ANALYSIS AND EXTENSION OF THE

RENORMALIZATION GROUP TREATMENT OF FLUID TURBULENCE

by

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ABSTRACT

The Navier-Stokes equations driven by a random stirring force have been treated with the Renormalization Group (RNG) methods by Yakhot and Orszag and other authors to obtain theoretical predictions of various constants of turbulence without empirically adjusted parameters. The analysis contains many approximations, some of which may be justified when the ratio of the resolution cutoff wavenumber (Λ_c) to the wavenumber under consideration (k) is very large. However, when this ratio approaches one (local interactions) many of the approximations required by RNG are no longer valid. Various methods attempting to extend RNG to include the local interactions have failed to produce results that could be validated. These attempts have been outlined and discussed in the first part of this study. In part two of this work, further attempt is made to extend the RNG method to produce turbulence models valid near the cutoff, specifically an eddy viscosity as a function of the wavenumber ratio (k/Λ_c) with a cusp up behavior for $k/\Lambda_c \longrightarrow 1$. General properties of the partial averaging operation have been presented and modified to allow different methods of averaging subgrid Fourier triads. Three methods of deriving an eddy viscosity function have been proposed. The results do not match the most likely form of eddy viscosity obtained in other studies. The validity of the temporal approximations made in the RNG are analyzed.

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Notation

1. General Remarks

The summation convention is used throughout where repeated tensor indices are taken to be summed without the summation symbol being needed.

Ensemble averages are denoted by Dirac brackets <..>. The partial ensemble average introduced in chapter 8 is denoted as <..>>.

If $a(x,t)$, $a(k,t)$ or $a(k,\omega)$ is an instantenous value of the variable a then:

$\overline{a}$ is the filtered variable or a variable invariant under partial averaging.

a' is the fluctuating portion of the variable with respect to the partial or ensemble averaging.

Bold font indicates a vector (eg. $\mathbf{x}$), the same variable in standard font indicates the vector magnitude (eg. x) the component of a vector is the vector magnitude with a subscript (eg. x_α).

Variable units are indicated in square brackets. The symbol [1] indicates a dimensionless variable.

2. Variables in x-t space

$\mathbf{x}$, $\mathbf{r}$ position vectors in configuration space [L]

t time [t]

τ, τ', τ'' time variables in correlation tensors and convolution integrals [t]

$\tau_{ADV}(k)$ advective time constant of an eddy at k [t]

$\tau_D(k)$ dynamic time constant of an eddy at k [t]

$\mathbf{u}(x,t)$ velocity vector [L/t]

$p(x,t)$ hydrostatic pressure [mass/(Lt^2)]

$f(x,t)$ stirring force per unit mass in configuration space [L/t^2]

$T_{\alpha\beta}$ Reynolds stress tensor [L^2/t^2]

$Q_{\alpha\beta}(r)$ two point velocity correlation [L^2/t^2]

$Q_{\alpha\beta}(r,\tau)$ two point, two time velocity correlation $[L^2/t^2]$

$\overline{G}(x-y)$ Kernel of a convolution filter integral $[1]$

ε ensemble averaged dissipation rate $[L^2/t^3]$

3. Variables in k-t Space

Note: Coefficients of a discrete Fourier summation have the same dimensions as the corresponding variable in x-t space. Variables transformed using a Fourier integral in three spatial dimensions get a factor of L^3 multiplying their x-t units. A Fourier transform from time domain to frequency cause a factor of t to multiply the x-t units.

$k, j,$ wavenumber vectors $[L^{-1}]$

$u(k,t)$ spatial Fourier velocity vector $[L^4/t]$

$p(k,t)$ spatial Fourier hydrostatic pressure $[massL^2/t^2]$

$f(k,t)$ stirring force per unit mass in configuration space $[L/t^2]$

$u_{tp}^{>k}(k,t)$ component of $u^>(k,t)$ due to the triple products $[L^4/t]$

$v_t,\ v(k,\Lambda_c)$ effective (eddy) viscosity $[L^2/t]$

$v_{YO}(k)$ Yakhot-Orszag eddy viscosity $[L^2/t]$

$\tilde{v}(l)$ dimensionless eddy viscosity $[1]$

$\tilde{v}_F(l)$ dimensionless eddy viscosity derived over fixed wavenumber averaging $[1]$

$\tilde{v}_{FL}(l)$ dimensionless eddy viscosity derived over fixed plus low
          wavenumber averaging $[1]$

$\tilde{v}_{FH}(l)$ dimensionless eddy viscosity derived over fixed plus high
          wavenumber averaging $[1]$

$I_F(l)$ Integral function of the trigonometric factors and stirring
          force for fixed wavenumber $[1]$

$I_{FL}'(l)$ Integral function of the trigonometric factors and stirring
          force for fixed plus low wavenumber > k $[1]$

$Q_{ij}(k,t)$ velocity correlation tensor in Fourier space $[L^5/t^2]$

$\overline{Q}_{ij}(k,t)$ correlation tensor of velocity components invariant under partial averaging $[L^5/t^2]$

$\overline{Q}'_{ij}(k,t)$ correlation tensor of velocity component invariant and fluctuating under partial averaging $[L^5/t^2]$

$Q'_{ij}(k,t)$ correlation tensor of velocity components affected by partial averaging $[L^5/t^2]$

$E(k,t)$ energy spectrum per unit mass $[L^3/t^2]$

$q(k,t)$ energy density spectrum per unit surface of a wavenumber sphere $[L^5/t^2]$

