD vacuum condensate into any calculation of $\mathcal{K}F_2(0)$ for constituent quarks, as such condensates are believed responsible for the generation of the constituent quark mass [23, 24, 25]. The contribution of quark-antiquark condensates to $\mathcal{S}(q^2)$ has recently been addressed by Elias and Sprague [26]. This contribution is seen not to alter the real part of $\mathcal{S}(0)$ at all, in which case there should be no further quark-antiquark-condensate contribution to the quark's anomalous magnetic moment. In the next section of this chapter we address the question of how the dimension-4 gluon condensate contributes to $\mathcal{K}F_2(0)$.

2.2 Gluon Condensate $\langle G^2 \rangle$ Contribution to the Vertex Correction

To evaluate the lowest-order gluon-condensate contribution to the vertex function, we must assume that residual normal-ordered terms in the Wick-Dyson expansion of (2.3) that are proportional to $\langle 0| : A_\tau(x) A_\rho(z) : |0\rangle$ do not vanish. In a purely perturbative theory, a normal-ordered product of fields always annihilates the vacuum. Thus the contribution one would get by replacing $\langle 0|T A_\tau(x) A_\rho(z)|0\rangle$ in (2.4) with $\langle 0| : A_\tau(x) A_\rho(z) : |0\rangle$ is no longer automatically zero if the vacuum $|0\rangle$ has non-perturbative content. According to Equation (59) of [27], we can obtain the covariant-gauge gluon condensate contribution to the vertex function (2.4) by replacing

$$\langle 0|T A_\tau(x) A_\rho(z)|0\rangle = -i g_{\tau\rho} \int \frac{d^4 k_2}{(2\pi)^4} \frac{e^{-ik_2 \cdot (x-z)}}{k_2^2} \tag{2.23}$$

with

$$\langle 0| : A_\tau(x)A_\rho(z) : |0\rangle = \frac{1}{8}\left[C(x-z)_\tau(x-z)_\rho + E(x-z)^2 g_{\tau\rho}\right] \tag{2.24}$$

where coefficients C and E are related to the dimension-four condensates [27]

$$C - 2E = \frac{1}{24}\langle G^2\rangle \tag{2.25}$$

$$5C + 2E = -\frac{1}{4}\langle\Omega| : (\partial\cdot A)^2 : |\Omega\rangle. \tag{2.26}$$

where $|\Omega\rangle$ is the non-perturbative vacuum. The gluon-condensate components of C and E are

$$C = \frac{\langle G^2\rangle}{144} + \cdots \tag{2.27}$$

$$E = -\frac{5\langle G^2\rangle}{288} + \cdots \tag{2.28}$$

We choose to work in the covariant gauge (as opposed to Fock-Schwinger or axial gauges) because only covariant gauge exhibits the explicit translational invariance required for factorization of the external lines from a 1PI vertex function Γ^γ. Substituting (2.24) in place of the internal-photon-line propagator (2.23) in (2.7), we find analogous to (2.4) and (2.9) that

$$\begin{aligned}\Delta G^g_\mu(p_2,p_1) = & \left[\frac{i(\not{p}_2+m)}{(p_2^2-m^2)}\right](-ieQ)(-ig_s)^2 Tr(\frac{\lambda^a}{2}\frac{\lambda^b}{2})\int\frac{d^4q_2}{(2\pi)^4}\int\frac{d^4q_3}{(2\pi)^4}\int\frac{d^4k_1}{(2\pi)^4}(i)^2 \\ & \cdot\gamma^\tau\left(\frac{\not{q}_2+m}{q_2^2-m^2}\right)\gamma^\sigma\left(\frac{\not{q}_3+m}{q_3^2-m^2}\right)\gamma^\rho\left(\frac{-ig_{\mu\sigma}}{k_1^2}\right) \\ & \cdot\int d^4x\int d^4y\int d^4z\, e^{i(p_2-q_2)\cdot x}e^{i(q_2-q_3-k_1)\cdot y}e^{i(q_3-p_1)\cdot z} \\ & \cdot\frac{1}{8}\left[C(x-z)_\tau(x-z)_\rho + E(x-z)^2 g_{\tau\rho}\right]\left[\frac{i(\not{p}_1-m)}{(p_1^2-m^2)}\right]\end{aligned} \tag{2.29}$$

where g_s is the strong coupling constant, $\frac{\lambda^a}{2}$ and $\frac{\lambda^b}{2}$ are SU(3) color matrices, and a and b indices correspond to the gluon color states. Since $Tr(\frac{\lambda^a}{2}\frac{\lambda^b}{2}) = \frac{1}{2}\delta^{ab}$, then

$$\sum_a Tr(\frac{\lambda^a}{2}\frac{\lambda^b}{2}) = \frac{1}{3}\sum_a\frac{1}{2}\delta^{aa} = \frac{4}{3} \tag{2.30}$$

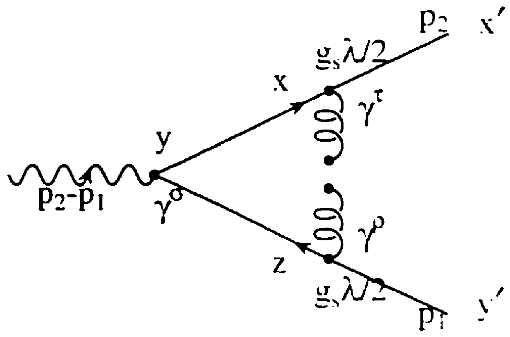

Figure 2.7: The leading gluon condensate contributions to the one loop vertex function in configuration space with the non-perturbative propagator replacing the photon propagator.

In order to evaluate integrals in (2.29), one defines the following coordinate transformation:

$$\bar{X} \equiv x - z, \quad \bar{Y} \equiv y, \quad \bar{Z} \equiv z + x \tag{2.31}$$

and the inverse of this transformation can be found as follows:

$$x = \frac{1}{2}(\bar{Z} + \bar{X}), \quad y = \bar{Y}, \quad z = \frac{1}{2}(\bar{Z} - \bar{X}) \tag{2.32}$$

The transformation between the integral elements are calculated through the Jacobian of this transformation.

$$\begin{aligned} d^4x\, d^4y\, d^4z &= \frac{\partial(x, y, z)}{\partial(\bar{X}, \bar{Y}, \bar{Z})} d^4\bar{X}\, d^4\bar{Y}\, d^4\bar{Z} \\ &= (\frac{1}{2})^4 d^4\bar{X}\, d^4\bar{Y}\, d^4\bar{Z} \end{aligned} \tag{2.33}$$

Changing variables x, y and z into $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ in the last two lines of Equation (2.29), one then can evaluate the configuration-space integrals as follows:

$$
\begin{aligned}
& \cdots \int d^4x \int d^4y \int d^4z \, e^{i(p_2-q_2)\cdot x} \, e^{i(q_2-q_3+k_1)\cdot y} \, e^{i(q_3-p_1)\cdot z} \\
& \cdot \frac{1}{8}\left[C(x-z)_\tau (x-z)_\rho + E(x-z)^2 g_{\tau\rho}\right] \cdots \\
= & \cdots \int d^4x \int d^4y \int d^4z \, e^{iq_2\cdot(-x+y)} e^{ip_2\cdot x} e^{iq_3\cdot(z-y)} e^{ik_1\cdot y} e^{-ip_1\cdot z} \\
& \cdot \frac{1}{8}\left[C(x-z)_\tau (x-z)_\rho + E(x-z)^2 g_{\tau\rho}\right] \cdots \\
= & \cdots (\frac{1}{2})^4 \int d^4\bar{X} \int d^4\bar{Y} \int d^4\bar{Z} e^{i(q_2-q_3+k_1)\cdot\bar{Y}} e^{i[\frac{1}{2}(p_1-p_2)-\frac{1}{2}(q_2-q_3)]\cdot\bar{X}} \\
& e^{i\left[\frac{1}{2}(p_2-p_1)-\frac{1}{2}(q_2-q_3)\right]\cdot\bar{Z}} \bar{X}^\alpha \bar{X}^\beta \frac{1}{8}\left[C g_{\alpha\tau} g_{\beta\rho} + E g_{\alpha\beta} g_{\tau\rho}\right] \cdots \\
= & \cdots (\frac{1}{2})^4 \int d^4\bar{X} \int d^4\bar{Y} \int d^4\bar{Z} e^{i(q_2-q_3+k_1)\bar{Y}} e^{i[\frac{1}{2}P-\frac{1}{2}(q_2+q_3)]\bar{X}} \\
& e^{i\left[\frac{1}{2}Q-\frac{1}{2}(q_2-q_3)\right]\bar{Z}} \bar{X}^\alpha \bar{X}^\beta \frac{1}{8}\left[C g_{\alpha\tau} g_{\beta\rho} + E g_{\alpha\beta} g_{\tau\rho}\right] \cdots \\
= & \cdots \frac{1}{8}\left(C g_{\alpha\tau} g_{\beta\rho} + E g_{\alpha\beta} g_{\tau\rho}\right) (\frac{1}{2})^4 (-\frac{\partial}{\partial P_\alpha}\frac{\partial}{\partial P_\beta}) \left\{(2\pi)^{12} \delta^4(k_1 - (q_3 - q_2)) \right. \\
& \left. \delta^4\left[\frac{1}{2}(q_2+q_3) - P\right] \delta^4\left[\frac{1}{2}(q_2-q_3) - Q\right]\right\} \cdots \qquad (2.34)
\end{aligned}
$$

where P and Q in above equation are defined as the new momentum variables in terms of fermion momentum p_1 and p_2.

$$
\begin{aligned}
P & \equiv \frac{1}{2}(p_2 + p_1) \\
Q & \equiv \frac{1}{2}(p_2 - p_1) \qquad (2.35)
\end{aligned}
$$

Substituting (2.34) into (2.11), we find the gluon condensate component of the vertex Green's function to be

$$
\begin{aligned}
\Delta G^g_\mu(p_2, p_1) = & \left[\frac{i(\not{p}_2 + m)}{(p_2^2 - m^2)}\right] (-ieQ)(-\frac{4}{3}g_s^2) \\
& \frac{1}{8}(C g_{\alpha\tau} g_{\beta\rho} + E g_{\alpha\beta} g_{\tau\rho}) \\
& \frac{\partial}{\partial P_\alpha}\frac{\partial}{\partial P_\beta} \left\{ \int \frac{d^4q_2}{(2\pi)^4} \int \frac{d^4q_3}{(2\pi)^4} \int \frac{d^4k_1}{(2\pi)^4} \right.
\end{aligned}
$$

$$\gamma^{\tau}\frac{i(\not{q}_2+m)}{(q_2^2-m^2)}\gamma^{\sigma}\frac{i(\not{q}_3+m)}{(q_3^2-m^2)}\gamma^{\rho}\frac{-ig_{\mu\sigma}}{k_1^2}$$
$$(\frac{1}{2})^4(2\pi)^{12}\delta^4\left[k_1-(q_3-q_2)\right]\delta^4\left[\frac{1}{2}(q_2+q_3)-P\right]$$
$$\delta^4\left[\frac{1}{2}(q_2-q_3)-Q\right]\bigg\}\frac{i(\not{p}_1+m)}{(p_1^2-m^2)} \quad (2.36)$$

We now evaluate the integrals inside the curly bracket in (2.36).

$$(\frac{1}{2})^4\int d^4q_2\int d^4q_3\int d^4k_1\gamma^{\tau}\frac{(\not{q}_2+m)}{(q_2^2-m^2)}\gamma^{\sigma}\frac{(\not{q}_3+m)}{(q_3^2-m^2)}\gamma^{\rho}\frac{-ig_{\mu\sigma}}{k_1^2}$$
$$\delta^4\left[k_1-(q_3-q_2)\right]\delta^4\left[\frac{1}{2}(q_2+q_3)-P\right]\delta^4\left[\frac{1}{2}(q_2-q_3)-Q\right]$$
$$=(\frac{1}{2})^4\int d^4q_2\int d^4q_3\gamma^{\tau}\frac{(\not{q}_2+m)}{(q_2^2-m^2)}\gamma^{\sigma}\frac{(\not{q}_3+m)}{(q_3^2-m^2)}\gamma^{\rho}\frac{-ig_{\mu\sigma}}{(q_3-q_2)^2}$$
$$\delta^4\left[\frac{1}{2}(q_2+q_3)-P\right]\delta^4\left[\frac{1}{2}(q_2-q_3)-Q\right]$$
$$=\int d^4q_3\gamma^{\tau}\frac{(2\not{P}-\not{q}_3)+m}{(2P-q_3)^2-m^2}\gamma^{\sigma}\frac{\not{q}_3+m}{q_3^2-m^2}\gamma^{\rho}\frac{-ig_{\mu\sigma}}{[2(q_3-P)]^2}\delta^4\left[q_3-(P-Q)\right]$$
$$=\gamma^{\tau}\frac{(\not{P}+\not{Q})+m}{(P+Q)^2-m^2}\gamma^{\sigma}\frac{(\not{P}-\not{Q})+m}{(P-Q)^2-m^2}\gamma^{\rho}\frac{-ig_{\mu\sigma}}{4Q^2} \quad (2.37)$$

We substitute (2.37) into (2.36) to find that

$$\begin{aligned}\Delta G_{\mu}^{g}(p_2,p_1) &= i\frac{(\not{p}_2+m)}{p_2^2-m^2}(-ieQ)(-\frac{4}{3}g_s^2)\frac{1}{8}(Cg_{\alpha\tau}g_{\beta\rho}-Eg_{\alpha\beta}g_{\tau\rho})\\ &\frac{\partial}{\partial P_\alpha}\frac{\partial}{\partial P_\beta}\left[\gamma^{\tau}\frac{(\not{P}+\not{Q}+m)}{(P+Q)^2-m^2}\gamma^{\sigma}\frac{(\not{P}-\not{Q}+m)}{(P-Q)^2-m^2}\gamma^{\rho}\frac{-ig_{\mu\sigma}}{(2Q)^2}\right]i\frac{(\not{p}_1+m)}{p_1^2-m^2}\\ &= \left[i\frac{(\not{p}_2+m)}{p_2^2-m^2}\right]\left[\frac{-ig_{\mu\sigma}}{(p_2-p_1)^2}\right](-ieQ)\Delta\Gamma_g^{\sigma}(P,Q)\left[i\frac{(\not{p}_1+m)}{p_1^2-m^2}\right]\end{aligned} \quad (2.38)$$

In the above equation, $\Delta\Gamma_g^{\sigma}(p_2,p_1)$ is the gluon condensate contribution to the truncated vertex Green's function, which is analogous to the purely perturbative photon correction (2.12). Therefore we then see from (2.38) that

$$\begin{aligned}\Delta\Gamma_g^{\sigma}(P,Q) &= \frac{-g_s^2}{6}(Cg_{\alpha\tau}g_{\beta\rho}+Eg_{\alpha\beta}g_{\tau\rho})\\ &\cdot\frac{\partial}{\partial P_\alpha}\frac{\partial}{\partial P_\beta}\left[\gamma^{\tau}\frac{(\not{P}+\not{Q}+m)}{(P+Q)^2-m^2}\gamma^{\sigma}\frac{(\not{P}-\not{Q}+m)}{(P-Q)^2-m^2}\gamma^{\rho}\frac{-ig_{\mu\sigma}}{(2Q)^2}\right]\end{aligned} \quad (2.39)$$

The renormalized form factors $F_1^g(q^2)$ and $\mathcal{K}F_2^g(q^2)$ can incorporate gluon condensate contributions by making the following transformation on factors in (2.18) and (2.19)

$$e^2Q^2\mathcal{R}(q^2) \rightarrow g_s^2\mathcal{R}^g(q^2) \quad \equiv \quad \langle\alpha_s G^2\rangle\mathcal{R}^g(q^2) \tag{2.40}$$

$$e^2Q^2\mathcal{S}(q^2) \rightarrow g_s^2\mathcal{S}^g(q^2) \quad \equiv \quad \langle\alpha_s G^2\rangle\mathcal{S}^g(q^2) \tag{2.41}$$

The gluon condensate contribution to the vertex correction can then be written as

$$\begin{aligned}
\bar{u}(p_2)\Gamma_g^\mu\, u(p_1) = \bar{u}(p_2)\Big[\Big(1 &+ \langle\alpha_s G^2\rangle\mathcal{R}^g(q^2) - g_s^2\Delta\mathcal{R}^g(q^2) \\
&+4\langle\alpha_s G^2\rangle\mathcal{S}^g(q^2) + g_s^2\Delta\mathcal{S}^g(q^2)\Big)\gamma^\mu \\
&-2\langle\alpha_s G^2\rangle\mathcal{S}^g(q^2)i\sigma^{\mu\nu}q_\nu/m\Big]\, u(p_1) \\
\equiv Z_g\, \bar{u}(p_2)\Big[F_1^g(q^2)\gamma^\mu &+ \mathcal{K}F_2^g(q^2)i\sigma^{\mu\nu}q_\nu/m\Big]\, u(p_1)
\end{aligned} \tag{2.42}$$

As before, the renormalization implicit in the last line of (2.42) is accomplished through the renormalization condition that $F_1^g(0) = 1$, in which case the (divergent) constant Z_g is given to order-g_s^2 by

$$\begin{aligned}
Z_g \quad = \quad & 1 + \langle\alpha_s G^2\rangle\mathcal{R}^g(0) + \mathcal{O}(g_s^2, \epsilon^2) \\
& +4\langle\alpha_s G^2\rangle\mathcal{S}^g(0) + \mathcal{O}(g_s^2, \epsilon^2)
\end{aligned} \tag{2.43}$$

To the leading order in g_s^2, one then finds that

$$F_1^g(q^2) = \frac{1 + \langle\alpha_s G^2\rangle\mathcal{R}^g(q^2) + 4\langle\alpha_s G^2\rangle\mathcal{S}^g(q^2) + \mathcal{O}(g_s^2, \epsilon^2)}{1 + \langle\alpha_s G^2\rangle\left[\mathcal{R}^g(0) + 4\mathcal{S}^g(0)\right]} \tag{2.44}$$

$$\mathcal{K}F_2^g(q^2) = \frac{-4\langle\alpha_s G^2\rangle\mathcal{S}^g(q^2) + \mathcal{O}(g_s^2, \epsilon^2)}{1 + \langle\alpha_s G^2\rangle\left[\mathcal{R}^g(0) + 4\mathcal{S}^g(0)\right]} \tag{2.45}$$

As before, we would like to interpret the $q^2 \rightarrow 0$ limit of Equation (2.45) as the gluon condensate contribution to the quark's anomalous magnetic moment.

2.3 Calculation of $\Delta\Gamma_g^\sigma(p_2, p_1)$

From Equation (2.39), we define the following variables:

$$F^{\alpha\beta\tau\sigma\rho} \equiv \frac{\partial}{\partial P_\alpha}\frac{\partial}{\partial P_\beta}\left[\gamma^\tau \frac{(\not{P} + \not{Q}) + m}{(P+Q)^2 - m^2}\gamma^\sigma \frac{(\not{P} - \not{Q}) - m}{(P-Q)^2 - m^2}\gamma^\rho\right] \tag{2.46}$$

$$D \equiv \left[(P+Q)^2 - m^2\right]\left[(P-Q)^2 - m^2\right] \equiv UV \tag{2.47}$$

$$U \equiv (P+Q)^2 - m^2 \tag{2.48}$$

$$V \equiv (P-Q)^2 - m^2 \tag{2.49}$$

$$p_2 \equiv P + Q \tag{2.50}$$

$$p_1 \equiv P - Q \tag{2.51}$$

$$H^\sigma = [(\not{P} + \not{Q}) + m]\,\gamma^\sigma\,[(\not{P} - \not{Q}) + m] = (\not{p}_2 - m)\gamma^\sigma(\not{p}_1 - m) \tag{2.52}$$

We substitute (2.47-2.52) into (2.46) to find that

$$\begin{aligned} F^{\alpha\beta\tau\sigma\rho} &= \frac{\partial}{\partial P_\alpha}\frac{\partial}{\partial P_\beta}\left[\frac{1}{D}\gamma^\tau H^\sigma\gamma^\rho\right] \\ &\equiv \gamma^\tau G^{\alpha\beta\sigma}\gamma^\rho \end{aligned} \tag{2.53}$$

We evaluate

$$G^{\alpha\beta\sigma} \equiv \frac{\partial}{\partial P_\alpha}\frac{\partial}{\partial P_\beta}\left(\frac{1}{D}H^\sigma\right) \tag{2.54}$$

from the quantities

$$A^\alpha \equiv \frac{\partial}{\partial P_\beta}\left(\frac{1}{D}\right) = -\frac{2}{D^2}(p_2^\beta V + p_1^\beta U). \tag{2.55}$$

$$\begin{aligned} B^{\alpha\beta} &\equiv \frac{\partial}{\partial P_\alpha}\frac{\partial}{\partial P_\beta}\left(\frac{1}{D}\right) \\ &= 2DA^\beta A^\alpha - \frac{2}{D^2}\left[g^{\alpha\beta}(U+V) - 2(p_2^\beta p_1^\alpha + p_1^\beta p_2^\alpha)\right]. \end{aligned} \tag{2.56}$$

to find that

$$
\begin{aligned}
G^{\alpha\beta\sigma} &= B^{\alpha\beta}H^{\sigma} + \frac{1}{D}\frac{\partial}{\partial P_\alpha}\frac{\partial}{\partial P_\beta}(P_\delta P_\lambda)\gamma^{\delta}\gamma^{\sigma}\gamma^{\lambda} \\
&= B^{\alpha\beta}H^{\sigma} + \frac{1}{D}(\gamma^{\beta}\gamma^{\sigma}\gamma^{\alpha} + \gamma^{\alpha}\gamma^{\sigma}\gamma^{\beta}) \qquad (2.57)
\end{aligned}
$$

We then see that the overall gluon-condensate contribution to the vertex correction is given by

$$
\begin{aligned}
\Delta\Gamma_g^{\sigma}(p_2, p_1) &= (\frac{-g_s^2}{6})(C g_{\alpha\tau}g_{\beta\rho} + E g_{\alpha\beta}g_{\tau\rho})\gamma^{\tau}G^{\alpha\beta\sigma}\gamma^{\rho} \\
&= (\frac{-g_s^2}{6}\gamma_\alpha G^{\alpha\beta\sigma}\gamma_\beta + E\gamma^{\tau}G^{\alpha\beta\sigma}g_{\alpha\beta}\gamma_\tau) \\
&= (\frac{-g_s^2}{6})\Bigg\{C\Bigg[\underbrace{\gamma_\alpha B^{\alpha\beta}H^{\sigma}\gamma_\beta}_{(1)} + \frac{1}{D}\underbrace{\gamma_\alpha(\gamma^{\beta}\gamma^{\sigma}\gamma^{\alpha} + \gamma^{\alpha}\gamma^{\sigma}\gamma^{\beta})\gamma_\beta}_{(2)}\Bigg] \\
&\quad + E\Bigg[g_{\alpha\beta}B^{\alpha\beta}\gamma^{\tau}H^{\sigma}\gamma_\tau + \frac{1}{D}\underbrace{g_{\alpha\beta}\gamma^{\tau}(\gamma^{\beta}\gamma^{\sigma}\gamma^{\alpha} + \gamma^{\alpha}\gamma^{\sigma}\gamma^{\beta})\gamma_\tau}_{(3)}\Bigg]\Bigg\} \qquad (2.58)
\end{aligned}
$$

In Equation (2.58). Terms (1), (2) and (3) are calculated separately as follows:

Term (1):

$$
\begin{aligned}
\gamma_\alpha B^{\alpha\beta}H^{\sigma}\gamma_\beta &= 2D\not{A}H^{\sigma}\not{A} - \frac{2}{D^2}[(U+V)\gamma^{\alpha}H^{\sigma}\gamma_\alpha - 2(\not{p}_1 H^{\sigma}\not{p}_2 - \not{p}_2 H^{\sigma}\not{p}_1)] \\
&= \frac{8}{D^3}(\not{p}_2 H^{\sigma}\not{p}_2 V^{-2} + \not{p}_1 H^{\sigma}\not{p}_1 U^{-2}) - \frac{4}{D^2}(\not{p}_2 H^{\sigma}\not{p}_1 - \not{p}_1 H^{\sigma}\not{p}_2) \\
&\quad - \frac{2}{D^2}(U+V)\gamma_\alpha H^{\sigma}\gamma_\alpha \qquad (2.59)
\end{aligned}
$$