$V(k)_{ADV}$ combined advective velocity $[L^4/t]$

$W(k)$ correlation of the stirring forces $[L^5/t^3]$

W constant coefficient in the correlation of the stirring forces
W maybe considered a function of $\in$ $[L^{6-\in}/t^3]$

v_0 characteristic velocity of the shell Λ-$\Delta\Lambda$ to Λ $[L/t]$

Λ_0 Kolmogorov wavenumber cutoff $[L^{-1}]$

Λ intermediate wavenumber cutoff, also written as $\Lambda(f)$ $[L^{-1}]$

Λ_C Grid wavenumber cutoff $[L^{-1}]$

Λ_L hypothetical low limit on partial averaging $[L^{-1}]$

Λ_P boundary between the production and inertial range $[L^{-1}]$

Λ_m maximum wavenumber of the flow $[L^{-1}]$

4. Variables in k-ω space

ω, Ω frequency $[t^{-1}]$
(Ω is also used to denote spatial domain in chapter 2)

ω_j frequency region typical for an eddy centered at wavenumber j $[t^{-1}]$

$\omega_{ADV}(k)$ advective frequency $[t^{-1}]$

$\omega_D(k)$ dynamic frequency $[t^{-1}]$

ω_Λ cutoff frequency of a hypothetical low-pass filter $[t^{-1}]$

$\omega^<$ frequencies lower than ω_Λ, unaffected by the low-pass filter $[t^{-1}]$

$\omega^>$ frequencies higher than ω_Λ, filtered out by the low-pass filter $[t^{-1}]$

$u(k,\omega)$ spatial and temporal Fourier velocity vector $[L^4]$

$p(k,\omega)$ spatial and temporal Fourier velocity vector $[massL^2/t]$

5. Constant properties of fluid and flow and proportionality coefficients

Units of constant properties will be invariant under Fourier transforms.

ν_0 kinematic, molecular viscosity of the fluid $[L^2/t]$

ρ density of the fluid $[mass/L^3]$

ε ensemble averaged dissipation rate $[L^2/t^3]$

U background advective velocity $[L/t]$

C_k Kolmogorov constant   [1],   YO value of 1.61 used throughout

C_{mg} RNG constant of proportionality   [1]

γ -   lower integration limit coefficient or
        Kolmogorov wavenumber constant of proportionality   [1]

D temporal constant   [1], 0.494 used throughout

6. Important Formulas and Mathematical symbols

$\delta_{\alpha\beta}$ Kronecker delta   [1]

$\delta(x)$ Dirac Delta function of a scalar $[x^{-1}]$

$\delta(x)$ Dirac Delta function of a vector $[x^{-3}]$

$$Q_{\alpha\beta}(r,\tau) = <u_\alpha(x,t)u_\beta(x+r,t+\tau> \quad [L^2/t^2]$$

$$<u_\alpha(k,t)u_\beta(k',t> = \delta(k+k')Q_{\alpha\beta}(k,t) \quad [L^8/t^2]$$

$$Q_{\alpha\beta}(k,t) = D_{\alpha\beta}(k)Q(k,t) \quad [L^5/t^2]$$

$$E(k) = C_k\varepsilon^{2/3}k^{-5/3} \quad [L^3/t^2]$$

$$v(k,\Lambda_c) = \varepsilon^{1/3}\Lambda_c^{-4/3}\ \tilde{v}(l) \quad [L^2/t]$$

$$Q_{ij}(k,t) = \overline{Q}_{ij}(k,t) + 2\ \overline{Q}'_{ij}(k,t) + Q'_{ij}(k,t) \quad [L^5/t^2]$$

$$\tau_k \approx D^{-1}\varepsilon^{-1/3}k^{-2/3} \quad [t]$$

$$D \approx 0.1904\ C_k^2$$

$$l = \frac{k}{\Lambda} \quad [1]$$

$$l_c = \frac{k}{\Lambda_c} \quad [1]$$

$$\Lambda(l) = \Lambda_0 e^{-l} \quad [L^{-1}]$$

$$k\bullet u(k,t) = 0$$

$$D_{\alpha\beta}(k) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \quad [1]$$

$$M_{\alpha\beta\gamma}(k) = \frac{1}{2i}P_{\alpha\beta\gamma}(k) \quad [L^{-1}]$$

$$P_{\alpha\beta\gamma}(k) = \left\{ k_\beta D_{\alpha\gamma}(k) + k_\gamma D_{\alpha\beta}(k) \right\} \quad [L^{-1}]$$

$$G(k,\omega) = \frac{1}{i\omega + k^2 v} \quad [t]$$

i imaginary number $(-1)^{0.5}$ [1]

$d = 3$ number of space dimensions [1]

λ bookkeeping parameter in the perturbation expansion of the Navier Stokes equation [1]

$\overline{\lambda}$ effective expansion parameter in the non-dimensionalized Navier-Stokes equations [1]

$\in \ = \ 4 \ + \ y \ - \ d$
expansion parameter for the Renormalization Group [1]

$$C_{mg} \ \varepsilon \ = \ \frac{2WS_d}{(2\pi)^d} \ (C_{mg} \ \text{value of 1.59 used in this work}) \ [L^2/t^3]$$

S_d - area of a d-dimensional sphere of unit radius, $S_3 \ = \ 4\pi$ [1]

1. Introduction

1.1 The need to model fluid turbulence

Turbulent flows occur in many geophysical and engineering applications. The prediction of weather and climate, the fluid flow around fast moving bodies. and the prediction of the rates of mixing and combustion are all examples of problems which require forecasting of turbulent flows. Turbulence is interesting from an academic as well as an applied perspective. since it is an integral part of many problems in physics and engineering. The non-linear. chaotic. multi-scale dynamics of turbulent flow has so far eluded analytical solution, and is of considerable academic interest as one of the unsolved problems of classical physics. Current trends in turbulence research include the study of coherent structures in turbulent flow. intermittency. backscatter, and the unpredictability problem.

Early theories and models of turbulence relied on experimental data for verification, but today computer simulation data is available as well. The exact computation of all scales of a turbulent flow is called a direct numerical simulation (DNS). Present day DNS computations are limited to Reynolds numbers of the order of 10^4. Speziale [1]. However, in many realistic flows the Reynolds numbers are much larger. For instance flow in the atmospheric boundary layer can have a Reynolds number of the order of 10^7, Stull [2]. Unfortunately, each time the Reynolds number doubles, DNS simulations require an order of magnitude increase in computer capability, as discussed by Yakhot and Orszag (refered to here as YO) [3]. For this reason, the direct simulation of turbulence will be limited in the forseeable future to moderate Reynolds numbers and carried out for the purpose of academic investigation. Realistic flows will be computed with the aid of turbulence models.

When turbulent flows are computed for the purpose of meteorological or engineering applications, the usual objective is to accurately simulate motion on scales comparable to the boundaries of the flow. The reason is that the largest scales of the turbulence carry most of the kinetic energy, govern bulk scalar transport. and impart mechanical stresses to structures such as buildings. airplane wings etc. These

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scales of motion are strongly affected by the boundaries of the flowfield. The smaller size fluctuations are somewhat more homogeneous and isotropic. and their behavior becomes independent from the flow geometry. For numerical simulation purposes, it is convenient to select a grid resolution in the range of the boundary independent scales, because the scales smaller than the grid (herein called the small scales) may then be represented by a universal (geometry independent) model. Small scales need to be modeled since they affect the resolved (herein also called large) scales through non-linear interactions. This interaction must be accounted for. particularly with regard to the rate of kinetic energy transfer. Thus, if the effects of the small scales can be modeled with an accuracy acceptable for the application of interest, this alternative is generally preferred. This leads to the Large Eddy Simulation methods (LES), where the turbulent scales are explicitly computed until a certain cutoff wavelength, while all the smaller scales are represented by a subgrid model. This model, which is a function of the resolved velocity scales. has output that is intended to simulate the effects of the unresolved scales on the equation of motion so that the large scale solution is acceptably close to the exact solution.

1.2 The search for a universal subgrid model and the Renormalized Group method

Most subgrid models contain empirical constants which are adjusted to suit the conditions of the particular flow field considered. The lack of universality of turbulence models reduces the user's confidence in the results since it is possible that certain features of the flow were not anticipated *a priori* and thus the model parameters were not adequately adjusted.

The application of the Renormalization Group Analysis (herein referred to as RNG) to turbulence modeling has been carried out by a number of researchers since the early 1970's and culminated with the YO [3] derivations of some fundamental constants for high Reynolds number, isotropic turbulence. The RNG values of the Kolmogorov's constant, turbulent Prandtl number, Batchelor constant, and the skewness factor were found to be in general agreement with experimental values and were derived without any empirically adjusted coefficients. The method also yielded a value of the

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Smagorinsky constant for use with LES computations, and the coefficients for the model kinetic energy and dissipation equations used in the RNG k-ε turbulence model. The latter model has been adopted for some computational fluid dynamics (CFD) commercial software and promoted as superior to the standard k-ε model for many flow configurations [4]. These facts show that the RNG method has provided solutions to at least some of the problems of turbulence theory and there is a potential for further progress. However, the procedure originated in other areas of physics and its adaptation to Navier-Stokes turbulence by YO has been made under very restrictive and sometimes contradictory assumptions.

The YO analysis is somewhat controversial in the research community because it contains some apparent mathematical inconsistencies and doubts exist about the convergence of the method in the Kolmogorov range. Also, the results are strictly valid only for the limit of the largest turbulent scales or equivalently, for wavenumbers close to zero. In view of the success of the RNG k-ε model in simulating many industrial flows, the problematic development of the theory has been subjected to some critical review by Smith and Reynolds [5], Lam [6], and others. However, no one has arrived at a successful reconciliation of the ad-hoc analysis with the correct answers it has produced, nor has any alternative procedure been proposed to avoid the contradictory analytical steps of YO.

Nothwithstanding the promising results to date, the performance of RNG in the context of turbulence has so far fallen short of its achievements in other areas of physics, such as critical phenomena and quantum electrodynamics, where agreement with experiments extended to ten significant digits in some cases (Smith and Woodruff [7]). The unorthodox steps required in the derivation of the RNG turbulence models caused a division of opinion in the research community as to the suitability of the method for the Navier-Stokes equations. The present study reviews the application of RNG to turbulence modeling and attempts to extend the range of the results. The detailed objectives of the present work are discussed in the next section.

1.3 Objectives of the present work

The objectives of the present work are to clarify and validate the RNG analysis and to extend the range of application of its results. The scope of the study is limited to the following stages:

1. A comprehensive review of the RNG method as applied to turbulence is presented. Assumptions and concepts are examined and clarified. This work summarizes the current state of efforts to extend the procedure to finite wavenumbers. The aim of these investigations has been to recover the cusp of the eddy viscosity at maximum resolved wavenumbers as predicted by other analytical theories.

2. The analytical tools of RNG have been developed to represent the effect of interactions between very different magnitudes of wavenumbers (non-local interactions). The current study examines whether these tools may be used to represent local interactions, between wavenumbers of similar magnitudes.

3. This study proposes to extend and modify the RNG method in order to recover the cusp of the eddy viscosity at maximum resolved wavenumbers. The results are compared to other theories.