Term (2):

$$
\gamma_\alpha(\gamma^{\beta}\gamma^{\sigma}\gamma^{\alpha} + \gamma_\alpha\gamma^{\sigma}\gamma^{\beta})\gamma_\beta = 20\gamma^{\sigma} \qquad (2.60)
$$

Term (3):

$$
g_{\alpha\beta}\gamma^{\tau}(\gamma^{\beta}\gamma^{\sigma}\gamma^{\alpha} + \gamma^{\alpha}\gamma^{\sigma}\gamma^{\beta})\gamma_\tau = 8\gamma^{\sigma} \qquad (2.61)
$$

By substituting (2.59).(2.60) and (2.61) into (2.58). we obtain:

$$\Delta\Gamma_g^\sigma(p_2, p_1) = (\frac{-g_s^2}{6})\Bigg\{\Big[Eg_{\alpha\beta}B^{\alpha\beta} - \frac{2C}{D^2}(U - V)\Big]\underbrace{\gamma^\tau H^\sigma \gamma_\tau}_{(4)}$$

$$+\frac{1}{D}(20C + 8E)\gamma_\sigma + \frac{8C}{D^3}\underbrace{(\not p_2 H^\sigma \not p_2 V^2 - \not p_1 H^\sigma \not p_1 U^2)}_{(5)}$$

$$+\frac{4C}{D^2}\underbrace{(\not p_2 H^\sigma \not p_1 + \not p_1 H^\sigma \not p_2)}_{(6)}\Bigg\} \quad (2.62)$$

In obtaining (2.62) we have used the following identities:

$$\gamma_\alpha \gamma^\beta \gamma^\sigma \gamma^\alpha = 4g^{\beta\sigma} \quad (2.63)$$

$$\gamma_\alpha \gamma^\sigma \not P \gamma^\alpha = 4P^\sigma \quad (2.64)$$

$$\gamma_\alpha \gamma^\sigma \gamma^\alpha = -2\gamma^\sigma \quad (2.65)$$

$$\gamma^\alpha \gamma_\alpha = 4 \quad (2.66)$$

We also use identities

$$\frac{(\not p_1 - m)}{p_1^2 - m^2}u(p_1) = \frac{1}{2m}u(p_1) \quad (2.67)$$

$$\bar{u}(p_2)\frac{(\not p_2 - m)}{p_2^2 - m^2} = \frac{1}{2m}\bar{u}(p_2) \quad (2.68)$$

which follow from the momentum-space Dirac equations for on-shell spinors:

$$(\not p - m)u(p_1) = 0 \quad (2.69)$$

$$\bar{u}(p_2)(\not p - m) = 0 \quad (2.70)$$

Terms (4), (5) and (6) in Equation (2.62) are calculated by using Equations (2.64-2.68):

Term (4):

$$\bar{u}(p_2)\gamma^\tau H^\sigma \gamma_\tau u(p_1) = \bar{u}(p_2)\Bigg\{\Big[4p_1 \cdot p_2 - \frac{UV}{2m^2} - (U + V)\Big]\gamma^\sigma$$

$$-\frac{2}{m}(p_2^\sigma V + p_1^\sigma U)\Big\} u(p_1)$$

$$\equiv \bar{u}(p_2)\left\{\mathcal{R}_4^g(q^2)\gamma^\sigma + \frac{2}{m}\mathcal{S}_4^g(q^2)(p_1^\sigma - p_2^\sigma)\right\} u(p_1) \qquad (2.71)$$

Term (5):

$$\begin{aligned}
\bar{u}(p_2)(\not{p}_2 H^\sigma \not{p}_2 V^{-2} + \not{p}_1 H^\sigma \not{p}_1 U^{-2})u(p_1) &= \bar{u}(p_2)\Big\{\Big[4m^2 p_1\cdot p_2(U^{-2} - V^{-2}) \\
&\quad - m^2(V^{-3} + U^{-3})\Big]\gamma^\sigma \\
&\quad - 2m(p_1^\sigma U^{-3} + p_2^\sigma V^{-3})\Big\} u(p_1) \\
&\equiv \bar{u}(p_2)\Big\{\mathcal{R}_5^g(q^2)\gamma^\sigma \\
&\quad + \frac{2}{m}\mathcal{S}_5^g(q^2)(p_1^\sigma - p_2^\sigma)\Big\} u(p_1) \qquad (2.72)
\end{aligned}$$

Term (6):

$$\begin{aligned}
\bar{u}(p_2)(\not{p}_2 H^\sigma \not{p}_1 + \not{p}_1 H^\sigma \not{p}_2)u(p_1) &= \bar{u}(p_2)\Big\{\Big[4m^4 + 4(p_1\cdot p_2)^2 - p_1\cdot p_2(U + V) \\
&\quad - \frac{UV}{(2m)^2}(2p_1\cdot p_2 - m^2)\Big]\gamma^\sigma \\
&\quad - \frac{2p_1\cdot p_2}{m}(p_2^\sigma V - p_1^\sigma U) + \frac{UV}{2m}(p_1^\sigma - p_2^\sigma)\Big\} u(p_1) \\
&\equiv \bar{u}(p_2)\Big\{\mathcal{R}_6^g(q^2)\gamma^\sigma \\
&\quad + \frac{2}{m}\mathcal{S}_6^g(q^2)(p_1^\sigma - p_2^\sigma)\Big\} u(p_1) \qquad (2.73)
\end{aligned}$$

Substituting (2.71), (2.72) and (2.73) into (2.62), we then find that

$$\begin{aligned}
\bar{u}(p_2)\Delta\Gamma^\sigma(p_2, p_1)u(p_1) &= \bar{u}(p_2)(-\frac{g_s^2}{6})\Big\{\Big[E g_{\alpha\beta}B^{\alpha\beta} - \frac{2C}{D^2}(U + V)\Big] \\
&\quad \Big[\mathcal{R}_4^g\gamma^\sigma + \frac{2}{m}\mathcal{S}_4^g(p_1^\sigma - p_2^\sigma)\Big] \\
&\quad + \frac{1}{D}(20C + 8E)\gamma^\sigma \\
&\quad + \frac{8C}{D^3}\Big[\mathcal{R}_5^g\gamma^\sigma + \frac{2}{m}\mathcal{S}_5^g(p_1^\sigma + p_2^\sigma)\Big] \\
&\quad + \frac{4C}{D^2}\Big[\mathcal{R}_6^g\gamma^\sigma + \frac{2}{m}\mathcal{S}_6^g(p_1^\sigma + p_2^\sigma)\Big]\Big\} u(p_1)
\end{aligned}$$

$$\equiv \bar{u}(p_2)\left[\mathcal{R}^g\gamma^\sigma + \frac{2}{m}\mathcal{S}^g(p_1^\sigma + p_2^\sigma)\right]u(p_1). \tag{2.74}$$

in which case

$$\begin{aligned}\mathcal{R}^g(q^2) &= (-\frac{g_s^2}{6})\left\{\left[Eg_{\alpha\beta}B^{\alpha\beta} - \frac{2C}{D^2}(U' + V')\right]\mathcal{R}_1^g \right.\\ &\left. +\frac{1}{D}(20C + 8E) + \frac{8C}{D^3}\mathcal{R}_5^g + \frac{4C}{D^2}\mathcal{R}_6^g\right\}.\end{aligned} \tag{2.75}$$

$$\begin{aligned}\mathcal{S}^g(q^2) &= (-\frac{g_s^2}{6})\left\{\left[Eg_{\alpha\beta}B^{\alpha\beta} - \frac{2C}{D^2}(U' + V')\right]\mathcal{S}_1^g \right.\\ &\left. +\frac{8C}{D^3}\mathcal{S}_5^g + \frac{4C}{D^2}\mathcal{S}_6^g\right\}\end{aligned} \tag{2.76}$$

We see from Equation (2.75) that

$$\begin{aligned}g_{\alpha\beta}B^{\alpha\beta} &= 2DA^\alpha A_\alpha - \frac{2}{D}[4(U' + V') - 2p_1\cdot p_2] \\ &= \frac{8}{D^3}(p_2^2V'^2 + p_1^2U'^2) - \frac{8}{D^2}[(U' - V') - p_1\cdot p_2]\end{aligned} \tag{2.77}$$

Substituting (2.77) into (2.75) and (2.76), we obtain

$$\begin{aligned}\mathcal{R}^g(q^2) &= (-\frac{g_s^2}{6})\left\{\left[\frac{8E}{D^3}U'^2(p_1^2 + p_2^2) - \frac{8E}{D^2}(2U' - p_1\cdot p_2)\right.\right.\\ &\left. -\frac{2C}{D^2}2U'\right](4p_1\cdot p_2 - \frac{U'^2}{2m^2} + 2U') - \frac{1}{D}(20C + 8E)\\ &+\frac{4C}{D^2}\left[4m^4 + 4(p_1\cdot p_2)^2 + 2p_1\cdot p_2U' - \frac{U'^2}{4m^2}(2p_1\cdot p_2 - m^2)\right]\\ &\left.+\frac{8C}{D^3}2m^2(4U'^2p_1\cdot p_2 + U'^3)\right\}\end{aligned} \tag{2.78}$$

$$\begin{aligned}\mathcal{S}^g(q^2) &= (-\frac{g_s^2}{6})\left\{\left[\frac{8E}{D^3}U'^2(p_1^2 + p_2^2) - \frac{8E}{D^2}(2U' - p_1\cdot p_2) - \frac{2C}{D^2}(2U')\right](-U')\right.\\ &\left.+\frac{8C}{D^3}(-m^2U'^3) + \frac{4C}{D^2}(-p_1\cdot p_2U' + \frac{U'V'}{4})\right\}\end{aligned} \tag{2.79}$$

When $q^2 = (p_2 - p_1)^2 = 0$, then $p_1\cdot p_2 = 0$ became $p_1^2 = p_2^2 = m^2$ on shell. Noting further that $U' = V' = 0$ on shell we find that the gluon -condensate corrections to the vertex function lead to divergent contributions to both $\mathcal{R}(0)$ *and* $\mathcal{S}(0)$:

$$\mathcal{R}^g(0) = \lim_{U\to 0}\left\{96(E+C)\frac{m^4}{U^4} - 9(C-4E)\frac{1}{U^2} - 8(C-2E)\frac{m^2}{U^3} - 2(4E-C)\frac{1}{U m^2}\right\} \quad (2.80)$$

$$\mathcal{S}^g(0) = \lim_{U\to 0}\left\{-\frac{4m^2}{U^3}(4E+3C) + \frac{1}{U^2}(16E+5C)\right\} \quad (2.81)$$

2.4 Discussion

Unlike the purely perturbative case in QED, the gluon condensate contributions to $\mathcal{S}(q^2)$, as calculated from the vertex diagram alone, are divergent on shell. To understand this result completely, it is necessary to review the renormalization procedure.

In purely perturbative QED, the anomalous magnetic moment contribution $\mathcal{S}(0)$ to the electromagnetic vertex is explicitly finite. Feynman [14] and Schwinger [15] calculated the anomalous magnetic moment $\mathcal{K}F_2(0)$ in QED to be

$$\begin{aligned}\mathcal{K}F_2(0) &= \frac{\mathcal{S}(0)}{\mathcal{R}(0)+4\mathcal{S}(0)} \\ &= \frac{e^2/8\pi}{1+e^2(Div)} \\ &= \frac{e^2}{8\pi^2} + \mathcal{O}(e^4)\end{aligned} \quad (2.82)$$

where

$$\mathcal{R}(0) = 1 + e^2(Div) \quad (2.83)$$

$$\mathcal{S}(0) = \frac{\alpha}{2\pi} = \frac{e^2}{8\pi^2} \quad (2.84)$$

$$(Div) \equiv Divergent\,Constant \quad (2.85)$$

In the non-perturbative QCD, the gluon condensate contributions to $\mathcal{R}(q^2)$ arising from the vertex diagram alone are even more divergent. The remaining gluon condensate contributions to $\mathcal{R}(q^2)$ arise from self-energy insertions that are calculated by Bagan et al [27], and these contributions do not cancel the vertex-diagram divergences contributing to $\mathcal{R}(q^2)$. Even though $\mathcal{S}(q^2)$ is divergent, we see from (2.80) and (2.81) that the gluon condensate contributions to $\mathcal{R}$ and $\mathcal{S}$ satisfy

$$\frac{\mathcal{S}^g(0)}{\mathcal{R}^g(0)} \equiv \lim_{q^2\to 0} \frac{\mathcal{S}^g(q^2)}{\mathcal{R}^g(q^2)} = 0 \tag{2.86}$$

The condensates are supposed to be RG-invariant structures. Thus the true dimension-4 gluon condensate should be $\langle\beta(\alpha_s)G^2\rangle$, instead of just $\langle G^2\rangle$. Similarly, the true dimension-4 quark condensate is $\langle m_q\bar{q}q\rangle$ instead of just $\langle\bar{q}q\rangle$. To the lowest order, the QCD β function is proportional to α_s^2, so the gluon condensate can be defined to be $\langle\alpha_s G^2\rangle \equiv \langle 0| : g_s^2 G^a_{\mu\nu}G^{a,\mu\nu} : |0\rangle$. Thus, two powers of coupling constant g_s are absorbed in the definition of the condensate, in which case $\mathcal{R}^g(q^2)$ and $\mathcal{S}^g(q^2)$ are *order-unity* in the perturbation series. Including both gluon condensate contributions in addition to other contributions (perturbative + other condensates) we find that

$$\begin{aligned}
\mathcal{R}(0) &= 1 + \langle\alpha_s G^2\rangle(Div) \\
&\quad + g_s^2(Div) + e^2(Div) + \cdots \\
&= 1 + \mathcal{R}^g(0) + \mathcal{O}(e^2, g_s^2)
\end{aligned} \tag{2.87}$$

$$\begin{aligned}
\mathcal{S}(0) &= \langle\alpha_s G^2\rangle(Div) + (e^2 + \frac{4}{3}g_s^2)(\frac{1}{8\pi^2}) + \cdots \\
&= \mathcal{S}^g(0) + \mathcal{O}(e^2, g_s^2)
\end{aligned} \tag{2.88}$$

Therefore the lowest-order gluon-condensate contribution to the quark's anomalous magnetic moment is found via (2.86) to be

$$\mathcal{K}F_2(0) = \frac{\mathcal{S}(0)}{\mathcal{R}(0) + 4\mathcal{S}(0)}$$

$$= \frac{\mathcal{S}^g(0) + \mathcal{O}(e^2, g_s^2)}{1 + \mathcal{R}^g(0) + 4\mathcal{S}^g(0) + \mathcal{O}(e^2, g_s^2)} = 0 \tag{2.89}$$

All other condensate and purely-perturbative contributions to $\mathcal{K}F_2(0)$ are suppressed by g_s^2 or e^2.

Thus, it appears that the presence of a QCD-vacuum gluon condensate precludes the possibility of an anomalous magnetic moment occurring at all. We have also calculated the gluon condensate contribution to self-energy diagrams which contribute only to $\mathcal{R}(0)$, and not to $\mathcal{S}(0)$ [for detailed calculations see Appendix A], but these contributions do not alter Equation (2.86). The main consequence of this equation, even though both $\mathcal{R}(0)$ and $\mathcal{S}(0)$ are divergent, appears to be the complete absence of an anomalous magnetic moment, a direct consequence of divergent gluon-condensate contributions (which do not occur in purely-perturbative QED). This result suggests that constituent quarks act like massive Dirac fermions ($g = 2$), as in the naive constituent quark model.

To summarize, we find for gluon-condensate contributions to the QED vertex that two factors of g_s are absorbed into the condensate itself, which has a known numerical value ($0.045 GeV^4$) even larger than that corresponding to a (non-coupling-constant-suppressed) QCD scale (Λ^4). Because two powers of g_s are absorbed into the definition of the condensate, the coefficient of $\langle \alpha_s G^2 \rangle$ in the vertex corrections $\mathcal{R}$ and $\mathcal{S}$ are order-unity. Thus the gluon-condensate contribution to the anomalous magnetic moment is

$$\lim_{q^2 \to 0} \frac{\mathcal{S}(q^2)}{\mathcal{R}(q^2) + 4\mathcal{S}(q^2)} = 0 \tag{2.90}$$

Because all other condensate and perturbative contributions to $\mathcal{R}$ and $\mathcal{S}$ involve two additional powers of g_s, this relation will be perturbatively unaltered by adding such contributions to the gluon condensate contribution. Thus, if the gluon condensate is non-zero, the quark's anomalous magnetic moment appears to be zero!

We have used a single constant quark mass throughout these calculations. This is consistent with a dynamical, rather than just a Lagrangian, quark mass, as is argued by Elias and Scadron [24] on grounds of gauge invariance. The Feynman rule QED and QCD vertices remain unaffected by the use of a single dynamical mass, as long as that mass is taken to be constant rather than momentum-dependent [28]. Thus, the quark magnetic moment appears not to change at all from its naive Dirac value $eQ/2m$, where m is about 300 MeV.

Chapter 3

Extracting the Instanton Contribution to Finite Energy Sum Rules

3.1 Approximate Instanton Contribution to Finite Energy Sum Rules

In the instanton liquid model [29], the direct single-instanton contribution to Laplace sum rules based on the pseudo-scalar(p) current correlation function is

$$
\begin{aligned}
R_0^p(s) &= \frac{1}{\pi}\int_0^\infty Im[(\Pi^p(t))_{inst}]e^{-st}dt \\
&= \left(\frac{4\pi^2 n_c\rho^2}{3m_*^2}\right)\frac{3\rho^2}{8\pi^2 s^3}e^{-\rho^2/2s}\left[K_0\left(\frac{\rho^2}{2s}\right)+K_1\left(\frac{\rho^2}{2s}\right)\right] \\
&= \frac{3\rho^2}{8\pi^2 s^3}e^{-\rho^2/2s}\left[K_0\left(\frac{\rho^2}{2s}\right)+K_1\left(\frac{\rho^2}{2s}\right)\right]
\end{aligned}
\tag{3.1}
$$

where ρ is the instanton size ($\approx 1/600MeV$), s is the Borel parameter ($s \equiv 1/M^2$), and K_0, K_1 are the modified Bessel functions of the second kind. $\Pi^p(q^2)$ denotes the correlator of appropriate light-quark pseudo-scalar current $i\bar{q}\gamma_5 q$. In the instanton liquid model the quantity n_c parameterizes the instanton density and m_* is the self-consistent dynamical mass.

The pseudoscalar correlation function is defined as

$$\Pi^p(q^2) = i\int d^4x\, e^{ip\cdot x}\langle 0|T\left(j(x)j(0)\right)|0\rangle \tag{3.2}$$

where $j(x) = \bar{\psi}(x)\gamma_5\psi(x)$ is the pseudoscalar current with ψ being the quark field. In Equation (3.1) $(\Pi^p(t))_{inst}$ is the portion of the pseudoscalar correlation function which arises purely from instantons.

The finite energy sum rules (see Section 4.1.2) we wish to obtain are

$$F_k^p(s_0) \equiv \frac{1}{\pi}\int_0^{s_0} Im[(\Pi(t))_{inst}]t^k dt \tag{3.3}$$

To evaluate (3.3), we see that $R_0(s)$ in (3.1) is itself a Laplace transform:

$$R_0^p(s) = \mathcal{L}\left[\frac{1}{\pi}Im(\Pi^p(t))_{inst}\right] \tag{3.4}$$

$$\mathcal{L}[f(t)] \equiv \int_0^\infty f(t)e^{-st}dt \tag{3.5}$$

From (3.3) and (3.5) we see that

$$\frac{d}{dt}F_k^p(t) = \mathcal{L}^{-1}[R_0^p(s)]t^k \tag{3.6}$$

Upon taking the Laplace transform of both sides of (3.6) and noting from (3.3) that $F_k(0) = 0$, we obtain [30]

$$F_k^p(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\left(-\frac{d}{ds}\right)^k R_0^p(s)\right] \tag{3.7}$$

Approximate expressions for the inverse Laplace transforms (3.7) in terms of elementary trigonometric functions may be obtained via asymptotic expansion methods in the complex plane. We begin with the asymptotic expansion [31]

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}}e^{-z}\left[\sum_{k=0}^{n-1}(\frac{1}{2z})^k\frac{\Gamma(\nu+k-\frac{1}{2})}{k!\Gamma(\nu-k-\frac{1}{2})} + \mathcal{O}(z^{-n})\right] \tag{3.8}$$

Calculating K_0 and K_1 from Equation (3.8), we can find

$$K_0(z) + K_1(z) \sim (\frac{\pi}{2z})^{\frac{1}{2}}e^{-z}\sum_{n=0} a_n z^{-n} \tag{3.9}$$

where

$$a_0 = 2,\ a_1 = \frac{1}{4},\ a_3 = -\frac{3}{64},\ a_4 = \frac{15}{512},\ \cdots \tag{3.10}$$

Given the inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{e^{-a/s}}{s^{1/2}}\right] = \frac{1}{\sqrt{\pi}} t^{-1/2} cos(2t^{1/2}a^{1/2}) \tag{3.11}$$

we repeatedly differentiate both sides of (3.11) with respect to a to find the following identities:

$$\mathcal{L}^{-1}\left[\frac{e^{-a/s}}{s^{3/2}}\right] = \frac{1}{\sqrt{\pi}} a^{-1/2} sin(2t^{1/2}a^{1/2}) \tag{3.12}$$

$$\mathcal{L}^{-1}\left[\frac{e^{-a/s}}{s^{5/2}}\right] = -\frac{1}{\sqrt{\pi}}\left[-\frac{1}{2}a^{-3/2} sin(2t^{1/2}a^{1/2}) + t^{1/2}a^{-1} cos(2t^{1/2}a^{1/2})\right] \tag{3.13}$$

$$\mathcal{L}^{-1}\left[\frac{e^{-a/s}}{s^{7/2}}\right] = \frac{1}{\sqrt{\pi}}\left[\left(\frac{3}{4}a^{-5/2} - ta^{-3/2}\right) sin(2t^{1/2}a^{1/2}) - \frac{3}{2}t^{1/2}a^{-2} cos(2t^{1/2}a^{1/2})\right] \tag{3.14}$$

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{e^{-a/s}}{s^{9/2}}\right] &= \frac{1}{\sqrt{\pi}}\left[\left(\frac{15}{8}a^{-7/2} - 3ta^{-5/2}\right) sin(2t^{1/2}a^{1/2})\right. \\ &\quad \left. + \left(-\frac{15}{4}t^{1/2}a^{-3} + t^{3/2}a^{-2}\right) cos(2t^{1/2}a^{1/2})\right]\end{aligned} \tag{3.15}$$

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{e^{-a/s}}{s^{11/2}}\right] &= \frac{1}{\sqrt{\pi}}\left[\left(\frac{105}{16}a^{-9/2} - \frac{45}{4} - t^2a^{-5/2}\right) sin(2t^{1/2}a^{1/2})\right. \\ &\quad \left. + \left(-\frac{105}{8}t^{1/2}a^{-4} + 5t^{3/2}a^{-3}\right) cos(2t^{1/2}a^{1/2})\right]\end{aligned} \tag{3.16}$$

Through the application of (3.11), (3.12), (3.13), (3.14), (3.15) and (3.16), we can calculate F_0^p as:

$$\begin{aligned}F_0^p &= \mathcal{L}^{-1}\left[\frac{R_0^p(s)}{s}\right] \\ &= \frac{3\rho}{4\pi^{\frac{3}{2}}}\left[\mathcal{L}^{-1}(s^{-\frac{7}{2}}e^{-\frac{\rho^2}{s}}) + \frac{1}{4\rho^2}\mathcal{L}^{-1}(s^{-\frac{5}{2}}e^{-\frac{\rho^2}{s}})\right. \\ &\quad \left. -\frac{3}{32\rho^4}\mathcal{L}^{-1}(s^{-\frac{3}{2}}e^{-\frac{\rho^2}{s}}) + \frac{15}{128\rho^6}\mathcal{L}^{-1}(s^{-\frac{1}{2}}e^{-\frac{\rho^2}{s}}) + \cdots\right]\end{aligned}$$

$$= \frac{3\rho}{4\pi^2}\Big[At\,sin(2\rho\sqrt{t}) + B\sqrt{t}cos(2\rho\sqrt{t}) + C sin(2\rho\sqrt{t}) + D\frac{1}{\sqrt{t}}cos(2\rho\sqrt{t}) + \cdots\Big] \quad (3.17)$$

where

$$A = -\rho^{-3},\ B = -\frac{7}{4}\rho^{-4},\ C = \frac{25}{32}\rho^{-5},\ D = \frac{15}{128}\rho^{-6} \quad (3.18)$$

Finally, we find that F_0^p finite energy sum rule can be expressed in terms of elementary trigonometric functions:

$$F_0^p(s_0) = \frac{3}{4\pi^2\rho^4}\Big\{sin(2\rho s_0^{1/2})\Big[-\rho^2 s_0 + \frac{25}{32} + \mathcal{O}(\frac{1}{\rho^2 s_0})\Big] + cos(2\rho s_0^{1/2})\Big[-\frac{7}{4}\rho s_0^{1/2} - \frac{15}{128\rho s_0^{1/2}} - \mathcal{O}(\frac{1}{\rho^3 s_0^{3/2}})\Big]\Big\} \quad (3.19)$$

Given an instanton size $1/\rho \equiv 600 MeV$. Equation (3.19) is seen to oscillate slowly as s_0 increases past 1 GeV^2, going from positive to negative as s_0 increases past 2.9 GeV^2. Since the purely-perturbative contribution is also positive and quadratic in s_0 [4, 32], we see the effect of instanton contributions is to enhance the size of field-theoretic contributions to F_0^p at low s_0, but to diminish somewhat the magnitude of field-theoretic contributions for values of the continuum threshold chosen to be above 2.9 GeV^2. The same methods as above leads to the following instanton contribution to F_1^p :

$$\begin{aligned} F_1^p &= \mathcal{L}^{-1}\left[\frac{1}{s}\left(-\frac{dR_0^p(s)}{ds}\right)\right] \\ &= \frac{3\rho}{16\pi^{\frac{3}{2}}}\Big[-4\rho^2\mathcal{L}^{-1}(s^{-\frac{11}{2}}e^{-\frac{\rho^2}{s}}) + 9\mathcal{L}^{-1}(s^{-\frac{9}{2}}e^{-\frac{\rho^2}{s}}) - \frac{15}{8\rho^2}\mathcal{L}^{-1}(s^{-\frac{7}{2}}e^{-\frac{\rho^2}{s}}) \\ &\quad -\frac{21}{32\rho^3}\mathcal{L}^{-1}(s^{-\frac{5}{2}}e^{-\frac{\rho^2}{s}}) + \frac{405}{512\rho^5}\mathcal{L}^{-1}(s^{-\frac{3}{2}}e^{-\frac{\rho^2}{s}}) \\ &\quad -\frac{3465}{2048\rho^6}\mathcal{L}^{-1}(s^{-\frac{1}{2}}e^{-\frac{\rho^2}{s}}) + \cdots\Big] \end{aligned} \quad (3.20)$$

Through the application of equations from (3.11) to (3.16), finite energy F_1^p sum rule can be rewritten as:

$$F_1^p(t) = \frac{3\rho}{16\pi^2}\Big[At^2 sin(2\rho\sqrt{t}) + Bt^{3/2}cos(2\rho\sqrt{t}) + Ct\,sin(2\rho\sqrt{t}) + Dt^{1/2}cos(2\rho\sqrt{t}) + Esin(2\rho\sqrt{t}) + Ft^{-1/2}cos(2\rho\sqrt{t}) + \cdots\Big] \quad (3.21)$$

where

$$A = -4\rho^{-3},\ B = -11\rho^{-4},\ C = \frac{129}{8}\rho^{-5}$$
$$D = \frac{531}{32}\rho^{-6},\ E = -\frac{3843}{512}\rho^{-7},\ F = -\frac{3465}{2048}\rho^{-8} \quad (3.22)$$

Again, we find that F_1^p finite energy sum rule can be expressed in terms of elementary trigonometric functions:

$$F_1^p(s_0) = \frac{3}{16\pi^2\rho^6}\Big\{sin(2\rho s_0^{1/2})\Big[-4\rho^4 s_0^2 + \frac{129}{8}\rho^2 s_0 - \frac{3843}{512} + \mathcal{O}(\frac{1}{\rho^2 s_0})\Big] + cos(2\rho s_0^{1/2})\Big[-11\rho^3 s_0^{3/2} + \frac{531}{32}\rho s_0^{1/2} - \frac{3465}{2048\rho s_0^{1/2}} + \mathcal{O}(\frac{1}{\rho^3 s_0^{3/2}})\Big]\Big\} \quad (3.23)$$

Once again, the leading instanton contribution to F_1^p is seen to be lower-degree in s_0 than the $\mathcal{O}(s_0^3)$ purely-perturbative contribution.

3.2 Explicit Expressions for $F_k^p(t)$

In this section we present the derivation of explicit expressions for the instanton contributions to finite energy sum rules $F_k^p(t)$. We begin the derivation with the identity [33]

$$\frac{1}{2s}e^{-1/2s}K_0(1/2s) = -\pi\int_0^\infty J_0(x)Y_0(x)e^{-sx^2}x\,dx$$

$$
\begin{aligned}
&= \mathcal{L}\left[-\frac{\pi}{2}J_0(\sqrt{t})Y_0(\sqrt{t})\right] \\
&\equiv \mathcal{L}[f(t)] \qquad (3.24)
\end{aligned}
$$

where

$$f(t) \equiv -\frac{\pi}{2}J_0(\sqrt{t})Y_0(\sqrt{t}) \qquad (3.25)$$

We differentiate both sides of (3.24) with respect to s, and apply $K_0'(z) = -K_1(z)$:

$$\frac{d(LHS)}{ds} = -\frac{1}{2s^2}e^{-1/2s}K_0(\frac{1}{2s}) + \frac{1}{4s^3}e^{-1/2s}\left[K_0(\frac{1}{2s}) + K_1(\frac{1}{2s})\right] \qquad (3.26)$$

$$\frac{d(RHS)}{ds} = \mathcal{L}[-tf(t)] \qquad (3.27)$$

Equating (3.26) and (3.27), the following relation can be obtained:

$$
\begin{aligned}
H_0(s) &\equiv \frac{1}{(2s)^3}e^{-1/2s}[K_0(1/2s) - K_1(1/2s)] \\
&= \frac{1}{2s}\mathcal{L}[f(t)] + \frac{1}{2}\mathcal{L}[-tf(t)] \\
&= \mathcal{L}[h(t)] \qquad (3.28)
\end{aligned}
$$

Applying the convolution theorem.

$$
\begin{aligned}
\mathcal{L}[f * g] &= \mathcal{L}[f]\cdot\mathcal{L}[g]. \qquad (3.29) \\
(f * g) &= \int_0^t f(t-\tau)g(\tau)d\tau. \qquad (3.30)
\end{aligned}
$$

and knowing that $\mathcal{L}[1] = 1/s$, we find that

$$
\begin{aligned}
h(t) &= \mathcal{L}^{-1}[H_0(s)] \\
&= \mathcal{L}^{-1}\left[\frac{1}{2s}\mathcal{L}[f(t)] + \frac{1}{2}\mathcal{L}[-tf(t)]\right] \\
&= \frac{\pi}{4}tJ_0(\sqrt{t})Y_0(\sqrt{t}) + \mathcal{L}^{-1}\left[\frac{1}{2s}\mathcal{L}\left[-\frac{\pi}{2}J_0(\sqrt{t})Y_0(\sqrt{t})\right]\right] \\
&= \frac{\pi}{4}tJ_0(\sqrt{t})Y_0(\sqrt{t}) - \frac{\pi}{4}\int_0^t J_0(\sqrt{w})Y_0(\sqrt{w})dw \qquad (3.31)
\end{aligned}
$$

Comparing (3.28) with (3.1), we see that

$$R_0^p = \frac{3}{\pi^2\rho^4} H_0(s/\rho^2), \tag{3.32}$$

using the rescaling relation

$$G(s/\rho^2) = \rho^2 \mathcal{L}[g(\rho^2 t)], \quad \text{for } G(s) = \mathcal{L}[g(t)]. \tag{3.33}$$

We find via (3.7) and (3.32) that

$$F_k^p(t) = \frac{3}{\pi^2 \rho^{4+2k}} o_k(\rho^2 t) \tag{3.34}$$

where

$$o_k(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\left(-\frac{d}{ds}\right)^k H_0(s)\right] = \int_0^t \tau^k h(\tau) d\tau \tag{3.35}$$

We then find from substitution of (3.31) into (3.35) that

$$\begin{aligned} o_k(t) &= \frac{\pi}{4}\int_0^t d\tau \tau^k \left[\tau J_0(\sqrt{\tau})Y_0(\sqrt{\tau}) - \int_0^\tau dw J_0(\sqrt{w})Y_0(\sqrt{w})\right] \\ &= \frac{\pi}{4(k+1)}\int_0^t [(k+2)\tau^{k+1} - t_0^{k+1}] J_0(\sqrt{\tau})Y_0(\sqrt{\tau}) d\tau \end{aligned} \tag{3.36}$$

Substitution of (3.36) into (3.34) yields a closed-form expression for the instanton contribution (3.3) to finite energy sum rules:

$$F_k^p(s_0) = \frac{3}{4\pi(k+1)}\int_0^{s_0} [(k+2)w^{k+1} - s_0^{k+1}] J_0(\rho\sqrt{w})Y_0(\rho\sqrt{w}) dw \tag{3.37}$$

Applying a change of variables in (3.37), using the identity

$$\int x J_0(x) Y_0(x) dx = \frac{1}{2}x^2[J_0(x)Y_0(x) + J_1(x)Y_1(x)] \tag{3.38}$$

and performing an integration by parts results in the expression

$$F_k^p(s_0) = -\frac{3}{4\pi}\int_0^{s_0} w^{k+1} J_1(\rho\sqrt{w})Y_1(\rho\sqrt{w}) dw \tag{3.39}$$

The integrand is easily seen to be smaller than the leading perturbative contribution. From comparison of (3.39) and (3.3) it is also possible to make the identification

$$\frac{1}{\pi}Im[\Pi^p(w)]_{inst} = -\frac{3}{4\pi}wJ_1(\rho\sqrt{w})Y_1(\rho\sqrt{w}) \tag{3.40}$$

For $k = 0$ and $k = 1$ we can find $F_0(s_0)$ and $F_1(s_0)$ via changing variables $\rho\sqrt{w} \equiv z$:

$$\begin{aligned} F_0^p(s_0) &= -\frac{3}{4\pi}\int_0^{s_0} wJ_1(\rho\sqrt{w})Y_1(\rho\sqrt{w})dw \\ &= -\frac{3}{2\pi\rho^4}\int_0^{\rho\sqrt{s_0}} z^3J_1(z)Y_1(z)dz \end{aligned} \tag{3.41}$$

$$\begin{aligned} F_1^p(s_0) &= -\frac{3}{4\pi}\int_0^{s_0} w^2J_1(\rho\sqrt{w})Y_1(\rho\sqrt{w})dw \\ &= -\frac{3}{2\pi\rho^6}\int_0^{\rho\sqrt{s_0}} z^5J_1(z)Y_1(z)dz \end{aligned} \tag{3.42}$$

Performing the definite integration over z we find

$$\begin{aligned} F_0^p(z) &= -\frac{3}{2\pi\rho^4}\int_0^{\rho\sqrt{s_0}} z^3J_1(z)Y_1(z)dz \\ &= -\frac{z^4}{4\pi\rho^4}[J_1(z)Y_1(z) - J_2(z)Y_2(z)] \end{aligned} \tag{3.43}$$

The following identity is employed above:

$$\int z^3J_1(z)Y_1(z)dz = \frac{z^4}{6}[J_1(z)Y_1(z) - J_2(z)Y_2(z)] \tag{3.44}$$

Using Equation (3.44) and

$$\int z^5J_2(z)Y_2(z)dz = \frac{z^6}{10}[J_2(z)Y_2(z) + J_3(z)Y_3(z)]. \tag{3.45}$$

we integrate by parts to derive the following identity:

$$\int z^5J_1(z)Y_1(z)dz = \frac{z^6}{8}[J_1(z)Y_1(z) + \frac{4}{5}J_2(z)Y_2(z) - \frac{1}{5}J_3(z)Y_3(z)] \tag{3.46}$$

Therefore, we find that

$$F_1^p(z) = -\frac{3}{16\pi\rho^6} z^6 [J_1(z)Y_1(z) + \frac{4}{5} J_2(z)Y_2(z) - \frac{1}{5} J_3(z)Y_3(z)] \tag{3.47}$$

Substituting $z = \rho\sqrt{w}$ back into (3.43) and (3.47), F_0^p and F_1^p are expressed as follows in terms of s_0:

$$F_0^p(s_0) = -\frac{s_0^2}{4\pi}[J_1(\rho\sqrt{s_0})Y_1(\rho\sqrt{s_0}) + J_2(\rho\sqrt{s_0})Y_2(\rho\sqrt{s_0})] \tag{3.48}$$

$$\begin{aligned} F_1^p(s_0) &= -\frac{3s_0^3}{16\pi\rho^6}[J_1(\rho\sqrt{s_0})Y_1(\rho\sqrt{s_0}) + \frac{4}{5} J_2(\rho\sqrt{s_0})Y_2(\rho\sqrt{s_0}) \\ &\quad -\frac{1}{5} J_3(\rho\sqrt{s_0})Y_3(\rho\sqrt{s_0})] \end{aligned} \tag{3.49}$$

We can show in (3.48) and (3.49) that

$$\begin{aligned} &\lim_{s_0 \to 0} F_0^p(0) \to 0 \\ &\lim_{s_0 \to 0} F_1^p(0) \to 0 \end{aligned} \tag{3.50}$$

Chapter 4

Finite Energy Sum Rules and Subcontinuum Resonances in the Scalar Channel

4.1 Finite Energy Sum Rule Phenomenology

4.1.1 The Correlation Functions and Dispersion Relations

The correlation functions of local operators are defined as the Fourier transforms of the vacuum expectation value of the time ordered product of a local current $J_\mu(x)$ times its hermitian conjugate. i.e.

$$\Pi_{\mu\nu}(q^2) = i \int d^4x\, e^{iq\cdot x} \langle 0|T\left(J_\mu(x) J_\nu(0)^\dagger\right)|0\rangle \qquad (4.1)$$

where the current $J_\mu(x)$ is one of the Noether currents associated with global gauge transformations of flavor degrees of freedom. like a vector current $\bar{q}\gamma^\mu q$. or an axial-vector current $\bar{q}\gamma^\mu\gamma_5 q$. It has been shown by Källén and Lehmann [34. 35] during 1950's that two-point correlation functions obey dispersion relations. The dispersion relation follows from the analyticity properties of $\Pi(q^2)$ as a complex function of q^2. the only energy-momentum invariant which appears in a two-point correlation functions. In general $\Pi(q^2)$ is an analytic function in the complex q^2-plane but for a cut in the real axis $0 \leq q^2 \leq \infty$. as illustrated in Fig. 4.1. With $J_\mu(x)$ a current with specific quantum numbers. the imaginary part of the correlation function is

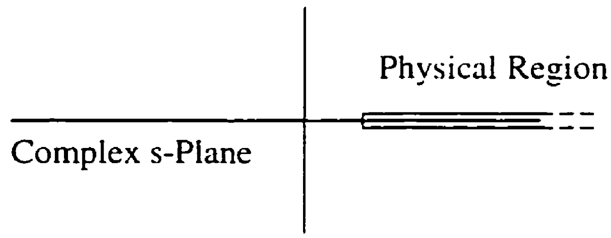

Figure 4.1: Representation of the complex s-plane.

then directly related to the total cross section for the production from the vacuum of hadronic states with those quantum numbers. For example, with the electromagnetic hadronic current light quarks.

$$J^\mu(x) = \frac{2}{3}\bar{u}(x)\gamma^\mu u(x) - \frac{1}{3}\bar{d}(x)\gamma^\mu d(x) - \frac{1}{3}\bar{s}(x)\gamma^\mu s(x) \tag{4.2}$$

the relation to the total e^+e^- annihilation cross-section into hadrons is

$$\sigma(q^2)_{e^+e^-\to hadrons} = \frac{4\pi^2\alpha}{q^2}e^2\frac{1}{\pi}Im\Pi_{em}(q^2). \tag{4.3}$$

4.1.2 Finite Energy Sum Rules

The finite energy sum rules are defined to be the integrals [32]

$$F_k(s_0) \equiv \frac{1}{2\pi i}\int_{C(s_0)} ds\, s^k\, Im\Pi(s) \tag{4.4}$$

with $k = 0, 1, 2, \cdots$. The contour $C(s_0)$ is an open circle of radius s_0 in the complex s-plane that does not cross the real s-axis as shown in Fig. 4.2. For the hadronic contribution to the FESR's $F_k^h(s_0)$, the contour $C(s_0)$ can be distorted into a line running below and above the physical singularities on the positive real s-axis (Fig. 4.3):

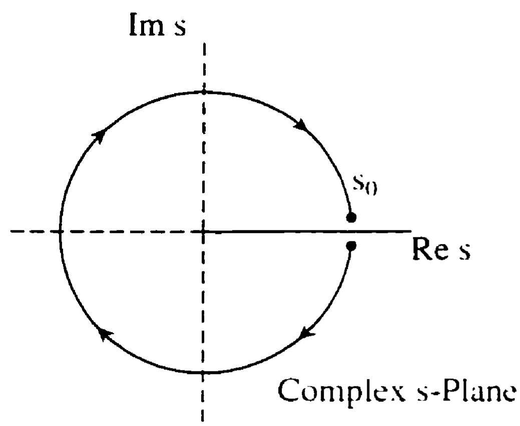

Figure 4.2: Contour of integration $C(s_0)$ in the complex s-plane.

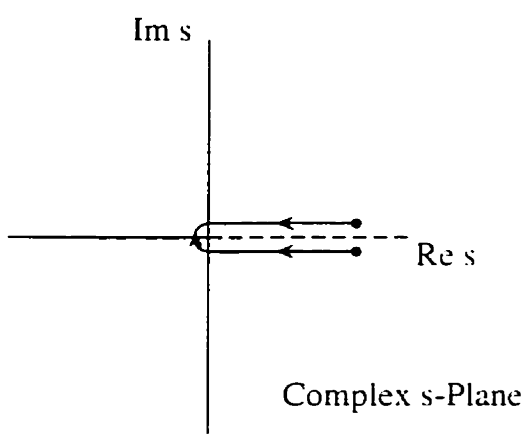

Figure 4.3: Distorted contour of integration in the complex s-plane.

$$F_k^h(s_0) = \int_0^{s_0} ds\, s^k \frac{1}{\pi} Im\Pi^h(s) \tag{4.5}$$

Finite energy sum rules are calculable quantities in QCD, provided that the upper limit s_0 is sufficiently large. In general non- perturbative $1/q^2$ power corrections and instanton corrections can also contribute to this integral. As one increases the k-power in the finite energy sum rule, one becomes more and more sensitive to the detailed high energy behavior of the hadronic spectral function.

One of the successful example of finite energy sum applications is the determination of the QCD coupling constant from the hadronic tau decays. From the phenomenological point of view, the quantity which can be measured by experiments is the hadronic tau decay branching ratio R_τ. It is related to the contribution of hadrons to the spectral function, i.e., the imaginary part of the correlation function.

$$\begin{aligned} R_\tau &\equiv \frac{\Gamma(\tau \to \nu_\tau + hadrons)}{\Gamma(\tau^- \to \nu_\tau e^- \bar{\nu}_e)} \\ &\sim \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} (1 - \frac{s}{m_\tau^2})^2 Im(\Pi(s)) \end{aligned} \tag{4.6}$$

R_τ has been calculated explicitly by Diberder and Pich [36] as follows:

$$R_\tau = N_c \left(|V_{ud}|^2 + |V_{us}|^2\right)^2 S_{ew} \left\{1 + \delta'_{ew} + \delta^{(0)} + \delta_{np}\right\} \tag{4.7}$$

where $S_{ew} = 1.0194$ and $\delta'_{ew} = 0.0010$ are the contributions from the leading and next-to-leading electroweak corrections, and where

$$\delta^{(0)} = \frac{\alpha_s(m_\tau^2)}{\pi} + 5.2023 \left(\frac{\alpha_s(m_\tau^2)}{\pi}\right)^2 + 26.366 \left(\frac{\alpha_s(m_\tau^2)}{\pi}\right)^3 + \mathcal{O}\left(\frac{\alpha_s(m_\tau^2)}{\pi}\right)^4 \tag{4.8}$$

is the result of the perturbative QCD calculation in the chiral limits. The remaining factor $\delta_{np} \equiv -0.016 \pm 0.005$ includes the estimated effects of small quark mass-corrections and non-perturbative power corrections. The experimental value of $\alpha_s(m_\tau^2) = 0.370 \pm 0.033$ [3] is in good agreement with the theoretical value.

4.2 Perturbative Contribution to F_0 and F_1 Scalar Sum-Rules

In this section we extract $\Pi(Q^2)$ of scalar correlation function from the three-loop order QCD radiative corrections of the $\mathcal{D}$ function [37]. This $\mathcal{D}$ function is related to the total hadronic decay width of a scalar Higgs boson which is determined by the imaginary part of the correlation function of the quark scalar currents.