1.4 Outline of the thesis

This thesis is divided into two parts with the objective of clearly differentiating between the published theory and the results of this investigation. The method of presentation is to first discuss the general concepts of a given topic and then to give the mathematical details.

Part 1 summarizes the necessary background material. The general principles of turbulence modeling are discussed in chapter 2. The concepts of the RNG application to turbulence are described in chapter 3. Chapter 4 presents the RNG method as adapted by YO [3]. Chapter 5 outlines the attempts of Zhou, Vahala, and Hossain [8]

(herein referred to as ZVH) and Zhou and Vahala [9], [10] (herein referred to as ZV) to apply a discrete version of RNG to finite wavenumbers and to recover the cusp in the eddy viscosity curve. Chapter 6 presents the work of Carati [11] who shows that the continuous limit of RNG applied to finite wavenumbers leads to results which no longer represent the physics of the Navier-Stokes equations.

Part 2 contains the contributions of this study. A detailed introduction is given in chapter 7. Chapter 8 presents some general properties of the partial averaging operator and the expansion series used in the RNG method. Derivations of three different versions of the RNG equations for eddy viscosity at finite wavenumbers are presented in chapter 9. The eddy viscosity equations are solved numerically in chapter 10. Some theoretical arguments for validation of the proposed models and comparison with published literature are also presented in this chapter. Chapter 11 contains general conclusions and recommendations for future work.

2. Fundamentals of turbulent flow computations

The fundamental mathematical formulation of the turbulent flow problem is discussed in this chapter, both in physical space (x,t) and in Fourier space (k,t) or (k,ω). A brief description is included of the large range of scales of motion or eddy sizes in turbulent flow and of the Kolmogorov energy cascade from the larger to the smaller scales. All modeling approaches split the flow scales into the resolved motion (which is computed) and the small scale motion (which is averaged and modeled). The different methods of averaging the turbulent scales and modeling their effect on the resolved scales are introduced in this chapter.

2.1 Flow equations

The scope of this study is restricted to Newtonian, constant property fluids. Accordingly, the flow is described by the incompressible version of the continuity equation and the Navier-Stokes equations:

$$\frac{\partial u_\alpha}{\partial x_\alpha} = 0 \qquad\qquad 2.1\text{-}1$$

$$\frac{\partial u_\alpha}{\partial t} + u_\beta \frac{\partial u_\alpha}{\partial x_\beta} = -\frac{1}{\rho}\frac{\partial p}{\partial x_\alpha} + v_0 \frac{\partial^2 u_\alpha}{\partial x_\beta^2} + f_\alpha \qquad\qquad 2.1\text{-}2$$

Here $u_\alpha(x,t)$ is a component of the velocity vector $u(x,t)$, and $p(x,t)$ is the pressure. The variables ρ and v_0 are the fluid density and the kinematic viscosity, respectively, both assumed constant in space and time. Also, $f_\alpha(x,t)$ is the α component of a body force per unit mass, $f(x,t)$. A well posed problem must include the appropriate boundary and initial conditions.

Analytical solutions to the above equations have been obtained for cases of laminar flow. However, flows with high Reynolds numbers are transitional or fully turbulent and impossible to solve analytically. An approximate solution may be sought

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by modeling some or all of the turbulent scales of the flow. For the purpose of modeling using the RNG method, an idealized problem is considered where there are no boundaries or initial conditions and the flow is due to the body force source term f(x,t). In the RNG procedure, f(x,t) is specified as a random 'stirring' force with a Gaussian probability distribution. As will be discussed in chapter 4, additional characteristics of this force are selected to obtain an approximate solution to equations 2.1-1 and 2.1-2 that is statistically 'similar' to turbulence. Some fundamentals of obtaining turbulence statistics are discussed below.

2.2 Description of turbulence through an ensemble average

Consider a turbulent flow experiment repeated a large number of times with a set of initial and boundary conditions that were as close as possible to the same for each realization of the flow. For each realization, velocity and pressure were recorded. The mean, variance, and higher order moments of each variable are computed from this population. The quantities $<u(x,t)>$, $<p(x,t)>$, $<u'^2(x,t)>$, etc. are called the *ensemble averages*. The turbulent variables, $u'(x,t)$ and $p'(x,t)$ vary each time the experiment is repeated, presumably due to the nonlinear amplifications of small differences in the initial and boundary conditions. These quantities are called quasi-random, or chaotic components of the flow because it is speculated that if it were possible to maintain the initial and boundary conditions exactly the same, the behavior of the turbulent variables would also be exactly the same, so they are deterministic in principle but not in practice. This idea is equivalent to the belief that the deterministic Navier-Stokes equations are sufficient to recover real turbulence, Frisch [16].

Therefore, it is not the spatial or temporal averaging but the 'ensemble' averaging which provides the distinction between the deterministic. mean variables, and the random, turbulent flow components, a point often obscured in introductory textbooks. However, in cases where the mean variables do not vary with space. the spatial average is equivalent to the ensemble, and when the mean variables do not

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vary with time, the temporal average is equivalent to the ensemble. Osborne Reynolds pioneered the following decomposition:

$$u(x,t) = <u(x,t)> + u'(x,t)$$ 2.2-1

where $<..>$ indicates the ensemble average. An equivalent decomposition is applied to the pressure.