The mathematical form of the $\mathcal{D}$ is given as

$$\mathcal{D}_j(Q^2, m_j, \alpha_s) = Q^2 \frac{d}{dQ^2}\left[\frac{\Pi_j(Q^2)}{Q^2}\right] \tag{4.9}$$

where $\Pi_j(Q^2)$ is the correlation function of the scalar current

$$\Pi_j(Q^2 = -q^2) = i\int d^4x e^{iqx}\langle 0|T(J_j(x)J_j^{-}(0))|0\rangle \tag{4.10}$$

with the scalar current as follows:

$$J_j(x) = m_j \bar{q}_j q_j, \quad q_j = u, d, c, s, b, t. \tag{4.11}$$

where $q_j = u, d, s, c, b, t$ are quarks. m_j are their corresponding masses. In ref. [37] the general expression obtained from the $\mathcal{D}$ function has the form

$$\begin{aligned}\mathcal{D}_j(\frac{Q^2}{\mu^2}, m_j, \alpha_s) &= \frac{d(R)m_j^2}{8\pi^2}\left[1 + \frac{\alpha_s}{\pi}\left(d_{10} - d_{11}ln(\frac{Q^2}{\mu^2})\right)\right.\\ &\left. +(\frac{\alpha_s}{\pi})^2\left(d_{20} + d_{21}ln(\frac{Q^2}{\mu^2}) + d_{22}ln^2(\frac{Q^2}{\mu^2})\right)\right]\end{aligned} \tag{4.12}$$

where μ the renormalization subtraction point, and the analytical expressions for the coefficients d_{ij} are as follows:

$$\begin{aligned}d_{10} &= \frac{17}{4}C_F, \;\; d_{11} = -\frac{3}{2}C_F,\\ d_{20} &= \frac{C_F}{16}\left[\left(\frac{893}{4} - 62\zeta(3)\right)C_A - (65 - 163\zeta(3))TN_f\right.\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{691}{4} - 36\zeta(3)\right) C_F\bigg) . \\
d_{21} &= \frac{C_F}{16}\left(-\frac{284}{3}C_A + \frac{88}{3}TN_f - 105C_F\right) . \\
d_{22} &= \frac{C_F}{16}(11C_A - 4TN_f + 18C_F). \qquad (4.13)
\end{aligned}
$$

In Equation (4.13) the Casimir operators for the adjoint and defining representations of the group $SU(N)$ are $C_A = N$ and $C_F = [(N^2-1)/2]/N$. $d(R)$ is the dimension of the representation R. $\zeta(3)$ is the Riemann Zeta function. $T = 1/2$ corresponds to the normalization condition T^a of SU(N) via the relation $Tr(T^aT^b) = T\delta^{ab}$ for the generators of SU(N) group. and N_f is the number of flavors. For the standard representation of SU(3) group, one has $d(R) = 3$. $C_A = 3$. and $C_F = 4/3$. Therefore the coefficients in (4.13) take the following forms:

$$
\begin{aligned}
d_{10} &= \frac{17}{3}. \quad d_{11} = -2 \\
d_{20} &= \frac{10801}{144} - \frac{39}{2}\zeta(3) - \left[\frac{65}{24} - \frac{2}{3}\zeta(3)\right] N_f \\
d_{21} &= -\frac{106}{3} + \frac{11}{9}N_f. \quad d_{22} = \frac{19}{4} - \frac{1}{6}N_f \qquad (4.14)
\end{aligned}
$$

We solve the differential equation (4.9) by multiplying an integrating factor $e^{-\int dQ^2 \frac{1}{Q^2}} = \frac{1}{Q^2}$. Therefore the scalar correlation function has the form

$$
\Pi_j(Q^2) = Q^2 \int dQ^2 \frac{1}{Q^2} D_j(Q^2) \qquad (4.15)
$$

Using the expression of the $\mathcal{D}$ function (4.12) and the integration results that

$$
\int dQ^2 \frac{ln(Q^2/\mu^2)}{Q^2} = \frac{1}{2} ln^2(Q^2/\mu^2). \qquad (4.16)
$$

$$
\int dQ^2 \frac{ln^2(Q^2/\mu^2)}{Q^2} = \frac{1}{3} ln^3(Q^2/\mu^2). \qquad (4.17)
$$

we obtain the scalar correlation functions $\Pi_j(Q^2)$ as follows:

$$
\Pi_j(Q^2) = \frac{3m_j^2}{8\pi^2} Q^2\, ln(\frac{Q^2}{\mu^2}) \left[1 + \left(\frac{\alpha_s}{\pi}\right) \left(e_{10} + e_{11} ln(\frac{Q^2}{\mu^2})\right)\right.
$$

$$+\left(\frac{\alpha_s}{\pi}\right)^2\left(e_{20}+e_{21}ln(\frac{Q^2}{\mu^2})+e_{22}ln^2(\frac{Q^2}{\mu^2})\right)\Bigg] \tag{4.18}$$

where the coefficients of e_{ij} can be written in terms of d_{ij}

$$e_{10}=d_{10},\quad e_{11}=\frac{d_{10}}{2},\quad e_{20}=d_{20}$$
$$e_{21}=\frac{d_{21}}{2},\quad e_{22}=\frac{d_{22}}{3} \tag{4.19}$$

Defining $-Q^2\equiv s$ and using $m_u=m_d=m$, we obtain the scalar spectral function via the following relations

$$Im\left[ln\left(\frac{Q^2}{\mu^2}\right)\right]=-\pi \tag{4.20}$$

$$Im\left[(ln\left(\frac{Q^2}{\mu^2}\right))^2\right]=-2\pi ln(\frac{s}{\mu^2}) \tag{4.21}$$

$$Im\left[(ln\left(\frac{Q^2}{\mu^2}\right))^3\right]=-3\pi\left(ln(\frac{s}{\mu^2})\right)^2+\pi^3 \tag{4.22}$$

in which case, the scalar spectral function is

$$\begin{aligned}\frac{1}{\pi}Im\left[\Pi^{pert}(s)\right] &= \frac{3m^2s}{8\pi^2}\Bigg\{1+\left(\frac{\alpha_s}{\pi}\right)\left(e_{10}-2e_{11}ln(\frac{s}{\mu^2})\right)\\ &+\left(\frac{\alpha_s}{\pi}\right)^2\Bigg[\left(e_{20}+2e_{21}ln(\frac{s}{\mu^2})+3e_{22}ln^2(\frac{s}{\mu^2})\right)\\ &-e_{22}\pi^2\Bigg]\Bigg\}\end{aligned} \tag{4.23}$$

Substituting Equation (4.23) into the definition of the finite energy sum rule, we use the following integration relations

$$\int_0^{s_0} s\,ln(\frac{s}{\mu^2})\,ds=\frac{1}{2}s_0^2\left[ln(\frac{s_0}{\mu^2})-\frac{1}{2}\right] \tag{4.24}$$

$$\int_0^{s_0} s\,ln^2(\frac{s}{\mu^2})\,ds=\frac{1}{2}s_0^2\left[ln^2(\frac{s_0}{\mu^2})-ln(\frac{s_0}{\mu^2})+\frac{1}{2}\right] \tag{4.25}$$

$$\int_0^{s_0} s^2\,ln(\frac{s}{\mu^2})\,ds=\frac{1}{3}s_0^3\left[ln(\frac{s_0}{\mu^2})-\frac{1}{3}\right] \tag{4.26}$$

$$\int_0^{s_0} s^2\, ln^2(\frac{s}{\mu^2})\, ds = \frac{1}{3}s_0^3 \left[ln^2(\frac{s_0}{\mu^2}) - \frac{2}{3}ln(\frac{s_0}{\mu^2}) - \frac{2}{9}\right] \tag{4.27}$$

to find that the perturbative contributions to F_0 and F_1 are

$$\begin{aligned} F_0^{pert}(s_0) &= \int_0^{s_0} \frac{1}{\pi} Im\left[\Pi^{pert}(s_0)\right] ds \\ &= \frac{3m^2 s_0^2}{16\pi^2}\left\{1 + \left(\frac{\alpha_s}{\pi}\right)\left[(e_{10} - e_{11}) + 2e_{11}\, ln(\frac{s_0^2}{\mu^2})\right]\right. \\ &+ \left(\frac{\alpha_s}{\pi}\right)^2\left[(e_{20} - e_{21}) + (2e_{21} - 3e_{22})\, ln(\frac{s_0^2}{\mu^2})\right. \\ &\left.\left. + 3e_{22}\, ln(\frac{s_0}{\mu^2}) + (\frac{3}{2} - \pi^2)e_{22}\right]\right\} \end{aligned} \tag{4.28}$$

$$\begin{aligned} F_1^{pert}(s_0) &= \int_0^{s_0} \frac{1}{\pi} Im\left[\Pi^{pert}(s_0)\right] s\, ds \\ &= \frac{m^2 s_0^3}{8\pi^2}\left\{1 + \left(\frac{\alpha_s}{\pi}\right)\left[(e_{10} - \frac{2}{3}e_{11}) + 2e_{11}\, ln(\frac{s_0^2}{\mu^2})\right]\right. \\ &+ \left(\frac{\alpha_s}{\pi}\right)^2\left[(e_{20} - \frac{2}{3}e_{21}) + 2(e_{21} - e_{22})\, ln(\frac{s_0^2}{\mu^2})\right. \\ &\left.\left. + 3e_{22}\, ln(\frac{s_0}{\mu^2}) + (\frac{2}{3} - \pi^2)e_{22}\right]\right\} \end{aligned} \tag{4.29}$$

4.3 Non-Perturbative Contribution to F_0 and F_1 Scalar Sum-Rules

Within the frame work of perturbative QCD. the perturbative QCD results are modified by *non-perturbative* effects at short distances. In the *physical* vacuum we can evaluate these non-perturbative effects in the correlation functions calculated at large Q^2-values. Those values appear as inverse power corrections in Q^2 by using Wilson's Operator Product Expansion (OPE) in the physical vacuum. The power corrections appear as the product of Wilson coefficients times the universal non-zero vacuum expectation value of gauge invariant operators called condensates. Therefore the correlation function can be expressed as the sum of both the perturbative and non-perturbative contribution

$$\Pi(Q^2) = C_{pert}(Q^2) + C_Q(Q^2)\langle m\bar{q}q\rangle + C_M(Q^2)\langle \bar{q}G\cdot\sigma q\rangle + C_{G^2}(Q^2)\langle \alpha_s G^2\rangle$$
$$+ C_{G^3}(Q^2)\langle g_s G^3\rangle + C_{(\bar{q}q)^2}(Q^2)\langle \alpha_s(\bar{q}q)^2\rangle + \cdots \tag{4.30}$$

where $C_{pert}(Q^2)$ is the contribution from the perturbative effects, $C_Q(Q^2)$, $C_M(Q^2)$, $C_{G^2}(Q^2)$, $C_{G^3}(Q^2)$ and $C_{(\bar{q}q)^2}(Q^2)$ are Wilson coefficients. Their corresponding condensates are defined as follows:

$$\langle m\bar{q}q\rangle \equiv \langle 0| : m\bar{q}_\alpha(0)q_\alpha(0) : |0\rangle \tag{4.31}$$
$$\langle \bar{q}G\cdot\sigma q\rangle \equiv \frac{1}{2}g_s\langle 0| : \bar{q}_\alpha(0)\sigma_{\mu\nu}(\lambda^a)_{\alpha\beta}q_\beta(0)G_a^{\mu\nu}(0) : |0\rangle \tag{4.32}$$
$$\langle \alpha_s G^2\rangle \equiv \langle 0| : \alpha_s G^a_{\mu\nu}(0)G_a^{\mu\nu}(0) : |0\rangle \tag{4.33}$$
$$\langle g_s G^3\rangle \equiv g_s f_{abc}\langle 0| : G^a_{\mu\nu}(0)G^{\nu\rho b}(0)G_\rho^{\mu c}(0) : |0\rangle \tag{4.34}$$
$$\langle \alpha_s(\bar{q}q)^2\rangle = \alpha_s\left(\langle m\bar{q}q\rangle\right)^2/m^2 \tag{4.35}$$

These quantities are respectively called the quark condensate, the quark gluon mixed condensate, the two-gluon condensate, the three-gluon condensate and the four quark condensate. Their corresponding Feynman diagrams are shown from Fig. 4.4 to Fig. 4.8. Our focus in this chapter in on the scalar correlation function, for which the relevant coefficients in (4.30) have been calculated as follows in ref. [38]:

$$C_Q = -\frac{(1-v)(1+2v)}{m^2(1+v)} \tag{4.36}$$
$$C_M = \frac{(1-v^2)}{2m^3(1+v)} \tag{4.37}$$
$$E_{G^2} = \frac{[(v^2-1)(3+v^2)X(v) + 2(3-v^2)]}{32\pi q^2 v^2} \tag{4.38}$$

$$E_{G^3} = \frac{1}{5760\pi q^4 v^6(1-v^2)}[15(1-v^2)^2(23+19v^2+5v^4+v^6)X(v)$$
$$-2(345-290v^2-216v^4+50v^6+15v^8)] \tag{4.39}$$

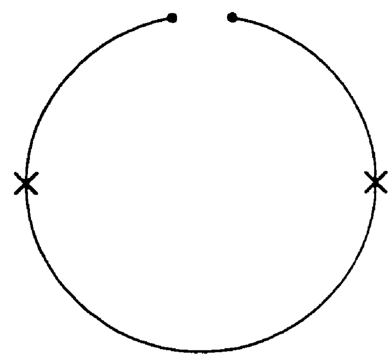

Figure 4.4: Feynman diagram representation of quark condensate contributions

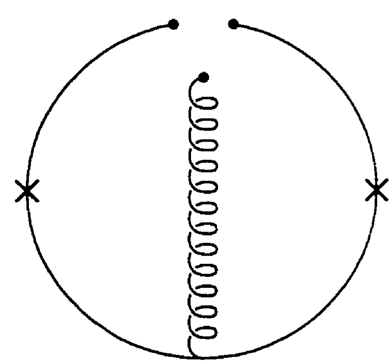

Figure 4.5: Feynman diagram representation of contributions from the mixed quark-gluon condensate

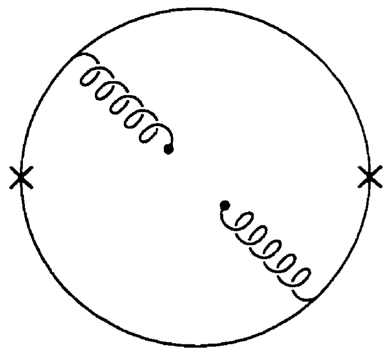

Figure 4.6: Feynman diagram representation of two-gluon condensate contributions

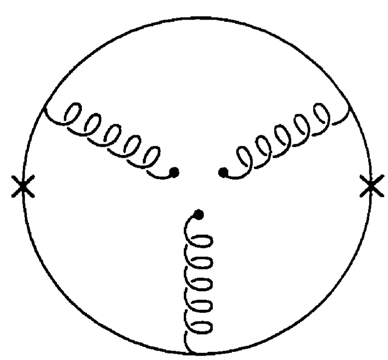

Figure 4.7: Feynman diagram representation of three-gluon condensate contributions

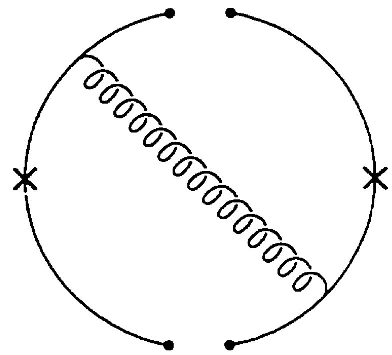

Figure 4.8: Feynman diagram representation of contributions from the four-quark condensate

where for $q^2 < 0$.

$$v = \sqrt{1 - \frac{4m^2}{q^2}} \tag{4.40}$$

$$X(v) = \frac{1}{v} ln \left(\frac{v+1}{v-1} \right) \tag{4.41}$$

Since the light quark mass is very small, an expansion in powers of light quark mass will be appropriate. The expansions of (4.36), (4.37), (4.38) and (4.39) in powers of light quark mass are obtained as follows:

$$C_Q = 3\frac{m^2}{Q^2} - 2\frac{m^4}{Q^4} + 3\frac{m^6}{Q^6} - 6\frac{m^8}{Q^8} + \mathcal{O}(\frac{m^{10}}{Q^{10}}) \tag{4.42}$$

$$C_M = \frac{m}{Q^4} - 3\frac{m^3}{Q^6} + 9\frac{m^5}{Q^8} + \mathcal{O}(\frac{m^7}{Q^{10}}) \tag{4.43}$$

$$\begin{aligned} E_{G^2} \;=\; & \frac{1}{24\pi}\left[-\frac{3}{Q^2} + 12\frac{m^2}{Q^4} ln(\frac{m^2}{Q^2}) + 18\frac{m^2}{Q^4} - 60\frac{m^4}{Q^6} ln(\frac{m^2}{Q^2}) \right. \\ & -96\frac{m^4}{Q^6} + 288\frac{m^6}{Q^8} ln(\frac{m^2}{Q^2}) + 444\frac{m^6}{Q^8} - 1988\frac{m^8}{Q^{10}} \\ & \left. -1320\frac{m^8}{Q^{10}} ln(\frac{m^2}{Q^2}) + \mathcal{O}(\frac{m^{10}}{Q^{12}}) \right] \end{aligned} \tag{4.44}$$

$$E_{G^3} = \frac{1}{360\pi}\Big[-\frac{3}{m^2Q^2} - 28\frac{1}{Q^4} + 180\frac{m^2}{Q^6}ln(\frac{m^2}{Q^2}) - 192\frac{m^2}{Q^6}$$
$$-2040\frac{m^4}{Q^8}ln(\frac{m^2}{Q^2}) - 4508\frac{m^4}{Q^8} - 32692\frac{m^6}{Q^{10}}$$
$$+16440\frac{m^6}{Q^{10}}ln(\frac{m^2}{Q^2}) + \mathcal{O}(\frac{m^8}{Q^{12}})\Big] \quad (4.45)$$

For two and three gluon condensates, the correct coefficients C_{G^2} and C_{G^3} are obtained from the naive coefficients E_{G^2} and E_{G^3} in ref. [38]:

$$C_{G^2} = E_{G^2} + \frac{1}{12\pi}C_Q - \frac{m}{2\pi}ln(\frac{m^2}{\mu^2})C_M$$
$$= \frac{1}{8\pi Q^2} + \frac{7m^2}{12\pi Q^4} - \frac{m^2}{2\pi Q^4}ln(\frac{Q^2}{\mu^2}) - \mathcal{O}(m^4) \quad (4.46)$$

$$C_{G^3} = E_{G^3} + \frac{1}{360\pi m^2}C_Q + \frac{1}{12\pi m}C_M$$
$$= \frac{135m^2}{120\pi Q^6} - \frac{m^2}{2\pi Q^6}ln(\frac{Q^2}{\mu^2}) - \mathcal{O}(m^4) \quad (4.47)$$

The dimension six quark condensate is obtained via the vacuum saturation hypothesis [39]

$$C_{(\bar{q}q)^2} = -\frac{176\pi}{27Q^4} + \mathcal{O}(m^2) \quad (4.48)$$

Finally we have

$$\Pi^{cond}(s) = C_Q\langle m\bar{q}q\rangle + C_M\langle \bar{q}G\cdot\sigma q\rangle - C_{G^2}\langle\alpha_s G^2\rangle$$
$$+C_{G^3}\langle g_sG^3\rangle + C_{(\bar{q}q)^2}\langle\alpha_s(\bar{q}q)^2\rangle + \cdots \quad (4.49)$$

Using the relations

$$\int_c \frac{lnQ^2}{Q^{2n}}dQ^2 = \frac{(-1)^n 2\pi i}{(n-1)s_0^{(n-1)}},\ n \geq 2 \quad (4.50)$$

We then find that $[Q^2 = -s]$

$$F_0^{cond}(s_0) = 3\langle m\bar{q}q\rangle + \frac{1}{8\pi}\langle\alpha_s G^2\rangle$$

$$
\begin{aligned}
& -\frac{m^2}{2\pi}\langle\alpha_s G^2\rangle\frac{1}{2\pi i}\int_{C(s_0)}\frac{lnQ^2}{Q^4}ds \\
& -\frac{m^2}{2\pi}\langle\alpha_s G^3\rangle\frac{1}{2\pi i}\int_{C(s_0)}\frac{lnQ^2}{Q^6}ds \\
= \; & 3\langle m\bar{q}q\rangle + \frac{1}{8\pi}\langle\alpha_s G^2\rangle - \frac{m^2}{2\pi}\langle\alpha_s G^2\rangle\frac{1}{s_0} \\
& -\frac{m^2}{2\pi}\langle\alpha_s G^3\rangle\frac{-1}{2s_0^2} \qquad (4.51)
\end{aligned}
$$

Since higher dimensional condensates are all suppressed by additional powers of the light quark mass. the leading order non-perturbative contribution is given by

$$
F_0^{cond}(s_0) = 3\langle m\bar{q}q\rangle + \frac{1}{8\pi}\langle\alpha_s G^2\rangle + \mathcal{O}(m^2) \qquad (4.52)
$$

Similarly

$$
\begin{aligned}
F_1^{cond}(s_0) \; = \; & 2m^2\langle m\bar{q}q\rangle - \frac{7m^2}{12\pi}\langle\alpha_s G^2\rangle - m\langle qG\cdot\sigma q\rangle \\
& +\frac{m^2}{2\pi i}ln(\frac{s_0}{\mu^2})\langle\alpha_s G^2\rangle + \frac{m^2}{2\pi}\langle\alpha_s G^3\rangle\frac{1}{s_0} \\
& +\frac{176\pi}{27}\langle\alpha_s(\bar{q}q)^2\rangle \\
= \; & \frac{176\pi}{27}\langle\alpha_s(\bar{q}q)^2\rangle + \mathcal{O}(m) \qquad (4.53)
\end{aligned}
$$

4.4 One Resonance Finite Energy Sum Rule Fit

Hadron properties can be extracted by relating phenomenological and field theoretical expression for integrals over the scalar current correlation functions. In the previous two sections we presented the field theory contributions to Finite Energy Sum Rules. The phenomenological expressions are generally extracted through the narrow resonance approximation.

In the narrow resonance approximation. hadronic contributions to the imaginary part of current-current correlation functions are proportional to δ-function at the resonance mass

$$Im[\Pi^h(s)] = \sum_r \pi g_r \delta(s - m_r^2) + \Theta(s - s_0) Im[\Pi^p(s)] \tag{4.54}$$

The summation r in (4.54) is over all scalar resonances, [i.e., whose quantum numbers are consistent with the choice of currents in the current correlation function] such that m_r^2 is less that s_0. Above this hadron-continuum threshold, the hadronic contribution $\Pi^h(s)$ to the correlation function is assumed to be the same as the contribution $\Pi^p(s)$ from perturbative QCD.

Substituting the narrow resonance approximation (4.54) into $F_k^h(s_0)$, one finds that

$$\begin{aligned} F_k^h(s_0) &= \frac{1}{\pi}\int_0^{s_0} ds s^k Im[\Pi^h(s)] \\ &= \sum_r g_r m_r^{2k} \equiv \sum_r [F_k^h(s_0)]_r \end{aligned} \tag{4.55}$$

The field theoretical expressions for the $F_0(s_0)$ and $F_1(s_0)$ finite energy sum rules are obtained through (4.28), (4.52), (3.48), (4.29), (4.53) and (3.49):

$$\begin{aligned} F_0(s_0) &= F_0^{pert}(s_0) + F_0^{cond}(s_0) + F_0^{inst}(s_0) \\ &= \frac{3s_0^2}{16\pi}\left[1 + \frac{20}{3}\left(\frac{\alpha_s}{\sqrt{s_0}}\right) + 49.8223\left(\frac{\alpha_s}{\sqrt{s_0}}\right)^2 + 302.1104\left(\frac{\alpha_s}{\sqrt{s_0}}\right)^3\right] \\ &\quad + 3\langle m\bar{q}q\rangle + \frac{1}{8\pi}\langle\alpha_s G^2\rangle \\ &\quad - \frac{s_0^2}{4\pi}\left[J_1(\rho\sqrt{s_0})Y_1(\rho\sqrt{s_0}) + J_2(\rho\sqrt{s_0})Y_2(\rho\sqrt{s_0})\right] \end{aligned} \tag{4.56}$$

$$\begin{aligned} F_1(s_0) &= F_1^{pert}(s_0) + F_1^{cond}(s_0) + F_1^{inst}(s_0) \\ &= \frac{s_0^3}{8\pi^2}\left[1 + \frac{19}{3}\left(\frac{\alpha_s}{\sqrt{s_0}}\right) + 43.3640\left(\frac{\alpha_s}{\sqrt{s_0}}\right)^2 + 215.8457\left(\frac{\alpha_s}{\sqrt{s_0}}\right)^3\right] \\ &\quad + \frac{176\pi}{27}\langle\alpha_s(\bar{q}q)^2\rangle \end{aligned}$$

$$-\frac{3s_0^3}{80\pi}[5J_1(\rho\sqrt{s_0})Y_1(\rho\sqrt{s_0}) + 4J_2(\rho\sqrt{s_0})Y_2(\rho\sqrt{s_0})$$

$$-J_3(\rho\sqrt{s_0})Y_3(\rho\sqrt{s_0})] \quad (4.57)$$

Corresponding expressions for the resonance contributions to F_0 and F_1 are obtained from (4.55):

$$F_0(s_0) = \sum_r g_r = F_0^{pert}(s_0) + F_0^{cond}(s_0) - F_0^{inst}(s_0) \quad (4.58)$$

$$F_1(s_0) = \sum_r g_r m_r^2 = F_1^{pert}(s_0) + F_1^{cond}(s_0) + F_1^{inst}(s_0) \quad (4.59)$$

Using (4.58) and (4.59), the upper bound on the σ mass can be obtained:

$$\frac{F_1(s_0)}{F_0(s_0)} = \frac{\sum_r g_r m_r^2}{\sum_r g_r} \geq m_\sigma^2 \quad (4.60)$$

This upper bound can be identified with the σ mass itself, if the σ is the only resonance contributing to the sums (4.58) and (4.59) the assumption implicit in a one-resonance fit. Such a fit has been obtained from the field-theoretical expressions (4.56) and (4.57) in unpublished work by K.B. Sprague. His calculation indicates that $m_\sigma^2 < 1GeV^2$ provided that $s_0 < 1.72GeV^2$. But other work [21] has shown that the perturbative contribution to FESR's converge very slowly, if at all, if $\sqrt{s_0} \leq 1.3GeV$, suggesting that a σ lighter than 1 GeV is unsupported by a one-resonance fit to finite energy sum rules.