The following properties result:

$$<u'(x,t)> = 0$$ 2.2-2

$$<<u(x,t)>u'(x,t)> = <u(x,t)><u'(x,t)> = 0$$ 2.2-3

$$< <u(x,t)><u(x,t)> > = <u(x,t)><u(x,t)>$$ 2.2-4

Equation 2.2-3 establishes that there is no correlation between the mean and the fluctuating flow variables. Equations 2.2-3 and 2.2-4 are strictly true only for ensemble average but not for the filtering operations which will be considered in section 2.3, Leonard [18]. Substituting the mean and the turbulent components into the Navier-Stokes equation and averaging term by term yields the governing equation for $<u_\alpha>$:

$$\frac{\partial <u_\alpha>}{\partial t} + <u_\beta> \frac{\partial <u_\alpha>}{\partial x_\beta} + \frac{\partial <u'_\alpha u'_\beta>}{\partial x_\beta} = -\frac{1}{\rho} \frac{\partial <p>}{\partial x_\alpha} + v_0 \frac{\partial^2 <u_\alpha>}{\partial x_\beta^2}$$ 2.2-5

where the mean of the body force has been taken as zero, $<f_\alpha(x,t)> = 0$. Equation 2.2-5 includes the variables $<u>$, $<p>$, and $<u'u'>$ and illustrates the famous closure problem of turbulence, since the quantity $<u'_\alpha u'_\beta>$ contains six unknown variables. The closure problem occurs if one wishes to solve for only the average quantities, since for three dimensional flow, there are four equations and ten unknowns in this case.

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The equation of motion for the turbulent velocity is obtained by subtracting equation 2.2-5 from the instantaneous Navier-Stokes equation yielding:

$$\frac{\partial u'_\alpha}{\partial t} + u'_\beta \frac{\partial \langle u_\alpha \rangle}{\partial x_\beta} + \langle u_\beta \rangle \frac{\partial u'_\alpha}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \left\{ u'_\alpha u'_\beta - \langle u'_\alpha u'_\beta \rangle \right\} =$$

$$-\frac{1}{\rho}\frac{\partial p'}{\partial x_\alpha} + v_0 \frac{\partial^2 u'_\alpha}{\partial x_\beta{}^2} + f_\alpha(x,t) \qquad\qquad 2.2\text{-}6$$

It should be noted that while the mean and the turbulent scales are uncorrelated (equation 2.2-3) they are coupled in individual flow realizations [12], so that the behavior of the turbulent variable affects the solution for the mean variable and vice versa.

The ensemble average of equation 2.2-6 is zero. If equation 2.2-6 is multiplied by $u'_\gamma(x,t)$ prior to averaging, an equation for the single-point, single-time second moment is obtained. This equation includes the triple velocity product $\langle u'_\alpha u'_\beta u'_\gamma \rangle$ and begins the so-called moment hierarchy - another form of the closure problem. If instead, equation 2.2-6 is multiplied by $u'_\gamma(x',t')$ prior to averaging, the two-point, two-moment hierarchy is generated. The second approach is necessary to study spatial and temporal correlations in turbulence. The single point moments are applied in engineering models while the two point correlations are useful in more fundamental studies of the physics of turbulence, [13].

For 'steady state', or 'stationary' turbulence, the probability distribution of the turbulent variables is constant with time, and the amplitudes of $u'(x,t)$ and $p'(x,t)$ are constrained within a statistical envelope. The probability distribution of $u'(x,t)$ at a point is close to Gaussian, but odd moments do exist: the third moment, $\langle u'u'u' \rangle$, is responsible for turbulent energy transfer. This point is crucial to understanding the limitations of the various quasi - Gaussian models of turbulence because a truly Gaussian variable cannot exchange energy between the mean and the turbulent flow and between the different scales of the turbulent flow. The original

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Quasi-Normal (QN) [14] closure, for example, assumed Gaussian behavior of the even velocity moments but relaxed the restriction of zero odd moments. This limitation affects the RNG model which is related to the QN and uses a Gaussian stirring force as an energy source but must allow the velocity to depart from a Gaussian behavior.

2.3 Estimation of turbulent flows with models using Reynolds decomposition

For many engineering requirements, only the average values of the flow rate, shear stress, pressure gradients, and energy loss are required. These variables may be obtained from the solution of the Reynolds-averaged Navier-Stokes equations (2.2-5) and the continuity equation provided that the effects of the single-point, single time, second moment $<u'_\alpha u'_\beta>$ are properly represented. Thus, the objective is to model $<u'_\alpha u'_\beta>$ as a function of the mean flow variables. The term $\dfrac{\partial <u'_\alpha u'_\beta>}{\partial x_\beta}$ is often moved to the right hand side of equation (2.2-5), so that the term $-\rho <u'_\alpha u'_\beta>$ is regarded as an additional stress term called the turbulent stress or the Reynolds stress tensor.

Two broad categories of turbulence modeling are the eddy viscosity models and the Reynolds stress models. Both methods will be discussed below but since the eddy viscosity concept forms the groundwork for the current work, the discussion of these models will be more detailed.

2.3.1 Eddy viscosity models

The oldest proposal for modeling the turbulent stresses was put forward by Boussinesq in 1877 [15]. In analogy to viscous stresses in laminar flows, the turbulent stress is assumed proportional to the gradient of the mean variables:

$$-\langle u'_\alpha u'_\beta \rangle = v_t \left(\frac{\partial \langle u_\alpha \rangle}{\partial x_\beta} + \frac{\partial \langle u_\beta \rangle}{\partial x_\alpha} \right) - \frac{2}{3} \langle E \rangle \, \delta_{\alpha\beta} \qquad\qquad 2.3\text{-}1$$

The proportionality coefficient, v_t, is the turbulent or eddy viscosity. The quantity E represents the kinetic energy of the turbulent fluctuations: $\langle E \rangle = \frac{1}{2}$ $(\langle u_1'^2 \rangle + \langle u_2'^2 \rangle + \langle u_3'^2 \rangle)$, often represented by the symbol k in the literature. Including the kinetic energy term in the eddy viscosity expression ensures the correct value of 2E for the sum of the normal stresses when $\alpha = \beta$. The eddy viscosity is a property of the turbulent flow and not of the fluid and therefore may vary significantly from one point in the flow to another, unlike the kinematic viscosity v_0 which is constant for isothermal flows.

The physical and mathematical interpretation of the eddy viscosity concept will be developed and refined in this and subsequent chapters as one of the key topics of the present work. A brief outline of the historical development of the eddy viscosity is given by Frisch, [16]. The idea originated in the nineteenth century with the work of Saint-Venant (1851) and his former student Boussinesq (1870). The analogy between the turbulent transport and molecular transport was conceived by Prandtl (1925) based on the role of the molecular viscosity as determined by the kinetic theory of gases.

The role of viscosity in the dissipation of the kinetic energy is crucial for the derivation of the RNG model since the eddy viscosity is a function of the average dissipation rate. If the momentum equation is converted into an equation for the kinetic energy, it is seen that the molecular viscosity appears in the diffusion term, $v_0 \frac{\partial^2 (u^2)}{\partial x^2}$, and in the dissipation term, $-2v_0 (\frac{\partial \langle u \rangle}{\partial x})^2$, the diffusion term being generally much smaller and usually neglected relative to the dissipation. Stull [2]. Thus, the role of the molecular viscosity is to diffuse momentum and to dissipate the kinetic energy of the fluid into heat. The turbulent or eddy viscosity has analogous functions with respect to the average flow, transporting momentum with mean flow gradients (equation 2.3-1) and dissipating the kinetic energy of the mean flow into the turbulent fluctuations. This analogy is expressed in equation 2.3-2:

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$$\varepsilon = \frac{v_0}{2} < \left\{ \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} \right\}^2 > = \frac{v_t}{2} < \left\{ \frac{\partial <u_\alpha>}{\partial x_\beta} + \frac{\partial <u_\beta>}{\partial x_\alpha} \right\}^2 > \qquad 2.3\text{-}2$$

where $\varepsilon(x,t)$ is the ensemble average dissipation rate. This parameter should really be written as $<\varepsilon>(x,t)$ but the established notation will be followed.

Prandtl proposed that the turbulent velocity fluctuations play the role of molecules, while a typical distance that a fluid particle travels (the mixing length) plays the role of the mean free path, so that $v_t = \mathcal{V} \ell_m$. The methods of determining the appropriate values of the velocity $\mathcal{V}$ and the mixing length, ℓ_m, are generally classified as zero equation, one equation, or two equation models.

The most widely used zero equation model is the Prandtl mixing length model. which has been successful in the prediction of two-dimensional, thin shear layers. One equation models usually involve the solution of a prognostic equation for the turbulent kinetic energy (k in the engineering literature) so that $v_t = k^{1/2} \bullet \ell_m$. and ℓ_m is empirically specified. The widely used k-ε model utilizes partial differential equations for the turbulent kinetic energy k and for the dissipation, ε. and the eddy

viscosity is obtained as $v_t = c_\mu \frac{k^2}{\varepsilon}$, where c_μ is an empirical parameter. A total of five empirical parameters must be assigned values for the k-ε model. [15]. The use of empirical parameters is typical of most turbulence models and implies that these models are not universal, but instead are tuned to the geometry of the problem. In contrast, the present work analyzes the RNG approach which allows the analytical determination of heretofore adjustable model parameters for the limited case of homogeneous, isotropic turbulence.

2.3.2 Reynolds stress models

The Reynolds stress models predict components of $<u_\alpha' u_\beta'>$ using transport equations for each stress component. The most advanced of these are the differential

Reynolds stress models (DSM) requiring the solution of six partial differential equations for the six components of the Reynolds stress tensor plus a length scale equation - usually determined via the dissipation rate, ε. The results are superior to the two equation eddy viscosity models for flows with abrupt changes of strain rate, separation, rotation, and strong curvature, [17] but at a much greater computational cost. The algebraic stress models (ASMs) solve algebraic equations for the stresses and have been used to yield good predictions of flows with secondary motions and for flows with rotation and curvature.

2.4 Description of turbulence through filtering

The ensemble average concept is limited to the theoretical study of turbulence. since it is not usually practical to repeat an experiment a very large number of times. Instead, it is preferable to use a temporal average with a sufficiently long sampling time T:

$$\overline{x}(t) = \frac{1}{T} \int_{t}^{t+T} x(t)dt \qquad 2.4\text{-}1$$

The operation 2.4-1 assumes that any turbulent time scale $\tau << T$. Any deterministic part of the variable which varies rapidly with time, say a periodic pressure fluctuation due to the propeller in the wind tunnel, will be filtered out along with the random, turbulent fluctuations. When the sampling time is limited. the integral 2.4-1 is not a proper statistical average but a low - pass filter. passing through fluctuations with frequencies lower than $\frac{2\pi}{T}$. In turbulence, the spatially large scales tend to have low frequencies relatively to the small and fast scales. Engineering applications deal with the large scales of the flow. whether deterministic or turbulent components, and therefore, a filtering operation is used instead of the ensemble average. An equivalent discussion applies to spatial filtering.