Chapter 5

Cancellation of Gluon Condensate Mass Singulartities in FESR's for the Scalar, Vector and Axial-Vector Correlation Functions

5.1 Real and Imaginary Parts of Gluon-Loop Integrals

The axial-vector correlation function is defined as

$$i\int d^4x\, e^{ip\cdot x}\langle 0|Tj_{\mu 5}(x)j_{\nu 5}(0)|0\rangle = [g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}]\,\Pi_T(p^2) + [\frac{p_\mu p_\nu}{p^2}]\,\Pi_L(p^2) \qquad (5.1)$$

with the axial-vector current being $j_{\mu 5}(x) = \bar{u}(x)\gamma_\mu\gamma_5 d(x)$. The purely perturbative contribution to the longitudinal component Π_L of the axial-vector correlation function can be obtained from ref. [38]

$$\begin{aligned}[\Pi_L(p^2)]_{pert} &\equiv C_{pert}(p^2)\\ &= \frac{3m^2}{2\pi^2}\left[-\frac{2}{n-4} - \gamma_E + ln\left(\frac{4\pi\mu^2}{m^2}\right) - I(p^2)\right]\end{aligned} \qquad (5.2)$$

where

$$I(p^2) \equiv \int_0^1 dx\, ln\left[1 - \frac{p^2}{m^2}x(1-x) - i\varepsilon\right],\quad \varepsilon > 0 \qquad (5.3)$$

For $p^2 < 0$, we follow Bagan et al [38] in defining the following function characterizing gluon loop contributions to correlation functions

$$X(v) \equiv \frac{1}{v}\ln\left(\frac{v+1}{v-1}\right) = \frac{I(p^2) - 2}{v^2} \qquad (5.4)$$

with

$$v = \sqrt{1 - \frac{4m^2}{p^2}} \tag{5.5}$$

If $p^2 < 4m^2$, the argument of the logarithm in (5.3) is positive definite, and the $i\varepsilon$ factor is irrelevant to the evaluation of the integral. It is straightforward to find that C_{pert} is real provided $p^2 < 4m^2$. Direct evaluation of the integral $I(p^2)$ yields the following results:

$$I(p^2) = -2 + \sqrt{1 - \frac{4m^2}{p^2}}\, ln\left(\frac{1 + \sqrt{1 - \frac{4m^2}{p^2}}}{\sqrt{1 - \frac{4m^2}{p^2}} - 1}\right), \quad p^2 < 0 \tag{5.6}$$

$$I(p^2) = -2 + 2\sqrt{\frac{4m^2}{p^2} - 1}\; tan^{-1}\left[\left(\frac{4m^2}{p^2} - 1\right)^{-1/2}\right], \quad 0 < p^2 < 4m^2 \tag{5.7}$$

One easily finds from either expression that $\lim_{p^2 \to 0} I(p^2) = 0$, and from the latter expression that $\lim_{p^2 \to (4m^2)^-} I(p^2) = -2$. The results (5.2) and (5.6) above are consistent with C_{pert} as calculated in ref. [38]. The relationship (5.4) between $X(v)$ and $I(p^2)$, the latter quantity defined via the integral (5.3), can be utilized to determine the real and imaginary part of $X(v)$ when $p^2 > 4m^2$, as shown below. We will need this information to determine gluon-condensate contributions to finite-energy sum rules.

If $p^2 > 4m^2$, the argument of the logarithm in the integrand of (5.3) can be factorized as follows:

$$I(p^2) = \int_0^1 dx\, ln\left[\frac{p^2}{m^2}(x - \tau_+ - i\varepsilon'')(x - \tau_- - i\varepsilon'')\right] \tag{5.8}$$

where

$$\tau_\pm \equiv \frac{1 \pm \sqrt{1 - \frac{4m^2}{p^2}}}{2} \tag{5.9}$$

and with

$$\varepsilon'' = \frac{m^2}{p^2}\frac{\varepsilon}{\tau_+ - \tau_-} > 0 \tag{5.10}$$

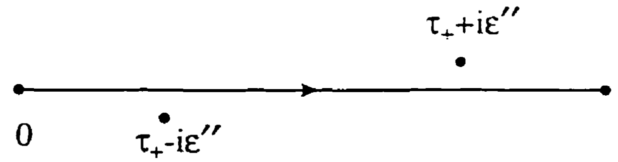

Figure 5.1: Integration contour along the real s-axis with the location of the $\tau^{\pm}$ singularities in the complex s-plane

Performing some integration by parts, one finds that

$$\begin{aligned} I(p^2) &= ln\left(\frac{p^2}{m^2}\right) + \int_0^1 dx\, ln(x - \tau_- - i\varepsilon'') - \int_0^1 dx\, ln(x - \tau_- + i\varepsilon'') \\ &= ln\left(\frac{p^2}{m^2}\right) + ln\left[(1-\tau_+)(1-\tau_-)\right] \\ &\quad -2 - (\tau_+ + i\varepsilon'')\int_0^1 \frac{dx}{x - \tau_+ - i\varepsilon''} \\ &\quad -(\tau_- - i\varepsilon'')\int_0^1 \frac{dx}{x - \tau_- + i\varepsilon''} \end{aligned} \quad (5.11)$$

Note that if $p^2 > 4m^2$, then $0 < \tau_- < \tau_+ < 1$, and , hence, that $\varepsilon'' > 0$. Consequently, the pole in (5.11) at $(\tau_+ + i\epsilon'')$ is above the real x axis, and the pole at $(\tau_- - i\varepsilon'')$ is below the real x axis, permitting the equivalent contours of Fig. 5.2 obtained from Fig. 5.1 to run below τ_+ and above τ_-. Using the contours of Fig.5.2, with C_+ and C_- assumed to be semi-circles of radius δ about $x = \tau_+$ and τ_-, respectively, one finds that $[C_\pm : z = \tau_\pm + \delta\exp^{i\theta_\pm}$; range of $\theta_+ : \pi \longrightarrow 2\pi$; range of $\theta_- : \pi \longrightarrow 0]$

$$\int_0^1 \frac{dx}{x - \tau_+ - i\varepsilon''} = \lim_{\delta\to 0}\left[\int_0^{\tau_+ - \delta}\frac{dx}{x - \tau_+} + \int_{\tau_+ + \delta}^1 \frac{dx}{x - \tau_+} + \int_{C_+}\frac{dz}{z - \tau_+}\right]$$

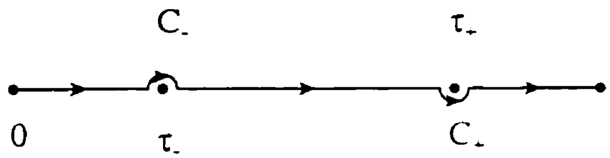

Figure 5.2: Distortion of the integration contour of Fig. 5.1 to accommodate the $\varepsilon'' \to 0$ limit

$$= ln\frac{\tau_-}{\tau_+} + i\pi \tag{5.12}$$

and

$$\int_0^1 \frac{dx}{x - \tau_- + i\varepsilon''} = ln\frac{\tau_+}{\tau_-} - i\pi \tag{5.13}$$

Substituting (5.12) and (5.13) into (5.11), we find for $p^2 > 4m^2$ that

$$I(p^2) = -2 + v\left[ln\left(\frac{1+v}{1-v}\right) - i\pi\right] \tag{5.14}$$

This result (5.14) for $p^2 > 4m^2$ implies that the real and imaginary parts are as follows:

$$ReX(v) = \frac{1}{v}ln\left(\frac{1+v}{1-v}\right) \tag{5.15}$$

$$ImX(v) = -\frac{\pi}{v} \tag{5.16}$$

Equation (5.15) and (5.16) will be utilized throughout the calculation in this chapter.

5.2 Evaluation of the Gluon Condensate Contribution to F_0^L

The "heavy-quark" (h.q.) two-gluon condensate contribution to Π_L, as defined in (5.1), is obtained from Appendix B.3 of ref. [38] as the sum of coefficients $[C_{1G^2}]_{h.q.}$ and $[C_{2G^2}]_{h.q.}$ for the axial-vector current function $\left[s \equiv p^2, v = \sqrt{1 - 4m^2/s}\right]$:

$$\left[\Pi_L(p^2)\right]_{G^2} = (C_{1G^2} + C_{2G^2})_{h.q.} \langle G^2 \rangle \tag{5.17}$$

where

$$[C_{1G^2}]_{h.q.} = \frac{\alpha_s}{48\pi s v^2}[3(1 - v^2)^2 X(v) - 6(1 + v^2)] \tag{5.18}$$

$$[C_{2G^2}]_{h.q.} = \frac{\alpha_s}{96\pi s v^4}[3(1 - v^2)^2(1 + v^2)X(v) - 2(3 - 2v^2 + 3v^4)] \tag{5.19}$$

We define

$$[C_{1G^2} + C_{2G^2}]_{h.q.} \equiv \alpha_s E_{G^2} = \alpha_s E_{pole} + \alpha_s C_x X(v) \tag{5.20}$$

and we find that pure-pole contributions and branch-singularity contributions are respectively given by

$$\alpha_s E_{pole} = -\frac{\alpha_s}{96\pi}\left[\frac{18}{s} + \frac{14}{s - 4m^2} + \frac{24m^2}{(s - 4m^2)^2}\right] \tag{5.21}$$

$$\alpha_s C_x X(v) = \frac{\alpha_s}{2\pi} m^4 \left[\frac{1}{s^3 v^4} - \frac{3}{s^3 v^2}\right] X(v) \tag{5.22}$$

The gluon condensate contribution to the finite energy sum rules F_0^L and F_1^L

$$\begin{aligned} F_0^L &= \frac{1}{2\pi i}\int_{C(s_0)} \Pi_L(s) ds \\ &= \frac{1}{2\pi i}\langle \alpha_s G^2 \rangle \int_{C(s_0)} ds\, C_{G^2}^L(-s) \end{aligned} \tag{5.23}$$

$$\begin{aligned} F_1^L &= \frac{1}{2\pi i}\int_{C(s_0)} \Pi_L(s) s ds \\ &= \frac{1}{2\pi i}\langle \alpha_s G^2 \rangle \int_{C(s_0)} ds\, s C_{G^2}^L(-s) \end{aligned} \tag{5.24}$$

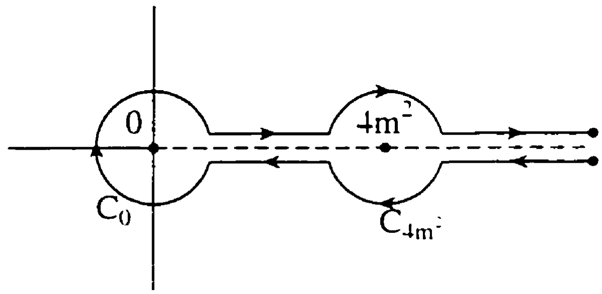

Figure 5.3: Distortion of the $C(s_0)$ contour [Fig. 4.2] for $\langle \alpha_s G^2 \rangle$ contributions to $F_{0,1}$ sum rules

can be obtained from the direct evaluation of the integrals

$$G_0 \equiv \int_{C(s_0)} E_{G^2} ds \tag{5.25}$$

$$G_1 \equiv \int_{C(s_0)} E_{G^2} s ds \tag{5.26}$$

with the contour $C(s_0)$ distorted as in Fig. 5.3 to encompass any pole singularities of E_{G^2} at $s = 0$ or $4m^2$ as well as the branch singularity for $s > 4m^2$. Using Equation (5.20) and (5.25), one finds that

$$\begin{aligned}
G_0 &= \int_{C(s_0)} E_{pole} ds + \int_{C(s_0)} C_x X(v) ds \\
&= \int_{C_0} E_{pole} ds + \int_{C_{4m^2}} E_{pole} ds \\
&\quad + \int_{C>4m^2} E_{pole} ds + \int_{C<4m^2} E_{pole} ds \\
&\quad + \int_{C_0} C_x X(v) ds + \int_{C_{4m^2}} C_x X(v) ds \\
&\quad + \int_{C>4m^2} C_x X(v) ds + \int_{C<4m^2} C_x X(v) ds \\
&= -2i\pi \int_{4m^2+\varepsilon}^{s_0} \frac{C_x}{v} ds
\end{aligned}$$

$$+ \int_{C_0} E_{pole} ds + \int_{C_{4m^2}} E_{pole} ds$$
$$+ \int_{C_0} C_x X(v) ds + \int_{C_{4m^2}} C_x X(v) ds \tag{5.27}$$

where the contours C_0 and C_{4m^2} are clockwise circles of radius ε about $s = 0$ and $s = 4m^2$ respectively. We see from (5.21) that

$$\int_{C_0} E_{pole} ds = \frac{3i}{8} \tag{5.28}$$

$$\int_{C_{4m^2}} E_{pole} ds = \frac{7i}{24} \tag{5.29}$$

The remaining three integrals in (5.27) are evaluated as follows. Using the expression for C_x in (5.22), we find that

$$-2i\pi \int_{4m^2+\varepsilon}^{s_0} \frac{C_x}{v} ds = -im^4[I_1 - 3I_2] \tag{5.30}$$

where

$$I_1 = \int_{4m^2+\varepsilon}^{s_0} \frac{1}{s^3 v^5} ds = \int_{4m^2+\varepsilon}^{s_0} s^{-1/2}(s - 4m^2)^{-5/2} ds \tag{5.31}$$

$$I_2 = \int_{4m^2+\varepsilon}^{s_0} \frac{1}{s^3 v^3} ds = \int_{4m^2+\varepsilon}^{s_0} s^{-3/2}(s - 4m^2)^{-3/2} ds \tag{5.32}$$

Both integrals can be evaluated via the trigonometric substitution $s = 4m^2 sec^2\theta$. One then finds that

$$I_1 = \frac{1}{8m^4} \int_{\theta_L}^{\theta_u} \frac{cos^3\theta}{sin^4\theta} d\theta = \left(-\frac{1}{24m^4} \frac{1}{sin^3\theta} + \frac{1}{8m^4} \frac{1}{sin\theta} \right) \Big|_{\theta_L}^{\theta_u} \tag{5.33}$$

$$I_2 = \frac{1}{8m^4} \int_{\theta_L}^{\theta_u} \frac{cos^3\theta}{sin^2\theta} d\theta = -\frac{1}{8m^4} \left(\frac{1}{sin\theta} + sin\theta \right) \Big|_{\theta_L}^{\theta_u} \tag{5.34}$$

where, using the parameterization of (5.5), we find that

$$\theta_u = sec^{-1} \left\{ \frac{s_0^{1/2}}{2m} \right\};$$
$$sin\theta_u = \sqrt{1 - \frac{4m^2}{s_0}} \equiv v_0 \tag{5.35}$$

and that

$$\theta_L = sec^{-1}\left(\frac{(4m^2+\varepsilon)^{1/2}}{2m}\right);$$
$$sin\theta_L = \left(\frac{\varepsilon}{4m^2+\varepsilon}\right)^{1/2} \tag{5.36}$$

Substituting (5.35) and (5.36) into (5.33) and (5.34), we find from (5.30) that

$$-2i\pi\int_{4m^2+\varepsilon}^{s_0}\frac{C_x}{v}ds = \frac{-i}{8}\left[-\frac{1}{3v_0^3}-\frac{2}{v_0}-3v_0\right]$$
$$-\frac{im^3}{3\varepsilon^{\frac{3}{2}}}-\frac{5im}{8\varepsilon^{\frac{1}{2}}}+\mathcal{O}(\varepsilon). \tag{5.37}$$

The integral around the origin is straightforward to obtain from (5.22) and (5.4). The integrand

$$C_rX(v) = \frac{1}{16\pi}[2+I(s)]\left[\frac{-3/2}{s-4m^2}+\frac{6m^2}{(s-4m^2)^2}+\frac{8m^2}{(s-4m^2)^3}-\frac{3}{2s}\right] \tag{5.38}$$

has a simple pole at $s=0$ because $I(0)=0$ as discussed in previous section:

$$\int_{C_0} C_rX(v)ds = -\frac{3i}{8} \tag{5.39}$$

Note that (5.39) exactly cancels (5.28), indicating that the origin can be excised from the contour of Fig. 5.3.

This cancellation is not peculiar to the channel we are in. We have verified that an identical cancellation occurs in the scalar, vector, and transverse-axial channels between the contributions of explicit $s=0$ poles in E_{G^2} [as in (5.28)] and the integrals of $C_rX(v)$ portions of E_{G^2} around C_0 [as in (5.39)]. Thus the quantum-field-theoretical singularities in G_0 and G_1 all occur for $s \geq 4m^2$ on the real s-axis for all of the above mentioned channels.

The divergence as $\varepsilon \to 0$ in (5.37) is canceled exactly by the integration of $C_rX(v)$, as given in (5.38), over the contour C_{4m^2} around $s=4m^2$, a cancellation which also occurs in the other three channels mentioned above. This cancellation is most easily seen by continuing the expression (5.7) to complex values of s in the vicinity of $s = 4m^2$:

$$\begin{aligned} I(s) + 2 &= 2\left[\frac{(4m^2 - s)}{s}\right]^{1/2} tan^{-1}\left(\frac{s}{4m^2 - s}\right)^{1/2} \\ &= \pi\left(\frac{4m^2 - s}{s}\right)^{1/2} - 2\left(\frac{4m^2 - s}{s}\right) - \frac{2}{3}\left(\frac{4m^2 - s}{s}\right)^2 + \cdots \end{aligned} \tag{5.40}$$

On the contour C_{4m^2}, $s = 4m^2 + \varepsilon e^{i\theta}$ with a clockwise rotation of θ from 2π to 0. When $s > 4m^2$, the correct(negative) sign of the imaginary part $[2iImI(s) \equiv I(s + i|\delta|) - I(s - i|\delta|)]$ is obtained by requiring that

$$(4m^2 - s)^{1/2} = -i\varepsilon^{1/2}e^{i\theta/2} \tag{5.41}$$

as

$$2iImI(s) = \lim_{\theta \to 0}\left[\pi\left(\frac{4m^2 - s}{s}\right)^{1/2}\right] - \lim_{\theta \to 2\pi}\left[\pi\left(\frac{4m^2 - s}{s}\right)^{1/2}\right] \tag{5.42}$$

with $s = s(\theta) = 4m^2 + |\varepsilon|e^{i\theta}$. Upon substitution of (5.40) into (5.38) one finds that

$$\begin{aligned} \int_{C_{4m^2}} C_x X(v)ds = -\frac{7i}{24} + \frac{1}{16}\Bigg[\frac{3}{2}\int_{C_{4m^2}} & s^{-1/2}(4m^2 - s)^{-1/2}ds \\ +6m^2\int_{C_{4m^2}} & s^{-1/2}(4m^2 - s)^{-3/2}ds \\ -8m^2\int_{C_{4m^2}} & s^{-1/2}(4m^2 - s)^{-5/2}ds \\ +\frac{3}{2}\int_{C_{4m^2}} & s^{-3/2}(4m^2 - s)^{1/2}ds\Bigg] \end{aligned} \tag{5.43}$$

The factor $-7i/24$ is just $-2\pi i$ times the aggregate residue at $s = 4m^2$ obtained from multiplication of (5.40)'s integer power of $(4m^2 - s)$ into (5.38). This pole contribution explicitly cancels the pole contribution (5.29). The remaining integrals in (5.43) result from multiplying the leading $\pi[(4m^2 - s)/s]^{1/2}$ term of (5.40) into

(5.38). These integrals are easily evaluated around the clockwise contour C_{4m^2} via (5.41):

$$\int_{C_{4m^2}} s^{-1/2}(4m^2 - s)^{-1/2} ds = \mathcal{O}(\varepsilon^{1/2}) \tag{5.44}$$

$$\int_{C_{4m^2}} s^{-1/2}(4m^2 - s)^{-3/2} ds = \frac{2i}{m\varepsilon^{1/2}} + \mathcal{O}(\varepsilon^{1/2}) \tag{5.45}$$

$$\int_{C_{4m^2}} s^{-1/2}(4m^2 - s)^{-5/2} ds = -\frac{2i}{3m\varepsilon^{3/2}} + \frac{i}{4m^2\varepsilon^{1/2}} + \mathcal{O}(\varepsilon^{1/2}) \tag{5.46}$$

$$\int_{C_{4m^2}} s^{-1/2}(4m^2 - s)^{-3/2} ds = \mathcal{O}(\varepsilon^{3/2}) \tag{5.47}$$

Substituting (5.44-5.47) into (5.43) we find that

$$\int_{C_{4m^2}} C_x X(v) ds = -\frac{7i}{24} + \frac{5im}{8\varepsilon^{1/2}} - \frac{im^2}{3\varepsilon^{3/2}} + \mathcal{O}(\varepsilon^{1/2}) \tag{5.48}$$

explicitly canceling the divergences in (5.37). Since all the $s = 0$ and $s = 4m^2$ pole terms contributing to G_0 are equal to the upper-bound contribution of the first integral on the right-hand side of (5.27), we find that

$$G_0 = -2i\pi \int^{s_0} \frac{C_x}{v} ds = \frac{i}{8}\left[\frac{1}{3v_0^3} - \frac{2}{v_0} - 3v_0\right] \tag{5.49}$$

$$v_0 \equiv \sqrt{1 - \frac{4m^2}{s_0}} \tag{5.50}$$

To obtain the full contribution of $\langle \alpha_s G^2 \rangle$ to the F_0 sum rule, we substitute (5.50) into (5.23) and use the following condensate mixing equation [38]

$$C_{G^2}(Q^2) = E_{G^2}(Q^2) + \frac{1}{12\pi} C_{\bar{q}q}(Q^2) - \frac{m_q}{2\pi} \ln\left(\frac{m_q^2}{\mu^2}\right) C_M(Q^2) \tag{5.51}$$

where the quark and quark-gluon condensate Wilson coefficients have the following expressions:

$$C_{\bar{q}q} = \frac{4}{3m^2}\frac{(1 - v)(1 + 2v)}{1 + v} \tag{5.52}$$

$$C_M = -\frac{1}{2m^3}\frac{(1 - v)^3}{v} \tag{5.53}$$

The gluon condensate contribution to $F_0^L(S_0)$ can then be found:

$$\left[F_0^L(s_0)\right]_{\langle\alpha_s G^2\rangle} = \langle\alpha_s G^2\rangle \left\{ \frac{1}{16\pi}\left[\frac{1}{3v_0^3} + \frac{2}{v_0} - 3v_0\right] - \frac{1}{3\pi} \right\}$$
$$= \langle\alpha_s G^2\rangle \left[\frac{m^4}{\pi s_0} + \frac{14m^6}{3\pi s_0^3} + \cdots\right] \tag{5.54}$$

Since $\langle\alpha_s G^2\rangle$ is chiral invariant, its contribution to F_0 in the longitudinal axial-vector channel vanishes in the $m = 0$ limit of Lagrangian chiral symmetry, as expected.

5.3 Evaluation of the Gluon Condensate Contribution to F_1^L

Consider first the integral G_1 (5.26), which can be evaluated via the following integrals arising from the distortion of C_{s_0} indicated in Fig. 5.3:

$$\begin{aligned} G_1 \;=\; & -2i\pi \int_{4m^2+\varepsilon}^{s_0} \frac{C_x}{v} s\,ds - \int_{C_0} E_{pole}\, s\,ds \\ & + \int_{C_{4m^2}} E_{pole}\, s\,ds + \int_{C_0} C_x X(v)\, s\,ds \\ & + \int_{C_{4m^2}} C_x X(v)\, s\,ds \end{aligned} \tag{5.55}$$

One sees from (5.21) that

$$\int_{C_0} E_{pole}\, s\,ds = 0 \tag{5.56}$$

$$\int_{C_{4m^2}} E_{pole}\, s\,ds = -\frac{5im^2}{3} \tag{5.57}$$

Using the expression for C_x in (5.22), we find that

$$-2i\pi \int_{4m^2+\varepsilon}^{s_0} \frac{C_x}{v} s\,ds = -im^4[I_3 + 3I_4] \tag{5.58}$$

where the integrals I_3 and I_4 are evaluated using (5.35) and (5.36), as in the previous section:

$$\begin{aligned} I_3 \;\equiv\; & \int_{4m^2+\varepsilon}^{s_0} \frac{1}{s^2 v^5} ds = -\frac{1}{6m^2} \frac{1}{\sin^3\theta}\Big|_{\theta_L}^{\theta_u} \\ =\; & -\frac{1}{6m^2 v_0^3} + \frac{4m}{3\varepsilon^{1/2}} + \frac{1}{2m\varepsilon^{1/2}} + \mathcal{O}(\varepsilon^{1/2}) \end{aligned} \tag{5.59}$$

$$\begin{aligned} I_4 &\equiv \int_{4m^2+\varepsilon}^{s_0} \frac{1}{s^2 v^3} ds = \frac{1}{2m^2}\left[-\frac{1}{\sin\theta}\right]_{\theta_\varepsilon}^{\theta_0} \\ &= -\frac{1}{6m^2 v_0^3} + \frac{1}{2m^2 v_0} + \frac{1}{m\varepsilon^{1/2}} - \mathcal{O}(\varepsilon^{1/2}) \end{aligned} \tag{5.60}$$

Substituting (5.59) and (5.60) into (5.58) we find that

$$-2i\pi \int_{4m^2+\varepsilon}^{s_0} \frac{C_x}{v} s\, ds = im^2\left[\frac{1}{6v_0^3} + \frac{3}{2v_0}\right] - \frac{4im^5}{3\varepsilon^{3/2}} - \frac{7im^3}{2\varepsilon^{1/2}} \tag{5.61}$$

Using (5.38), we find that $C_x X(v)s$ has no poles at $s = 0$ [note that $2 - I(0) = 2$], in which case

$$\int_{C_0} C_x X(v)\, s\, ds = 0 \tag{5.62}$$

Once again, we note that the origin can be excised entirely from the contour of Fig. 5.3. We have verified explicitly that integrals (5.56) and (5.62) are zero in the scalar, vector, and transverse axial channels as well.

As in the previous section, the divergence in (5.61) as $\varepsilon \to 0$ is exactly canceled by integration of $C_x X(v)s$ around the contour C_{4m^2}. From (5.38) we find that

$$C_x X(v)\, s = \frac{2}{\pi}[2 + I(s)]\left[\frac{m^4}{(s-4m^2)^2} - \frac{m^6}{(s-4m^2)^3}\right] \tag{5.63}$$

If we substitute (5.40) into (5.63) and integrate around C_{4m^2}, we easily separate a pure-pole contribution from an ε-dependent contribution involving half-integral powers of $(4m^2 - s)$:

$$\begin{aligned} \int_{C_{4m^2}} C_x X(v)\, s\, ds &= -\frac{5im^2}{3} + 2m^4 \int_{C_{4m^2}} s^{-1/2}(4m^2 - s)^{-3/2} ds \\ &\quad -2m^6 \int_{C_{4m^2}} s^{-1/2}(4m^2 - s)^{-5/2} ds \\ &= -\frac{5im^2}{3} + \frac{7im^3}{2\varepsilon^{1/2}} + \frac{4im^5}{3\varepsilon^{3/2}} + O(\varepsilon^{1/2}) \end{aligned} \tag{5.64}$$

The final line of (5.64) is obtained through use of (5.45) and (5.46). Not only are the ε-dependent terms in (5.61) canceled by the final line of (5.64), but the pure-pole contribution (5.57) also cancels against the pole term in (5.64). Thus we find that G_1 is equal to the upper-bound contribution of the first integral on the right-hand side of (5.26):

$$G_1 = -2i\pi \int^{s_0} \frac{C_x}{v} s\, ds = im^2 \left[\frac{1}{6v_0^3} - \frac{3}{2v_0}\right] \tag{5.65}$$

To obtain the full contribution of $\langle \alpha_s G^2 \rangle$ to the F_1 sum rule, we again substitute Equation (5.65) into (5.24) and utilize the identity (5.51). F_1^L is then found to be

$$\begin{aligned} \left[F_1^L(s_0)\right]_{\langle\alpha_s G^2\rangle} &= \frac{m^2}{2\pi}\langle \alpha_s G^2\rangle \left\{\left[\frac{1}{6v_0^3} + \frac{3}{2v_0}\right] - \frac{2}{3}\right\} \\ &= \frac{m^2}{2\pi}\langle \alpha_s G^2\rangle \left\{1 + \frac{4m^2}{s_0} - \frac{14m^4}{s_0^2} + \frac{160m^6}{3s_0^3} - \cdots\right\} \end{aligned} \tag{5.66}$$

5.4 Gluon Condensate Contribution to $F_{0,1}$ in Scalar, Vector and the Transverse Component of the Axial-Vector Channels

Utilizing the notation and conventions of Section 2 and Section 3, we obtain the following results from scalar, vector and the transverse component of the axial-vector correlation functions.

5.4.1 Scalar Channel

From Appendix B.1 of ref. [38], we quote the condensate coefficients of the scalar current as follows:

$$C_{\bar{q}q} = -\frac{1}{m^2}\frac{(1-v)(1+2v)}{1-v} \tag{5.67}$$

$$C_M = \frac{1}{2m^3}\frac{(1-v)^2}{1+v} \tag{5.68}$$

$$[C_{G^2}]_{h.q.} \equiv \alpha_s E_{G^2} = \alpha_s \left(E_{pole} + C_x X(v)\right) \tag{5.69}$$

where E_{pole} and C_x have the following expressions:

$$E_{pole} = \frac{(3 - v^2)}{16\pi s v^2} \tag{5.70}$$

$$C_x = -\frac{(1 - v^2)(3 + v^2)}{32\pi s v^2} \tag{5.71}$$

Referring to the contours of Fig. 5.3. we find that

$$\int_{C_0} E_{pole}\, ds = \frac{i}{8} \tag{5.72}$$

$$\int_{C_{4m^2}} E_{pole}\, ds = -\frac{3i}{8} \tag{5.73}$$

$$-2i\pi \int_{4m^2+\epsilon}^{s_0} \frac{C_x}{v} ds = \frac{i}{8}\left[-\frac{3}{v_0} - v_0 + \frac{6m}{\sqrt{\epsilon}}\right] \tag{5.74}$$

$$\int_{C_0} C_x X(v)\, ds = -\frac{i}{8} \tag{5.75}$$

$$\int_{C_{4m^2}} C_x X(v)\, ds = \frac{3i}{8}\left(1 - \frac{2m}{\sqrt{\epsilon}}\right) \tag{5.76}$$

Summing (5.71-5.76) we obtain

$$G_0 = \int_{C(s_0)} E_{G^2} ds = \frac{i}{8}\left[-\frac{3}{v_0} - v_0\right] \tag{5.77}$$

Integrating $C_{\bar{q}q}$ (5.68) and C_M (5.69) over the contour $C(s_0)$, we find that

$$\int_{C(s_0)} C_{\bar{q}q}\, ds = 6i\pi \tag{5.78}$$

$$\int_{C(s_0)} C_M\, ds = 0 \tag{5.79}$$

which implies via (5.51) and (5.24) that

$$[F_0(s_0)]_{\langle \alpha_s G^2 \rangle} = \frac{1}{16\pi}\left[-\frac{3}{v_0} + v_0 + 4\right] \langle \alpha_s G^2 \rangle \tag{5.80}$$

Unlike the case of F_0, the finite energy sum rule F_1 requires the use of (5.51) to eliminate a logarithmic mass singularity in G_1 obtained by summing the following five integrals:

$$\int_{C_0} E_{pole}\, s\, ds = 0 \tag{5.81}$$

$$\int_{C_{4m^2}} E_{pole}\, s\, ds = -\frac{3im^2}{2} \tag{5.82}$$

$$-2i\pi \int_{4m^2+\epsilon}^{s_0} \frac{C_x}{v}\, s\, ds = \frac{im^2}{2}\left[-\frac{3}{v_0} + 2ln(1 - v_0^2) - \frac{6m}{\sqrt{\epsilon}}\right] \tag{5.83}$$

$$\int_{C_0} C_x X(v)\, s\, ds = 0 \tag{5.84}$$

$$\int_{C_{4m^2}} C_x X(v)\, s\, ds = \frac{3im^2}{2} - \frac{3m^3}{\sqrt{\epsilon}} \tag{5.85}$$

We then find that

$$G_1 = \int_{C_{(s_0)}} E_{G^2}\, s\, ds = \frac{im^2}{2}\left[-\frac{3}{v_0} - 2ln\left(\frac{4m^2}{s_0}\right)\right] \tag{5.86}$$

which is not analytic in m at $m = 0$. However the results

$$\int_{C_{(s_0)}} C_{\bar{q}q}\, s\, ds = 4im^2\pi \tag{5.87}$$

$$\int_{C_{(s_0)}} C_M\, s\, ds = -2im\pi \tag{5.88}$$

used in conjunction with (5.51) and (5.24) eliminate the quark-mass from the logarithm:

$$[F_1(s_0)]_{(\alpha_s G^2)} = \frac{m^2}{2\pi}\left[-\frac{3}{2v_0} + \frac{1}{3} - ln\left(\frac{s_0}{4\mu^2}\right)\right] \alpha_s G^2 \tag{5.89}$$

5.4.2 Vector Channel

From Equation (II.19) of ref. [38], we find that

$$E_{pole} = -\frac{(3 - 2v^2 + 3v^4)}{48\pi s v^4} \tag{5.90}$$

$$C_x = \frac{(1 - v^2)^2(1 - v^2)}{32\pi s v^4} \tag{5.91}$$

We also quote the quark and quark-gluon condensate coefficients from Equation (II.13) and Equation (II.17) in ref. [38].

$$C_{\bar{q}q} = \frac{2}{3m^2}\frac{(1 - v)(2 + v)}{(1 + v)} \tag{5.92}$$

$$C_M = -\frac{1}{6m^3}\frac{(1 - v)^3}{v(1 + v)} \tag{5.93}$$

We then find F_0 (5.23) from G_0 (5.25), using (5.51) and the following equations:

$$\int_{C_0} E_{pole}\, ds = \frac{i}{8} \tag{5.94}$$

$$\int_{C_{4m^2}} E_{pole}\, ds = \frac{i}{24} \tag{5.95}$$

$$-2i\pi \int_{4m^2+\varepsilon}^{s_0} \frac{C_I}{v} ds = \frac{i}{8}\left[v_0 + \frac{1}{3v_0^3} - \frac{8m^3}{3\varepsilon^{3/2}} - \frac{m}{\varepsilon^{1/2}}\right] \tag{5.96}$$

$$\int_{C_0} C_I\, X(v)\, ds = -\frac{i}{8} \tag{5.97}$$

$$\int_{C_{4m^2}} C_I\, X(v)\, ds = -\frac{i}{24} + \frac{im^3}{3\varepsilon^{3/2}} + \frac{im}{8\varepsilon^{1/2}} \tag{5.98}$$

$$\int_{C_{(s_0)}} C_{\bar{q}q}\, ds = -4i\pi \tag{5.99}$$

$$\int_{C_{(s_0)}} C_M\, ds = 0 \tag{5.100}$$

$$G_0 = \int_{C_{(s_0)}} E_{G^2}\, ds = \frac{i}{8}\left[v_0 - \frac{1}{3v_0^3}\right] \tag{5.101}$$

$$[F_0(s_0)]_{\langle\alpha_s G^2\rangle} = \frac{1}{16\pi}\left[v_0 + \frac{1}{3v_0^3} - \frac{8}{3}\right] \langle\alpha_s G^2\rangle \tag{5.102}$$

Corresponding results for F_1 are listed below:

$$\int_{C_0} E_{pole}\, s\, ds = 0 \tag{5.103}$$

$$\int_{C_{4m^2}} E_{pole}\, s\, ds = \frac{2im^2}{3} \tag{5.104}$$

$$-2i\pi \int_{4m^2+\varepsilon}^{s_0} \frac{C_I}{v}\, s\, ds = \frac{im^2}{2}\left[\frac{1}{v_0} + \frac{1}{3v_0^3} - \frac{8m^3}{3\varepsilon^{3/2}} - \frac{3m}{\varepsilon^{1/2}}\right] \tag{5.105}$$

$$\int_{C_0} C_I\, X(v)\, s\, ds = 0 \tag{5.106}$$

$$\int_{C_{4m^2}} C_I\, X(v)\, s\, ds = -\frac{2im^2}{3} + \frac{4im^5}{3\varepsilon^{3/2}} + \frac{3im^3}{2\varepsilon^{1/2}} \tag{5.107}$$

$$\int_{C(s_0)} C_{\bar{q}q}\, s\, ds = -\frac{16}{3} i\pi m^2 \tag{5.108}$$

$$\int_{C_{(s_0)}} C_M\, s\, ds = 0 \tag{5.109}$$

$$G_1 = \int_{C_{(s_0)}} E_{G^2}\, s\, ds = \frac{im^2}{2}\left[\frac{1}{v_0} - \frac{1}{3v_0^3}\right] \tag{5.110}$$

$$[F_1(s_0)]_{\langle\alpha_s G^2\rangle} = \frac{m^2}{4\pi}\left[+\frac{1}{v_0} + \frac{1}{3v_0^3} - \frac{8}{9}\right] \langle\alpha_s G^2\rangle \tag{5.111}$$

5.4.3 Transverse Axial Channel

From Appendix B.3 of ref. [38], we have

$$[C_{1G^2}]_{h.q.} \equiv \alpha_s E_{G^2} = \alpha_s \left(E_{pole} + C_x X(v)\right) \tag{5.112}$$

where

$$E_{pole} = -\frac{(1-v^2)}{8\pi s v^2} \tag{5.113}$$

$$C_x = \frac{(1-v^2)^2}{16\pi s v^2} \tag{5.114}$$

We also quote the quark and quark-gluon condensate coefficients for the transverse axial-current:

$$C_{\bar{q}q} = \frac{2}{3m^2}\frac{(1-v)(1-2v)}{(1+v)} \tag{5.115}$$

$$C_M = -\frac{1}{6m^3}\frac{(1-v)^3}{(1+v)} \tag{5.116}$$

We then find that

$$\int_{C_0} E_{pole}\, ds = \frac{i}{4} \tag{5.117}$$

$$\int_{C_{4m^2}} E_{pole}\, ds = \frac{i}{4} \tag{5.118}$$

$$-2i\pi \int_{4m^2+\varepsilon}^{s_0} \frac{C_x}{v}\, ds = \frac{i}{4}\left[\frac{1}{v_0} - v_0 - \frac{2m}{\sqrt{\varepsilon}}\right] \tag{5.119}$$

$$\int_{C_0} C_x X(v)\, ds = -\frac{i}{4} \tag{5.120}$$

$$\int_{C_{4m^2}} C_x X(v)\, ds = -\frac{i}{4} + \frac{im}{2\sqrt{\varepsilon}} \tag{5.121}$$

As before, the contour-radius singularity as $\varepsilon \to 0$ cancels between (5.119) and (5.121):

$$G_0 = \int_{C_{(s_0)}} E_{G^2}\, ds = \frac{i}{4}\left[\frac{1}{v_0} - v_0\right] \tag{5.122}$$

Using the expressions for $C_{\bar{q}q}$ (5.116) and C_M (5.116) one finds the following results over integration contour $C(s_0)$:

$$\int_{C(s_0)} C_{\bar{q}q}\, ds = -4\pi i \tag{5.123}$$

$$\int_{C(s_0)} C_M\, ds = 0 \tag{5.124}$$

We find via (5.51) and (5.23) that

$$[F_0(s_0)]_{\langle\alpha_s G^2\rangle} = \frac{1}{8\pi}\left[\frac{1}{v_0} - v_0 - \frac{4}{3}\right]\langle\alpha_s G^2\rangle \tag{5.125}$$

Corresponding results for F_1 are listed below:

$$\int_{C_0} E_{pole}\, s\, ds = 0 \tag{5.126}$$

$$\int_{C_{4m^2}} E_{pole}\, s\, ds = im^2 \tag{5.127}$$

$$-2i\pi\int_{4m^2-\varepsilon}^{s_0} \frac{C_x}{v}\, s\, ds = im^2\left[\frac{1}{v_0} - \frac{2m}{\sqrt{\varepsilon}}\right] \tag{5.128}$$

$$\int_{C_0} C_x X(v)\, s\, ds = 0 \tag{5.129}$$

$$\int_{C_{4m^2}} C_x X(v)\, s\, ds = -im^2 + \frac{2im^3}{\sqrt{\varepsilon}} \tag{5.130}$$

$$\int_{C(s_0)} C_{\bar{q}q}\, s\, ds = -\frac{8im^2\pi}{3} \tag{5.131}$$

$$\int_{C(s_0)} C_M\, s\, ds = 0 \tag{5.132}$$

$$G_1 = \int_{C(s_0)} E_{G^2}\, s\, ds = \frac{im^2}{v_0} \tag{5.133}$$

$$[F_1(s_0)]_{\langle\alpha_s G^2\rangle} = \left(\frac{m^2}{2\pi v_0} - \frac{m^2}{9\pi}\right)\langle\alpha_s G^2\rangle \tag{5.134}$$

Chapter 6

Search for Anomalous $\tau\bar{\tau}\gamma$ Couplings

6.1 Motivation

The Standard Model (SM) continues to provide an excellent description of almost all aspects of existing experimental data [3], especially with the discovery of the top quark mass near 175 GeV by the Collider Detector at Fermilab(CDF) Collaboration at the Fermilab Tevatron [40]. On the other hand, with the recent announcement of discovery of evidence for neutrino mass by the Super-Kamiokande Collaboration on June 5th, 1998 [41], we know that physics beyond the Standard Model must occur. Among the possible ways new physics may manifest itself, one that has been getting increasing attention is the anomalous coupling of heavy flavor fermions to the conventional Standard Model gauge bosons, i.e., Z, W, γ and g. In the case of a neutral on-shell gauge boson, these anomalous couplings take the form of either electric or magnetic dipole form factors: electric dipole moments are inherently CP violating. These two types of new couplings represent the lowest dimensional nonrenormalizable operators which can be added to the usual Standard Model Lagrangian. Since the heaviest fermions are all the fermions in the third generation, t, b, τ, as well as charmed fermion c, the presumption has been that their couplings would be the most sensitive to the existence of this new high mass scale physics. Because many of these particles have been around for quite some time and much data have been accumulated about their properties, it seems quite natural to ask if the better-known heavy flavor fermions possess anomalous couplings or, at the very least, to ask what the limits are on such couplings from existing data. If such couplings were ever to be found we would certainly need to investigate and understand how they arose. Our approach to analyzing the effects of such hypothetical couplings is purely phenomenological. We do not seek here to address the possible origin of these anomalous fermion couplings should they exist.

In fact, the τ leptons have received some attention in this aspect [42, 43, 44, 45, 46, 47, 48, 50, 51], especially in regard to a possible CP violation associated with an electric dipole moment interaction with the Z. In our study, we concentrate on the possible anomalous couplings of the τ lepton to the photon.

6.2 Four-Body Radiative Decay Process $f_1 \to f_2 + f_3 + \bar{f}_4 + \gamma$

In this section we derive the squared matrix element in a four-body decay process $f_1 \to f_2 + f_3 + \bar{f}_4 + \gamma$ where f_1, f_2, f_3 and f_4 are fermions. We then apply this result to the radiative leptonic τ decay $\tau \to \mu\bar{\nu}_\mu\nu_\tau\gamma$.

The Feynman diagrams in momentum space for the four-body decay process $f_1 \to f_2 + f_3 + \bar{f}_4 + \gamma$ are shown in Fig. (6.1-6.4).

The Lorentz invariant matrix elements corresponding to Figs. 6.1(a), 6.2(b), 6.3(c) and 6.4(d) with the *standard model* couplings can be written as follows by applying

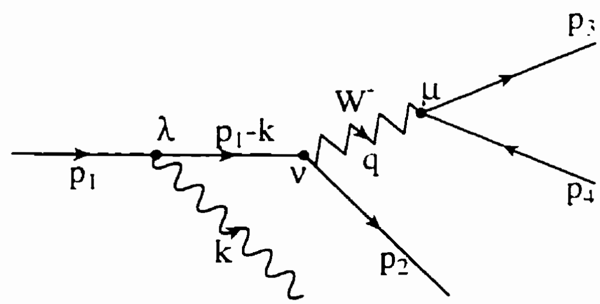

Figure 6.1: Feynman diagram (a) for the four body radiative decay $f_1 \rightarrow f_2 + f_3 + \bar{f}_4 + \gamma$

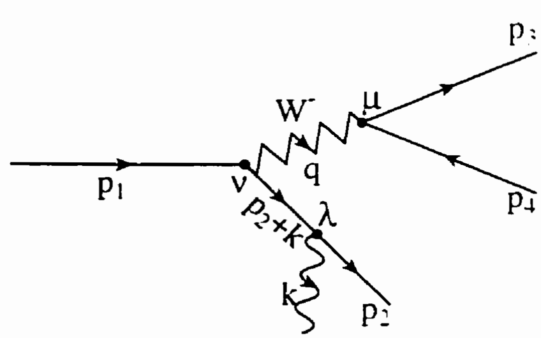

Figure 6.2: Feynman diagram (b) for the four body radiative decay $f_1 \rightarrow f_2 + f_3 + \bar{f}_4 + \gamma$

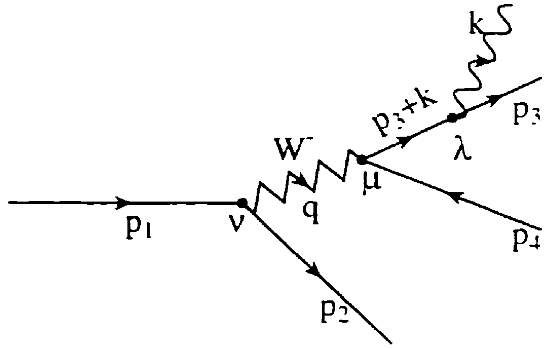

Figure 6.3: Feynman diagram (c) for the four body radiative decay $f_1 \rightarrow f_2 + f_3 + \bar{f}_4 + \gamma$

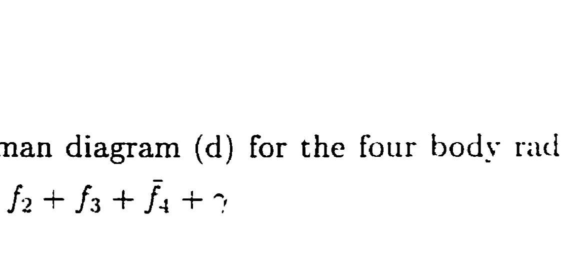

Figure 6.4: Feynman diagram (d) for the four body radiative decay $f_1 \rightarrow f_2 + f_3 + \bar{f}_4 + \gamma$

relevant Feynman Rules in the Appendix F of [52]:

$$\begin{aligned}
\mathcal{M}_a &= \bar{u}(p_3)(-i\frac{g}{\sqrt{2}}V_{34})\gamma_\mu \frac{1-\gamma_5}{2} v(p_4) \frac{(-g^{\mu\nu} + q^\mu q^\nu / M_W^2)}{q^2 - M_W^2} \\
& \quad \bar{u}(p_2)(-i\frac{g}{\sqrt{2}}V_{12})\gamma_\nu \frac{1-\gamma_5}{2} \frac{i}{(\not{p}_1 - \not{k}) - m_1} \varepsilon_\lambda^*(k)(-ieq_1\gamma^\lambda)u(p_1). \qquad (6.1) \\
q &\equiv p_3 + p_4
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_b &= \bar{u}(p_3)(-i\frac{g}{\sqrt{2}}V_{34})\gamma_\mu \frac{1-\gamma_5}{2} v(p_4) \frac{(-g^{\mu\nu} + q^\mu q^\nu / M_W^2)}{q^2 - M_W^2} \\
& \quad \bar{u}(p_2)(-ieq_2\gamma^\lambda)\varepsilon_\lambda^*(k) \frac{i}{(\not{p}_2 - \not{k}) - m_2} (-i\frac{g}{\sqrt{2}}V_{12})\gamma_\nu \frac{1-\gamma_5}{2} u(p_1). \qquad (6.2) \\
q &\equiv p_3 + p_4
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_c &= \bar{u}(p_3)(-ieq_3\gamma^\lambda)\varepsilon_\lambda^*(k) \frac{i}{(\not{p}_3 - \not{k}) - m_3} (-i\frac{g}{\sqrt{2}}V_{34})\gamma_\mu \frac{1-\gamma_5}{2} v(p_4) \\
& \quad \frac{(-g^{\mu\nu} + q^\mu q^\nu / M_W^2)}{q^2 - M_W^2} \bar{u}(p_2)(-i\frac{g}{\sqrt{2}}V_{12})\gamma_\nu \frac{1-\gamma_5}{2} u(p_1). \qquad (6.3) \\
q &\equiv p_3 + p_4 + k
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_d &= \bar{u}(p_3)(-i\frac{g}{\sqrt{2}}V_{34})\gamma_\mu \frac{1-\gamma_5}{2} \frac{i}{(-\not{p}_4 - \not{k}) - m_4} \varepsilon_\lambda^*(k)(-ieq_4\gamma^\lambda)v(p_4) \\
& \quad \frac{(-g^{\mu\nu} + q^\mu q^\nu / M_W^2)}{q^2 - M_W^2} \bar{u}(p_2)(-i\frac{g}{\sqrt{2}}V_{12})\gamma_\nu \frac{1-\gamma_5}{2} u(p_1). \qquad (6.4) \\
q &\equiv p_3 + p_4 + k
\end{aligned}$$

where the four-momenta p_1, p_2, p_3, p_4 and k correspond respectively to the decaying f_1 fermion, the f_2 fermion, the f_3 fermion, the $\bar{f}_4$ anti-fermion and the photon γ. The q_i's are the corresponding fermion electric charges in units of e. V_{ij} are the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements.