For these applications where all of the turbulent fluctuations are to be replaced with averaged quantities, the filtering takes place in the time domain. treating all flow motions with a time scale smaller than a given cutoff as turbulence. while the resolved motions are the 'average' flow. The relations 2.2-3 and 2.2-4 are assumed to hold at least approximately, implying a large separation of time scales or a spectral gap between the mean motion and the turbulent motion. Such spectral gaps often occur in atmospheric flows, [2], where the 'average' quantities still vary with time. For industrial flows such as pipe flows, the mean may often be time invariant.

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Alternatively, the filtering operation may be performed in the spatial domain. or. equivalently, in the spatial wavenumber domain, in preparation for the large eddy simulation method (LES). The scales with a wavelength smaller than the grid mesh Δx are eliminated by applying a low pass filter to the flow variables, $f(x,t)$:

$$\overline{f}(x,t) = \int_{\Omega} f(y,t)\overline{G}(x-y)dy \qquad 2.4-2$$

where Ω is the spatial domain of the flow.

The function $\overline{G}(x-y)$ defines the filter properties, and may be taken as independent of the position vector x for the case of uniform grids. The formal application of such filters to the Navier-Stokes equations for the purpose of LES has been introduced by Leonard [18]. The most common forms of $\overline{G}(x-y)$ include a box hat filter in physical space, a Gaussian filter in physical space, or a box hat filter in Fourier space. The last of these filters, also known as the sharp Fourier cutoff. will be used implicitly in the current work. The resulting filtered equation of motion is:

$$\frac{\partial \overline{u}_\alpha}{\partial t} + \overline{u}_\beta \frac{\partial \overline{u}_\alpha}{\partial x_\beta} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_\alpha} + \upsilon_0 \frac{\partial^2 \overline{u}_\alpha}{\partial x_\beta^2} + \overline{f}_\alpha + \frac{\partial T_{\alpha\beta}}{\partial x_\beta} \qquad 2.4-3$$

where the subgrid scale tensor is given by:

$$T_{\alpha\beta} = \overline{u}_\alpha \overline{u}_\beta - \overline{u_\alpha u_\beta} \qquad 2.4-4$$

Introducing the decomposition $u = \overline{u} + u'$; and substituting into the expression for $T_{\alpha\beta}$, the resulting expression is:

$$T_{\alpha\beta} = \overline{u}_\alpha \overline{u}_\beta - \overline{\overline{u}_\alpha \overline{u}_\beta} - \left(\overline{\overline{u}_\alpha u_\beta'} + \overline{u_\alpha' \overline{u}_\beta} + \overline{u_\alpha' u_\beta'} \right) \qquad 2.4-5$$

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where the term $\overline{\bar{u}_\alpha \bar{u}_\beta} - \overline{u_\alpha u_\beta}$ is known as the Leonard tensor, while $\overline{\bar{u}_\alpha u_\beta'}$ and $\overline{u_\alpha' \bar{u}_\beta}$ are called the 'cross' terms. The Leonard tensor is an explicit term that may be computed in terms of the filtered variables, but the remaining terms in equation 2.4-5 are unknown. The reason for the appearance of these extra terms is that the filtered quantity $\bar{u}$ undergoes a variation within the filter length scale, Δx, [18]. If the maximum wavelength of u' « filter wavelength « minimum wavelength of $\bar{u}$, relations 2.2-3 and 2.2-4 are approximately applicable, and the above equation reduces to $T_{\alpha\beta} = - \overline{u_\alpha' u_\beta'}$ as for ensemble averaging.

The Leonard 'cross' terms are relevant to the current investigation. The RNG method utilizes an approximation called the 'partial ensemble averaging' of the small scales, in which the scales smaller than an arbitrary cutoff are averaged, while the scales larger than the cutoff are assumed to be constant. This procedure is equivalent to a box hat filter with the Leonard terms neglected. As seen from the above discussion, this approximation is acceptable if the ratio of the wavelengths of the averaged and the 'constant' terms is small so that there is a large separation between the large, resolved scales and the small, averaged scales. Physically, small turbulence scales will go through many cycles relatively to a much larger eddy, thus approximating the ensemble average. However, a problem arises from the different treatment of terms whose scales of motion straddle the filter boundary, thereby leading to errors in the smallest resolved scales. Although it is difficult to estimate the magnitude of such errors, further discussion of this problem will be presented at various stages of this study.

2.5 Governing equations in Fourier space

The discussion of the physics of the different scales of the turbulent flow and of the Large Eddy Simulation methods is aided by introducing the Fourier transform of the velocity, pressure, and stirring force fields. The physical flow domain is assumed to be of infinite size so that the Fourier transform of the velocity $u_\alpha(x,t)$ is given by:

$$u(\mathbf{k},t) = \int_{-\infty}^{\infty} u(\mathbf{x},t)e^{i\mathbf{x} \cdot \mathbf{k}} \, d^3x \qquad\qquad 2.5\text{-}1$$

Expressions similar to 2.5-1 hold for the force and pressure. The wavenumber vector $\mathbf{k} = \{k_1, k_2, k_3\}$, the magnitude of the wavenumber, $|\mathbf{k}| = k$, ranges from $\frac{2\pi}{L} = \Lambda_L$ to a maximum wavenumber Λ_m which corresponds to the minimum scale of the flow. Here L is the side length of a 'periodicity box', [16], so that $u(\mathbf{x},t)$ are L-periodic. The case of unbounded domain will be recovered later by letting L $\longrightarrow$ ∞.