Since the W boson is heavy ($M_W \approx 80GeV$) compared to the mass of τ lepton, we can assume that $q^2 \ll M_W^2$. The fermion propagators are simplified in Equations (6.1-

6.4) as follows:

$$\frac{1}{(\not{p}_1 - \not{k}) - m_1} = \frac{(\not{p}_1 - \not{k}) + m_1}{-2p_1 \cdot k} \tag{6.5}$$

$$\frac{1}{(\not{p}_2 - \not{k}) - m_2} = \frac{(\not{p}_2 - \not{k}) + m_2}{-2p_2 \cdot k} \tag{6.6}$$

$$\frac{1}{(\not{p}_3 - \not{k}) - m_3} = \frac{(\not{p}_3 - \not{k}) + m_3}{-2p_3 \cdot k} \tag{6.7}$$

$$\frac{1}{(\not{p}_4 - \not{k}) - m_4} = \frac{(\not{p}_4 - \not{k}) + m_4}{-2p_4 \cdot k} \tag{6.8}$$

Substituting Equations (6.6-6.8) into Equations (6.1-6.4), then we can find the Feynman amplitude:

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}_a + \mathcal{M}_b + \mathcal{M}_c + \mathcal{M}_d \\
&= \frac{G_F e}{2\sqrt{2}} V_{34} V_{12} \Bigg[\underbrace{2B \cdot \varepsilon^* \bar{u}(p_3)\gamma_\mu(1-\gamma_5)v(p_4)\bar{u}(p_2)\gamma^\mu(1-\gamma_5)u(p_1)}_{A} \\
&\quad + \underbrace{\frac{q_1}{p_1 \cdot k}\bar{u}(p_3)\gamma_\mu(1-\gamma_5)v(p_4)\bar{u}(p_2)\gamma^\mu(1-\gamma_5)\not{\varepsilon}^*\not{k}u(p_1)}_{B} \\
&\quad + \underbrace{\frac{q_2}{p_2 \cdot k}\bar{u}(p_3)\gamma_\mu(1-\gamma_5)v(p_4)\bar{u}(p_2)\gamma^\mu(1-\gamma_5)\not{k}\not{\varepsilon}^*u(p_1)}_{C} \\
&\quad + \underbrace{\frac{q_3}{p_3 \cdot k}\bar{u}(p_3)\not{k}\not{\varepsilon}^*\gamma_\mu(1-\gamma_5)v(p_4)\bar{u}(p_2)\gamma^\mu(1-\gamma_5)u(p_1)}_{D} \\
&\quad - \underbrace{\frac{q_4}{p_4 \cdot k}\bar{u}(p_3)\gamma_\mu(1-\gamma_5)\not{\varepsilon}^*\not{k}v(p_4)\bar{u}(p_2)\gamma^\mu(1-\gamma_5)u(p_1)}_{E} \Bigg]
\end{aligned} \tag{6.9}$$

where G_F is the Fermi coupling constant:

$$\frac{G_F}{2\sqrt{2}} = \frac{g^2}{8M_W^2} \tag{6.10}$$

The momentum variable B appearing in term A is defined as follows:

$$B^\mu \equiv 2\left(\frac{q_1 p_1^\mu}{p_1 \cdot k} - \frac{q_2 p_2^\mu}{p_2 \cdot k} - \frac{q_3 p_3^\mu}{p_3 \cdot k} - \frac{q_4 p_4^\mu}{p_4 \cdot k}\right) \tag{6.11}$$

We list here the following trace and γ matrix identities which are used to calculate the squared matrix element. Those identities are derived from the basic trace theorems and γ matrix identities in Appendix A of ref. [13]:

$$Tr[R(L)\gamma^{\mu}\gamma^{\nu}] = 2g^{\mu\nu} \tag{6.12}$$

$$Tr[R(L)\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta}] = 2[g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} + (-)i\epsilon^{\mu\nu\alpha\beta}] \tag{6.13}$$

$$\begin{aligned} Tr[R(L)\gamma^{\mu}\gamma^{\alpha}\gamma^{\nu}\gamma^{\beta})Tr[R(L)\gamma_{\mu}\gamma_{\alpha'}\gamma_{\nu}\gamma_{\beta'}] &= Tr[R(L)\gamma^{\alpha}\gamma^{\mu}\gamma^{\beta}\gamma^{\nu}]Tr[R(L)\gamma_{\alpha'}\gamma_{\mu}\gamma_{\beta'}\gamma_{\nu}] \\ &= 16g^{\beta}_{\beta'}g^{\alpha}_{\alpha'} \end{aligned} \tag{6.14}$$

where

$$R = \frac{1+\gamma_5}{2}, \quad L = \frac{1-\gamma_5}{2} \tag{6.15}$$

We define operators a, $\bar{a}$, A and $\bar{A}$ as odd and even combinations of operators $\not{\phi}_1, \not{\phi}_2, \cdots, \not{\phi}_n$:

$$a = \not{\phi}_1\not{\phi}_2\cdots\not{\phi}_{2n-1} \tag{6.16}$$

$$\bar{a} = -\not{\phi}_{2n-1}\not{\phi}_{2n}\cdots a_1 \tag{6.17}$$

$$A = \not{A}_1\not{A}_2\cdots\not{A}_{2n} \tag{6.18}$$

$$\bar{A} = \not{A}_{2n}\not{A}_{2n-1}\cdots\not{A}_1 \tag{6.19}$$

where $n = 1, 2, 3, \cdots$. The trace identities in terms of $a, \bar{a}, A, \bar{A}$ can be derived as follows:

$$Tr[R(L)a\gamma^{\mu}]Tr[R(L)b\gamma_{\mu}] = -2Tr[R(L)a\bar{b}] \tag{6.20}$$

$$\begin{cases} Tr(Ra\gamma^{\mu})Tr(Lb\gamma_{\mu}) = 2Tr(Rab) \\ Tr(La\gamma^{\mu})Tr(Rb\gamma_{\mu}) = 2Tr(La\bar{b}) \end{cases} \tag{6.21}$$

$$\begin{cases} Tr(Ra\gamma^{\mu}b\gamma_{\mu}) = 2Tr(Ra\bar{b}) \\ Tr(RA\gamma^{\mu}B\gamma_{\mu}) = 2Tr(RA)Tr(LB) \end{cases} \tag{6.22}$$

$$\begin{cases} Tr[R(L)a_1\gamma^\mu a_2\gamma^\nu]Tr[R(L)a_3\gamma_\mu a_4\gamma_\nu] = 4Tr[R(L)a_1\bar{a}_3]Tr[R(L)a_2\bar{a}_4] \\ Tr[R(L)A_1\gamma^\mu A_2\gamma^\nu]Tr[R(L)A_3\gamma_\mu A_4\gamma_\nu] = 4Tr[R(L)A_1\bar{A}_3]Tr[R(L)A_2\bar{A}_4] \end{cases} \quad (6.23)$$

$$\begin{cases} Tr[R(L)a_1\gamma^\mu a_2\gamma^\nu]Tr[L(R)a_3\gamma_\mu a_4\gamma_\nu] = 4Tr[R(L)a_1a_4]Tr[L(R)a_3a_2] \\ Tr[R(L)A_1\gamma^\mu A_2\gamma^\nu]Tr[L(R)A_3\gamma_\mu A_4\gamma_\nu] = 4Tr[R(L)A_1A_4]Tr[L(R)A_3A_2] \end{cases} \quad (6.24)$$

$$\begin{cases} Tr[R(L)A_1\gamma^\mu A_2\gamma^\nu]Tr[L(R)a_3\gamma_\mu a_4\gamma_\nu] = 4Tr[R(L)A_1a_4\bar{A}_2a_3] \\ Tr[R(L)A_1\gamma^\mu A_2\gamma^\nu]Tr[L(R)a_3\gamma_\mu a_4\gamma_\nu] = 4Tr[R(L)A_1\bar{a}_3A_2a_4] \end{cases} \quad (6.25)$$

In order to calculate the spin-averaged matrix element squared, we must average over initial τ states for unpolarized τ leptons, and sum over the final muon and photon polarization states. For the neutrinos there is no averaging over initial neutrino helicities, since only left-handed neutrinos participate in the weak interaction (we assume here that $m_\nu = 0$). Similarly, there is no sum over final neutrino helicities. However, for convenience of calculation, we can in fact sum over both helicity states of both neutrinos since the $(1-\gamma_5)$ factors guarantee that right-handed neutrinos do not contribute to the matrix element squared. After we have done all the spin averaging and summations mentioned above, and the spin-averaged matrix element squared can be written as the products of traces:

$$\begin{aligned}
|\bar{\mathcal{M}}|^2 = & -G_F^2 e^2 |V_{12}V_{34}|^2 [B\cdot B\, Tr(\not{p}_1 R\gamma^\nu \not{p}_2\gamma^\mu L)Tr(\not{p}_4 R\gamma_\nu \not{p}_3\gamma_\mu L) \\
& +(\frac{q_1}{p_1\cdot k})^2 Tr(\not{p}_1\not{k}\gamma^\lambda R\gamma^\mu\not{p}_2\gamma^\nu L\gamma_\lambda\not{k})Tr(\not{p}_4 R\gamma_\mu \not{p}_3\gamma_\nu L) \\
& +(\frac{q_2}{p_2\cdot k})^2 Tr(\not{p}_1 R\gamma^\mu\gamma^\lambda\not{k}\not{p}_2\not{k}\gamma_\lambda\gamma^\nu L)Tr(\not{p}_4 R\gamma_\mu \not{p}_3\gamma_\nu L) \\
& +(\frac{q_3}{p_3\cdot k})^2 Tr(\not{p}_4 R\gamma_\mu\gamma^\lambda\not{k}\not{p}_3\not{k}\gamma_\lambda\gamma_\nu L)Tr(\not{p}_1 R\gamma^\mu \not{p}_2\gamma^\nu L) \\
& +(\frac{q_4}{p_4\cdot k})^2 Tr(\not{p}_4\not{k}\gamma^\lambda R\gamma_\mu\not{p}_3\gamma_\nu L\gamma_\lambda\not{k})Tr(\not{p}_1 R\gamma^\mu \not{p}_2\gamma^\nu L) \\
& +\frac{q_1}{p_1\cdot k} Tr(\not{p}_1 R\gamma^\mu\not{p}_2\gamma^\nu L\not{B}\not{k})Tr(\not{p}_4 R\gamma_\mu \not{p}_3\gamma_\nu L) \\
& +\frac{q_1}{p_1\cdot k} Tr(\not{p}_1\not{k}\not{B} R\gamma^\mu\not{p}_2\gamma^\nu L)Tr(\not{p}_4 R\gamma_\mu \not{p}_3\gamma_\nu L) \\
& +\frac{q_2}{p_2\cdot k} Tr(\not{p}_1 R\gamma^\mu\not{p}_2\not{k}\not{B}\gamma^\nu L)Tr(\not{p}_4 R\gamma_\mu \not{p}_3\gamma_\nu L)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{q_2}{p_2 \cdot k} Tr(\not{p}_1 R\gamma^\mu \not{B}\not{k}\not{p}_2\gamma^\nu L)Tr(\not{p}_4 R\gamma_\mu \not{p}_3\gamma_\nu L) \\
&+\frac{q_3}{p_3 \cdot k} Tr(\not{p}_1 R\gamma^\mu \not{p}_2\gamma^\nu L)Tr(\not{p}_4 R\gamma_\mu \not{B}\not{k}\not{p}_3\gamma_\nu L) \\
&+\frac{q_3}{p_3 \cdot k} Tr(\not{p}_1 R\gamma^\mu \not{p}_2\gamma^\nu L)Tr(\not{p}_4 R\gamma_\mu \not{p}_3\not{k}\not{B}\gamma_\nu L) \\
&-\frac{q_4}{p_4 \cdot k} Tr(\not{p}_1 R\gamma^\mu \not{p}_2\gamma^\nu L)Tr(\not{p}_4 R\gamma_\mu \not{p}_3\gamma_\nu L\not{B}\not{k}) \\
&-\frac{q_4}{p_4 \cdot k} Tr(\not{p}_1 R\gamma^\mu \not{p}_2\gamma^\nu L)Tr(\not{p}_4\not{k}\not{B} R\gamma_\mu \not{p}_3\gamma_\nu L) \\
&-\frac{q_1 q_4}{p_1 \cdot k p_4 \cdot k} Tr(\not{p}_1\not{k}\gamma^\lambda R\gamma^\mu \not{p}_2\gamma^\nu L)Tr(\not{p}_4 R\gamma_\mu \not{p}_3\gamma_\nu L\gamma_\lambda\not{k}) \\
&-\frac{q_1 q_4}{p_1 \cdot k p_4 \cdot k} Tr(\not{p}_1 R\gamma^\mu \not{p}_2\gamma^\nu L\gamma_\lambda\not{k})Tr(\not{p}_4 R\not{k}\gamma^\lambda R\gamma_\mu \not{p}_3\gamma_\nu L) \\
&+\frac{q_2 q_3}{p_2 \cdot k p_3 \cdot k} Tr(\not{p}_1 R\gamma^\mu\gamma_\lambda\not{k}\not{p}_2\gamma^\nu L)Tr(\not{p}_4 R\gamma_\mu \not{p}_3\not{k}\gamma^\lambda\gamma_\nu L) \\
&+\frac{q_2 q_3}{p_2 \cdot k p_3 \cdot k} Tr(\not{p}_1 R\gamma^\mu \not{p}_2\not{k}\gamma^\lambda\gamma^\nu L)Tr(\not{p}_4 R\gamma_\mu\gamma_\lambda\not{k}\not{p}_3\gamma_\nu L)\Bigg]
\end{aligned}
\tag{6.26}
$$

Applying Equations (6.12-6.25) to Equation (6.26), the squared matrix element $|\mathcal{M}|^2$ becomes

$$
\begin{aligned}
|\bar{\mathcal{M}}|^2 \;=\; & 32G_F^2 e^2 |V_{12}V_{34}|^2 \Bigg[2p_2 \cdot p_3(\frac{q_1^2}{p_1 \cdot k}p_4 \cdot k + \frac{q_4^2}{p_4 \cdot k}p_1 \cdot k) \\
& +2p_1 \cdot p_4(\frac{q_2^2}{p_2 \cdot k}p_3 \cdot k + \frac{q_3^2}{p_3 \cdot k}p_2 \cdot k) \\
& -4(q_2 q_3 p_1 \cdot p_4 - q_1 q_4 p_2 \cdot p_3) \\
& -p_2 \cdot p_3(\frac{q_1}{p_1 \cdot k} + \frac{q_4}{p_4 \cdot k})(B \cdot p_1 p_4 \cdot k - B \cdot p_4 p_1 \cdot k) \\
& -p_1 \cdot p_4(\frac{q_4}{p_4 \cdot k} + \frac{q_3}{p_3 \cdot k})(B \cdot p_3 p_2 \cdot k - B \cdot p_2 p_3 \cdot k) \\
& -\frac{1}{2}B \cdot B p_{2\cdot 3}\, p_1 \cdot p_4\Bigg]
\end{aligned}
\tag{6.27}
$$

Notice that we have neglected all fermion mass effects since these only contribute very small corrections at high energies. The squared matrix element which we obtained agrees with Equation (3) in [53].

With the charges of neutrinos being zero, we find from Equation (6.27) the squared matrix element for the radiative leptonic τ decay $\tau \to \mu\bar{\nu}_\mu\nu_\tau\gamma$:

$$
\begin{aligned}
|\bar{\mathcal{M}}_\tau|^2 \;=\; & 32G_F^2e^2\left\{2p_2\cdot p_3\frac{p_4\cdot k}{p_1\cdot k} - 2p_1\cdot p_4\frac{p_3\cdot k}{p_2\cdot k}\right. \\
& +\frac{p_2\cdot p_3}{p_1\cdot k}(B\cdot p_4 p_1\cdot k - C\cdot p_1 p_4\cdot k) \\
& -\frac{p_1\cdot p_4}{p_2\cdot k}(B\cdot p_3 p_2\cdot k - C\cdot p_2 p_3\cdot k) \\
& \left.-\frac{1}{2}C\cdot C p_2\cdot p_3 p_1\cdot p_4\right\} \qquad (6.28)
\end{aligned}
$$

where

$$C^\mu = 2(\frac{p_3^\mu}{p_3\cdot k} - \frac{p_1^\mu}{p_1\cdot k}) \qquad (6.29)$$

6.3 The Radiative Leptonic τ Decay with Anomalous Couplings

Consider the radiative leptonic τ decay $\tau \to \mu\bar{\nu}_\mu\nu_\tau\gamma$ with anomalous couplings of the τ lepton to the on shell photon. In general a photon may couple to a tau lepton through its electric charge, magnetic dipole moment, or electric dipole moment. We parameterize this coupling with the following matrix element:

$$\langle\tau(p_2)|J_\mu|\tau(p_1)\rangle = \bar{u}(p_2)\Gamma_\mu u(p_1) \qquad (6.30)$$

The most general Lorentz-invariant form of the vertex Γ_μ which describes the interaction between the τ lepton and on-shell photon is

$$\Gamma_\mu = \left(F_1(q^2)\gamma_\mu + \frac{F_2(q^2)}{2m_\tau i}\sigma_{\mu\nu}q^\nu + \frac{F_3(q^2)}{2m_\tau}\sigma_{\mu\nu}q^\nu\gamma_5\right) \qquad (6.31)$$

where $F_1(q^2)$ corresponds to the electric charge with $F_1(0) = 1$. $F_2(q^2)$ ($F_3(q^2)$) is the anomalous magnetic (electric) dipole moment. m_τ represents the mass of the τ lepton and q is the (out going) moment of the photon.

With the electric charges of neutrinos being zero ($q_2 = q_4 = 0$) and $q_1 = q_3 = -1$ in Equation (6.9), the C and E terms vanish. We replace the couplings of the photon γ^λ with the (SM + *anomalous*) couplings (6.31), i.e.,

$$\gamma^\lambda \rightarrow \left[\gamma^\lambda + \frac{F_2(q^2)}{2m_\tau i}\sigma^{\lambda\nu}q^\nu + \frac{F_3(q^2)}{2m_\tau}\sigma^{\lambda\nu}q_\nu\gamma_5\right] \tag{6.32}$$

The invariant amplitude with anomalous couplings to τ lepton can be written as follows:

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}_\tau + \mathcal{M}_A \\
&= \frac{G_F e}{2\sqrt{2}}V_{34}V_{12}\,[2B\cdot\varepsilon^*\bar{u}(p_3)\gamma_\mu(1-\gamma_5)v(p_4)\bar{u}(p_2)\gamma^\mu(1-\gamma_5)u(p_1) \\
&\quad + \frac{q_1}{p_1\cdot k}\bar{u}(p_3)\gamma_\mu(1-\gamma_5)v(p_4)\bar{u}(p_2)\gamma^\mu(1-\gamma_5)\not{\varepsilon}^*\not{k}u(p_1) \\
&\quad + \frac{q_3}{p_3\cdot k}\bar{u}(p_3)\not{k}\not{\varepsilon}^*\gamma_\mu(1-\gamma_5)v(p_4)\bar{u}(p_2)\gamma^\mu(1-\gamma_5)u(p_1) \\
&\quad + \frac{q_1}{p_1\cdot k}\bar{u}(p_3)\gamma_\mu(1-\gamma_5)v(p_4) \\
&\quad \bar{u}(p_2)\gamma^\mu(1-\gamma_5)(\not{\varepsilon}^*\not{k}-\not{k}\not{\varepsilon}^*)(\mathcal{F}_2+i\mathcal{F}_3)u(p_1)]
\end{aligned} \tag{6.33}$$

where

$$\mathcal{F}_2 \equiv \frac{F_2}{4m_\tau},\quad \mathcal{F}_3 \equiv \frac{F_3}{4m_\tau} \tag{6.34}$$

Again using the trace and γ matrix identities in previous section, and averaging over initial spin states and summing over the final spin states, we obtain the anomalous contribution to squared matrix element:

$$\begin{aligned}
|\bar{\mathcal{M}}|^2 &= |\bar{\mathcal{M}}_\tau + \bar{\mathcal{M}}_A|^2 \\
&= 32G_F^2e^2\left\{2\,p_2\cdot p_3\,\frac{p_4\cdot k}{p_1\cdot k} + 2\,p_1\cdot p_4\,\frac{p_3\cdot k}{p_2\cdot k}\right. \\
&\quad + \frac{p_2\cdot p_3}{p_1\cdot k}(B\cdot p_4\,p_1\cdot k - B\cdot p_1\,p_4\cdot k) \\
&\quad - \frac{p_1\cdot p_4}{p_2\cdot k}(B\cdot p_3\,p_2\cdot k - B\cdot p_2\,p_3\cdot k)
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{2} B \cdot B \, p_2 \cdot p_3 \, p_1 \cdot p_4 \\
& + \frac{F_2}{p_1 \cdot k \, p_3 \cdot k} [2 p_2 \cdot p_3 \, p_4 \cdot k \, p_3 \cdot k - p_1 \cdot k (p_2 \cdot p_3 \, p_4 \cdot k - p_3 \cdot p_4 p_2 \cdot k) \\
& - p_4 \cdot k (p_1 \cdot p_2 \, p_3 \cdot k - p_1 \cdot p_3 \, p_2 \cdot k)] \\
& + \frac{F_3}{p_1 \cdot k \, p_3 \cdot k} (p_1 \cdot k - p_4 \cdot k) \, \varepsilon^{\mu\nu\alpha\beta} p_{1\mu} \, p_{2\nu} \, p_{3\alpha} \, p_{4\beta} \Big\} \\
\equiv \; & 32 G_F^2 e^2 |A|^2
\end{aligned}
\tag{6.35}
$$

The expression for the photon spectrum in the radiative τ decay $\tau \rightarrow \mu\nu_\mu\nu_\tau\gamma$ using ref. [53] can be written as follows:

$$
\begin{aligned}
\frac{d\Gamma}{d\omega} = \; & \frac{\alpha G_F^2 \omega}{64\pi^5 m_\tau^2 (m_\tau - 2\omega)} \\
& \int_{-1}^{1} d(cos\theta_2') \int_{-1}^{1} d(cos\theta_k) \int_0^{2\pi} do \int_{m_3^2}^{m_{1k}^2} dm_{23}^2 \\
& \left\{ \frac{\lambda^{\frac{1}{2}}(m_{1k}^2, m_{23}^2, m_1^2) \lambda^{\frac{1}{2}}(m_{23}^2, m_2^2, m_3^2)}{m_{23}^2} |A|^2 \right\} .
\end{aligned}
\tag{6.36}
$$

where ω is the photon energy in the rest frame of the τ and

$$
\begin{aligned}
m_{1k}^2 &= m_\tau (m_\tau - 2\omega); \\
\lambda^{\frac{1}{2}}(x, y, z) &= (x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)^{\frac{1}{2}} \\
0 \le \omega &\le \frac{m_\tau^2 - m_3^2}{2m_\tau}, \; (m_2, m_1 \approx 0)
\end{aligned}
\tag{6.37}
$$

The detailed description of the variables relevant to the phase space can be obtained from [53] by using appropriate values for the charges, masses and colour factor. The above photon spectrum is reliable for "hard" photons only ($E_\gamma > 0.1 m_\tau$) because we are ignoring the one loop contributions needed to cancel the relevant infrared divergence. In ref [54] the same radiative decay process was studied in order to constrain the τ anomalous magnetic moment, but attention was given mostly to the end of the electron spectrum as opposed to the hard photon spectrum studied here.

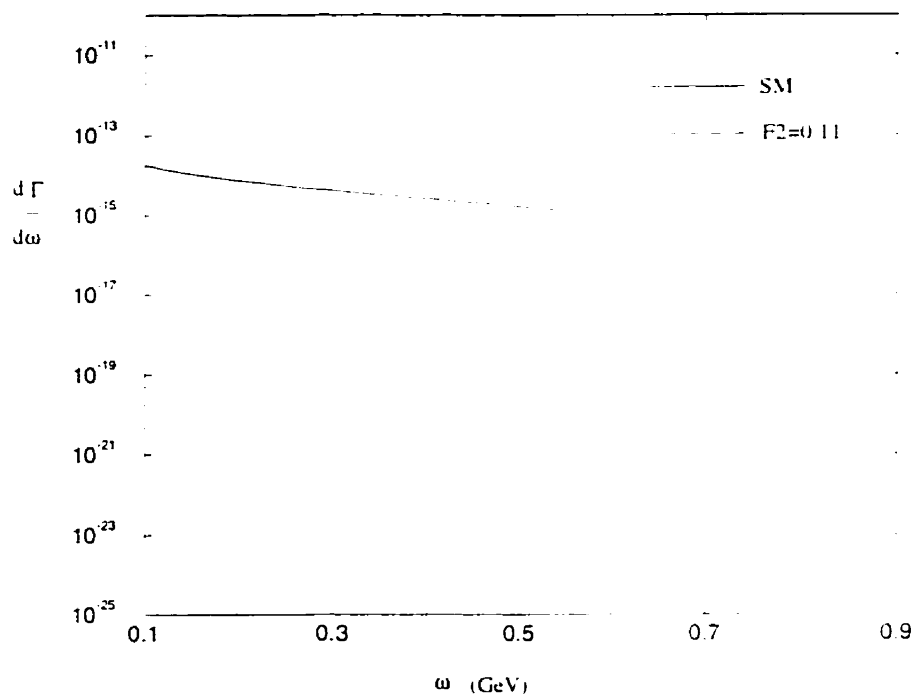

Figure 6.5: The photon spectrum for the process $\tau \to \mu\bar{\nu}_\mu\nu_\tau\gamma$. The continuous line corresponds to the Standard Model. The dashed line corresponds to the correction due to $F_2(0) = 0.11$.

In our numerical calculations we use $m_\tau = 1.784 GeV$ and $m_\mu = 0.106 GeV$ respectively. In Fig. 6.5 we plot our results for the hard photon spectrum in $\tau \to \mu\bar{\nu}_\mu\nu_\tau\gamma$ within the SM as well as the corrections due to $F_2(0) \neq 0$ if this had a value of 0.11 (present upper limits). Fig. 6.5 shows that the effects of a pure magnetic moment ($F_2(0) = 0.11, F_3(0) = 0$) are roughly two orders of magnitude smaller than those of the pure standard model contributions. Using the present static limits of $F_2(0) = 0.11$ [55] , we obtain the following ratio of integrated correction rate and

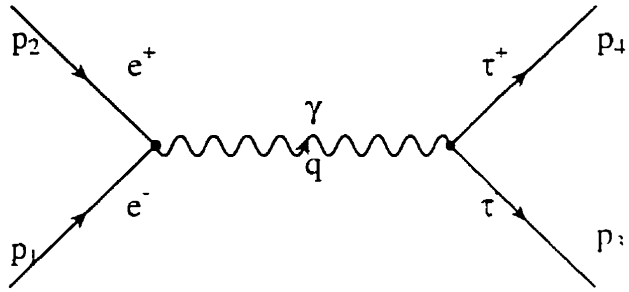

Figure 6.6: Feynman diagram for the exchange of photon γ in the $e^+e^- \to \tau^+\tau^-$ scattering process

integrated SM rate in the hard photon region from $0.2GeV$ to $0.8GeV$:

$$\int_{0.2}^{0.8} (\frac{d\Gamma}{d\omega})_{F_2=0.11}\, d\omega / \int_{0.2}^{0.8} (\frac{d\Gamma}{d\omega})_{SM}\, d\omega \approx 2.8 \times 10^{-3} \quad (6.38)$$

The branching ratio of the integrated SM rate for this region is 6.4×10^{-4}. In spite of the small branching ratio. F_2 effects may be marginally improved by measuring these decays at a tau factory.

6.4 Anomalous Couplings in the $e^+e^- \to \tau^+\tau^-$ Scattering Process

In the lowest order of electroweak theory the amplitude for $e^+e^- \to \tau^+\tau^-$ is expressed as the sum of the electromagnetic and the weak amplitude as shown in Figs.6.6 and 6.7. The scattering amplitude can be written according to Feynman rules [52] as follows:

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_\gamma + \mathcal{A}_{Z^0} \\ &= \frac{ig^{\mu\nu}}{q^2}\bar{v}_e(p_2)(-ie\gamma_\nu)u_e(p_1) \end{aligned}$$

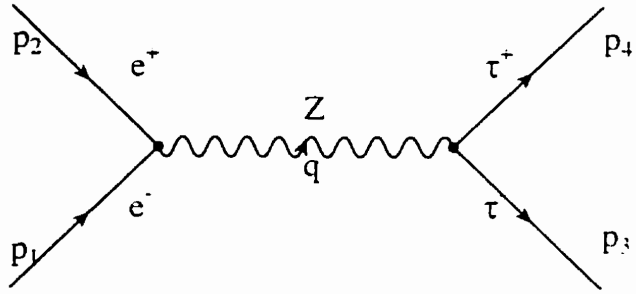

Figure 6.7: Feynman diagram for the exchange of Z^0 in the $e^+e^- \to \tau^+\tau^-$ scattering process

$$\begin{aligned}
& \bar{u}_\tau(p_3)\left\{-ie\left[\gamma_\mu + \frac{F_2(q^2)}{2m_\tau i}\sigma_{\mu\lambda}q^\lambda + \frac{F_3(q^2)}{2m_\tau}\sigma_{\mu\lambda}q^\lambda\gamma_5\right]\right\} v_\tau(p_4) \\
& + \frac{i\left[g^{\mu\nu} - \frac{q^\mu q^\nu}{M_Z^2}(1 + i\frac{\Gamma_Z M_Z}{q^2})\right]}{q^2 - M_Z^2 + i\Gamma_Z M_Z}\bar{v}_e(p_2)\left(\frac{-ig}{2cos\theta_W}\right)\gamma_\nu(C_V - C_A\gamma_5)u_e(p_1) \\
& \bar{u}_\tau(p_3)\left(\frac{-ig}{2cos\theta_W}\right)\gamma_\mu(C_V - C_A\gamma_5)v_\tau(p_4) \\
= \; & (-ie^2)\left\{\frac{1}{q^2}\bar{v}_e(p_2)\gamma^\mu u_e(p_1)\bar{u}_\tau(p_3)[\gamma_\mu + (\gamma_\mu \not{q} - \not{q}\gamma_\mu)(\mathcal{F}_2 - i\mathcal{F}_3\gamma_5)]v_\tau(p_4)\right. \\
& + \frac{1}{sin^2(2\theta_W)}\frac{1}{q^2 - M_Z^2 + i\Gamma_Z M_Z}\bar{v}_e(p_2)\gamma^\mu(C_V - C_A\gamma_5)u_e(p_1) \\
& \left.\bar{u}_\tau(p_3)\gamma_\mu(C_V - C_A\gamma_5)v_\tau(p_4)\right\} \qquad (6.39)
\end{aligned}$$

with

$$g sin\theta_W = e, \; \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \qquad (6.40)$$

$$\mathcal{F}_2 \equiv \frac{F_2}{4m_\tau}, \; \mathcal{F}_3 \equiv \frac{F_3}{4m_\tau} \qquad (6.41)$$

$$C_A = -\frac{1}{2}, \; C_V = -\frac{1}{2} + 2sin^2\theta_W \qquad (6.42)$$

M_Z is the mass of Z^0, and C_V and C_A denote the vector and axial vector coupling constants of the Z^0 to the electron and the final state τ currents. The definition of

θ_W is $sin^2\theta_W \equiv 1 - (M_W^2/M_Z^2) = 0.2259 \pm 0.0043$ [3].

In the present calculation we assume that only the photon has anomalous couplings to the τ. Under this assumption and ignoring terms of order m_τ^2/q^2, m_τ^2/M_Z^2, we obtain the following expression of the unpolarized differential cross section in center-of-mass system:

$$\frac{d\sigma}{dcos\theta} = \frac{\pi\alpha^2}{2s}\Big\{\left[1 + 2C_V^2C_2 + \chi^2(C_V^2 + C_A^2)^2\right](1 + cos^2\theta) + 4(C_A^2C_2 + 2\chi^2C_V^2C_A^2)cos\theta + \frac{q^2}{4m_\tau^2}(F_2^2 + F_3^2)sin^2\theta - \frac{F_2}{2}\left[8 + C_2(C_V^2 + C_A^2cos\theta)\right]\Big\} \quad (6.43)$$

where

$$C_2 = \frac{1}{sin^2 2\theta_W}Re\left(\frac{s}{s - M_Z^2 + i\Gamma_Z M_Z}\right) \quad (6.44)$$

$$\chi^2 = \frac{1}{sin^4 2\theta_W}\frac{s^2}{(s - M_Z)^2 + \Gamma_Z^2 M_Z^2} \quad (6.45)$$

We exhibit representative numerical results in Table 6.1. From Equation (6.43) we can see that the differential cross section can be used to distinguish between a CP conserving $F_2(q^2)$ and a CP violation $F_3(q^2)$ correction. This is due to the fact that there are no linear terms in F_3. Table 6.1 shows that at low energies($S \ll M_Z^2$), the F_2 linear terms dominate while F_3 has only quadratic terms. The present limits of $F_2 < 0.02$ and $F_3 < 0.025$ come from this region at PETRA [44, 55] . At LEP-I ($\sqrt{s} = M_Z$), there are no useful limits on F_2 and F_3 under our assumption due to the dominance of the Z intermediate state. Useful limits can be obtained by assuming correlated $\tau\bar{\tau}\gamma$ and $\tau\bar{\tau}Z$ anomalous couplings [45]. At higher energies (LEP 200), the F_2 and F_3 effects are most noticeable. We expect that experiments will be sensitive to F_2 and $F_3 \approx 0.006$ or even 0.003.

$cos\theta$	$\sqrt{s}$	$F_2 = 0.02$	$F_3 = 0.02$	$F_2 = 0.006$	$F_3 = 0.006$
0	$M_Z/4$	-6.4%	1.6%	-2.25%	0.15%
	M_Z	0.11%	0.15%	-0%	0.015%
	$2M_Z$	78.8%	85.3%	5.7%	7.7%
0.5	$M_Z/4$	-5.6%	1.0%	-1.90%	0.09%
	M_Z	0.05%	0.09%	-0%	0%
	$2M_Z$	28.4%	31.7%	1.86%	2.9%
0.8	$M_Z/4$	-4.7%	0.38%	-1.5%	0%
	M_Z	0%	0.03%	-0%	0%
	$2M_Z$	8.3%	10.7%	0.25%	0.96%

Table 6.1: Percentage deviation from the SM $e^+e^- \rightarrow \tau^+\tau^-$ cross section resulting from the non-SM values of F_2 and F_3 listed in the top row. Results are displayed for scattering angles θ and centre-of-mass energies $\sqrt{s}$ listed in the first two columns

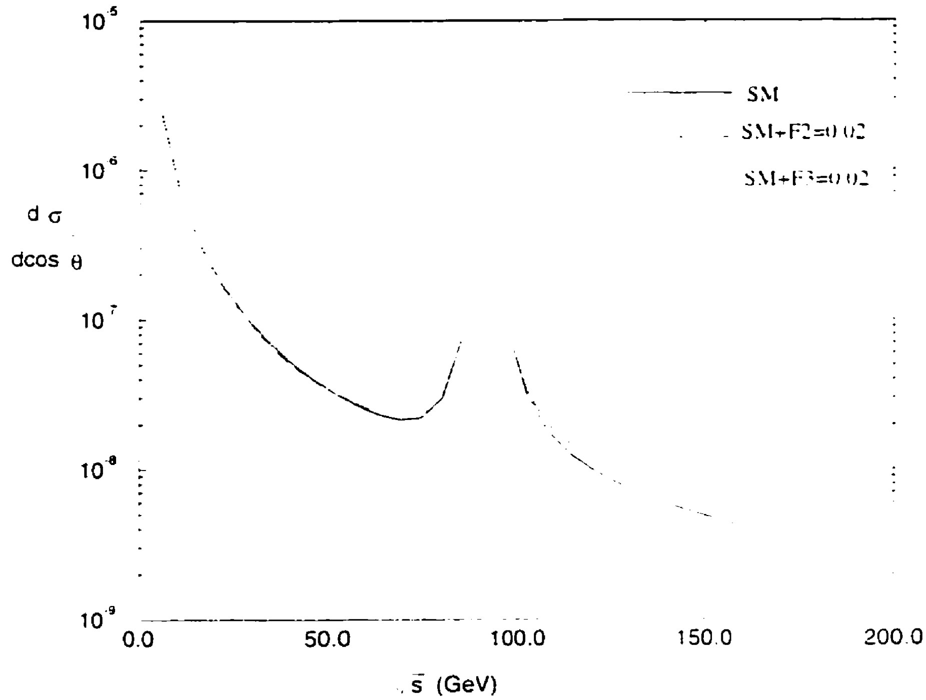

Figure 6.8: The unpolarized differential cross section with respect to the center of mass energy $\sqrt{s}$ in the scattering process $e^+e^- \to \tau^+\tau^-$

We also plot the unpolarized differential cross section in Fig. 6.8 at $cos\theta = 0$ to show that at high energy, $F_2(0)$ and $F_3(0)$ effects are most noticeable.

To summarize, the cross section can be useful to distinguish between $F_2(q^2)$ and $F_3(q^2)$ effects. The anomalous couplings effects of F_2 and F_3 should be most noticeable at LEP 200 energies and $cos\theta \ll 1$.

Chapter 7

Conclusions

In this thesis we have investigated both non perturbative QCD as well as the electroweak physics beyond the Standard Model. We began the investigation with the calculation of the two-gluon condensate's contribution to the quark electromagnetic form factor F_2. We found that the gluon condensate does not appear to contribute to the anomalous magnetic moment of quarks, once the renormalization procedure for the electromagnetic vertex is suitably redefined to account for divergent order-unity condensate contributions. In Appendix A, we have also shown that the self-energy contributions to the vertex involving $\langle \alpha_s G^2 \rangle$ do not contribute to the anomalous magnetic moment of quarks.

We obtained instanton contributions to finite energy sum rules F_0 and F_2 via the asymptotic expansions in Chapter 3. The results (3.19) and (3.23) are not meaningful unless $2\rho\sqrt{s_0} > 1$. Since s_0 has to be sufficiently large for finite energy sum rules to be applied, the asymptotic expansions are appropriate and useful. In the large s_0 limit, the leading perturbative contribution to F_0 (4.28) and F_2 (4.29) dominates the instanton contribution. However, for the value of s_0 near 1 GeV^2, the instanton contribution is shown to be larger than the perturbative contribution. Since purely-perturbative and QCD-vacuum condensate contributions to scalar-current correlation functions can not distinguish between isoscalar and isovector channels, the instanton component of the QCD vacuum is necessary to distinguish these degenerate states.

We calculated the perturbative and non-perturbative (QCD condensate and instanton) contributions to finite energy sum rules in the scalar channel. From a one-resonance fit to the first two finite energy sum rules , we found that the light sigma resonance can exist only if the continuum threshold s_0 is smaller than $1.72 GeV^2$, a value likely to be too low for convergence of the purely perturbative contributions to the sum rule [21].

The two gluon condensate contributions to the finite-energy sum rules $F_{0,1}$ are respectively calculated for the longitudinal component of the axial vector correlation functions. This contribution is shown to arise entirely from a net branch singularity when $s \geq 4m^2$. The cancellation of net pole contributions at $s = 0$, as well as the cancellation of infrared singularities arising from integration of the exact expression along the branch cut against those arising from integration around the branch cut terminus at $s = 4m^2$ is also demonstrated explicitly. We emphasize that all of these results including the singularity structure described above are applicable to the gluon condensate contributions to the finite energy sum rules in scalar, vector, and the transverse component of the axial-vector channels. The explicit cancellation of quark-mass singularities via operator mixing is also demonstrated for channels in which such singularities naively occur.

Finally we addressed deviations from the Standard Model in purely perturbative electroweak physics. We analyzed the hard photon spectrum in radiative leptonic τ decays $\tau \to \mu\bar{\nu}_\mu\nu_\tau\gamma$ in the presence of the possible $\tau\bar{\tau}\gamma$ anomalous couplings. We found that the present limits on the τ anomalous magnetic moment $F_2(0)$ could be marginally improved by measuring those decays. We find as well that the unpolarized differential cross section in the scattering process $e^-e^- \to \tau^+\tau^-$ is quite sensitive to both $F_2(0)$ and $F_3(0)$, especially at LEP II energies.

Appendix A

Gluon Condensate $\langle \alpha_s G^2 \rangle$ Contributions to $\mathcal{R}(0)$ from Self-Energy Diagrams

The Feynman amplitude of self-energy diagram contribution to the vertex function Γ^σ corresponding to Fig. A.1 and Fig. A.2 is

$$A_s = \bar{u}(p_2)\gamma_\mu \frac{1}{\not{p}_1 - m}\Sigma(p_1)u(p_1) + \bar{u}(p_2)\Sigma(p_2)\frac{1}{\not{p}_2 - m}\gamma_\mu u(p_1) \tag{A.1}$$

where $\Sigma(p_1)$ and $\Sigma(p_2)$ are the lowest-order gluon condensate contributions to the quark self-energy.

In fact the explicit covariant-gauge gluon-condensate contribution to the quark self-energy have been calculated by Bagan et al [27]:

$$\Sigma(p) = \frac{\pi\langle \alpha_s G^2 \rangle}{9(p^2 - m^2)^3}\left[(p^2 - 3m^2)(\not{p} - m) + mp^2\right] \tag{A.2}$$

We define the gluonic condensate portion of the quark self-energy $\Sigma(p)$ as

$$\Sigma(p) \equiv \mathcal{A}(p^2) - \mathcal{B}(p^2)(\not{p} - m) \tag{A.3}$$

and compare (A.2) and (A.3):

$$\mathcal{A}(p^2) = \frac{\pi\langle \alpha_s G^2 \rangle}{9(p^2 - m^2)^3} mp^2 \tag{A.4}$$

$$\mathcal{B}(p^2) = \frac{\pi\langle \alpha_s G^2 \rangle}{9(p^2 - m^2)^3}(3m^2 - p^2) \tag{A.5}$$

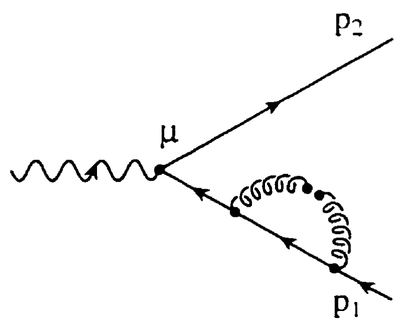

Figure A.1: Gluon-condensate contributions to self-energy corrections of the electromagnetic vertex function: diagram (a).

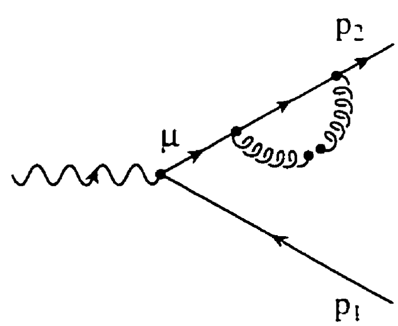

Figure A.2: Gluon-condensate contributions to self-energy corrections of the electromagnetic vertex function: diagram (b).

Substituting (A.4) and (A.5) into (A.1), we obtain the amplitude A_s:

$$
\begin{aligned}
A_s &= \bar{u}(p_2)\gamma_\mu \left[\frac{2m}{p_1^2 - m^2}\mathcal{A}(p_1^2) - \mathcal{B}(p_1^2)\right] u(p_1) \\
&+ \bar{u}(p_2)\gamma_\mu \left[\frac{2m}{p_2^2 - m^2}\mathcal{A}(p_2^2) - \mathcal{B}(p_2^2)\right] u(p_1) \\
&\equiv \bar{u}(p_2)\left[\mathcal{R}_s(q^2)\gamma_\mu + \frac{2\mathcal{S}_s(q^2)}{m}(p_1 + p_2)_\mu\right] u(p_1) \qquad \text{(A.6)}
\end{aligned}
$$

Comparing the last two lines in the above equation (A.6), we can find that the gluon condensate contribution to the $\mathcal{R}_s(q^2)$ is as follows:

$$
\mathcal{R}_s(q^2) = 2m\left(\frac{\mathcal{A}(p_1^2)}{p_1^2 - m^2} + \frac{\mathcal{A}(p_2^2)}{p_2^2 - m^2}\right) - \left(\mathcal{B}(p_1^2) + \mathcal{B}(p_2^2)\right) \qquad \text{(A.7)}
$$

$$
\mathcal{S}_s(q^2) = 0 \qquad \text{(A.8)}
$$

Using the following on-shell conditions

$$
p_1^2 = p_2^2 = m^2 \qquad \text{(A.9)}
$$

$$
p_2^2 - m^2 = p_1^2 - m^2 = U \qquad \text{(A.10)}
$$

the $\mathcal{R}_s(0)$ becomes

$$
\mathcal{R}_s(0) = \lim_{U \to 0}\left\{\frac{\pi\langle\alpha_s G^2\rangle}{9} 2\left(2\frac{m^4}{U^4} - \frac{m^3}{U^3}\right)\right\} \qquad \text{(A.11)}
$$

From the above calculations we can conclude that the gluon condensate does not contribute to $\mathcal{S}_s(0)$ but only to $\mathcal{R}_s(0)$. The sum of (A.11) and (2.80) is the net gluon-condensate contribution to $\mathcal{R}$, which remains sufficiently divergent on-shell to ensure that (2.86) remains valid.

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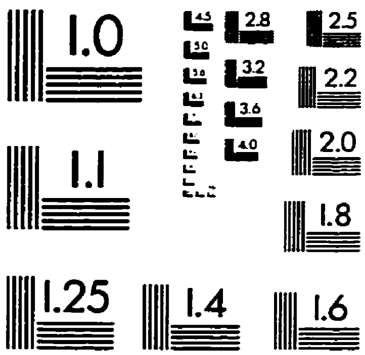
IMAGE EVALUATION TEST TARGET (QA-3)



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