It is noted that the Fourier transform integral 2.5-1 diverges since $u(\mathbf{x},t)$ does not vanish at infinity, [20]. Therefore, $u(\mathbf{k},t)$ exists only as a generalized function or a distribution, [14]. However, the Fourier transform of the correlation $\langle u_\alpha(\mathbf{x},t)u_\beta(\mathbf{x+r},t+\tau)\rangle = Q_{\alpha\beta}(\mathbf{r},\tau)$ is well defined, since Q goes to zero as $r \longrightarrow \infty$ or as $\tau \longrightarrow \infty$ so the energy and force spectra are bounded functions. These considerations yield the requirement that the L $\longrightarrow$ ∞ limit is taken after the averaging operation. [13].

In the present work, the Fourier modes of the velocity field are referred to interchangeably as the flow scales or the turbulent 'eddies'. The implied meaning is that large scales or eddies contain a range of Fourier coefficients at small wavenumbers, while small scales and eddies contain a range of large wavenumbers. A more precise definition of an idealized eddy in terms of its wavenumber content is given by Tennekes and Lumely [21].

The Fourier transformed continuity equation is:

$$\mathbf{k} \cdot u(\mathbf{k},t) = 0 \qquad\qquad 2.5\text{-}2$$

Equation 2.5-2 shows that the Fourier velocity coefficient is orthogonal to its wavenumber vector for incompressible flows. This property leads to important simplifications during the Fourier analysis of the flow equations and is used extensively in this work.

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The details of Fourier transforming the Navier Stokes equations are given in references [13] and [14]. The final result is:

$$\left(\frac{\partial}{\partial t} + \upsilon k^2\right)u_\alpha(\mathbf{k},t) = -i k_\beta \frac{1}{(2\pi)^4}\int d^3j \ u_\alpha(\mathbf{k}-\mathbf{j},t)u_\beta(\mathbf{j},t) - i k_\alpha \frac{1}{\rho}p(\mathbf{k},t) + f_\alpha(\mathbf{k},t) \qquad 2.5\text{-}3$$

where $k^2 = |\mathbf{k}|^2 = k_1^2 + k_2^2 + k_3^2$.

The continuity equation 2.5-2 is used to eliminate pressure from the momentum equation 2.5-3. This goal is accomplished in two steps presented in detail by McComb [13]. First, the linear velocity terms are eliminated from 2.5-3 to obtain a Poissons equation for the pressure. Then 2.5-3 is multiplied by k_α, summed over α, and rearranged with the use of equation 2.5-2 to obtain:

$$i k^2 \frac{1}{\rho} \ p(\mathbf{k},t) = -i k_\alpha k_\beta \frac{1}{(2\pi)^4}\int d^3j u_\alpha(\mathbf{k}-\mathbf{j},t)u_\beta(\mathbf{j},t) \qquad 2.5\text{-}4$$

Equation 2.5-4 is a Poisson expression for the pressure in terms of the nonlinear velocity term. The second step is to multiply equation 2.5-4 by $\dfrac{-k_\alpha}{k^2}$ and substitute the expression into 2.5-3. Collecting like terms and renaming dummy index variables as appropriate yields the solenoidal form of the momentum equations:

$$\left(\frac{\partial}{\partial t} + \upsilon_0 k^2\right)u_\alpha(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k})\frac{1}{(2\pi)^4}\int d^3j \ u_\gamma(\mathbf{k}-\mathbf{j},t)u_\beta(\mathbf{j},t) + f_\alpha(\mathbf{k},t) \qquad 2.5\text{-}5$$

where

$$M_{\alpha\beta\gamma}(\mathbf{k}) = \frac{1}{2i}P_{\alpha\beta\gamma}(\mathbf{k}), \quad P_{\alpha\beta\gamma}(\mathbf{k}) = \left\{ k_\beta D_{\alpha\gamma}(\mathbf{k}) + k_\gamma D_{\alpha\beta}(\mathbf{k}) \right\} \qquad 2.5\text{-}6$$

and

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$$D_{\alpha\beta}(k) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}$$

$$2.5\text{-}7$$

The RNG analysis makes extensive use of ensemble averaging and the properties of tensor quantities. If the flow field is homogeneous, the Fourier transform of the covariant velocity tensor is:

$$\langle u_\alpha(k)u_\beta(k')\rangle = \int d^3x \int d^3r \; e^{i(k\cdot x + k'\cdot r)} \langle u_\alpha(x)u_\beta(x+r)\rangle = \qquad 2.5\text{-}8a$$

$$\int d^3x \int d^3r \; e^{i(k\cdot x + k'\cdot r)} \langle u_\alpha(0)u_\beta(0+r)\rangle = Q_{\alpha\beta}(k)\delta(k+k') \qquad 2.5\text{-}8b$$

where $Q_{\alpha\beta}(k) = \langle u_\alpha(k)u_\beta(-k)\rangle$ and the homogeneity property was used in substituting 0 for x. If the flow field is further constrained to be isotropic (as well as incompressible) the spectral tensor $Q_{\alpha\beta}(k)$ may be written as:

$$Q_{\alpha\beta}(k) = D_{\alpha\beta}(k)Q(k) \qquad 2.5\text{-}9$$

where the scalar $Q(k)$ is only a function of the wavenumber magnitude, k.

The momentum equation 2.5-5 shows that every Fourier mode is coupled to every other mode in the Navier-Stokes equations and thus presents a difficult nonlinear problem. The process of energy transfer between different scales in the flow field is called the energy cascade. The main concepts of the energy cascade will be reviewed next.

2.6 Kolmogorov's energy cascade in turbulent flow

In turbulent flows there is a wide, continuous range of scales of motions, ranging from the size of the flow boundaries down to the dissipation length scale, also known

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as Kolmogorov's length scale, Λ_0^{-1}. The range of scales increases with Reynolds number since the dissipation scale decreases. Due to the rolling and swirling appearance of the flow, the turbulence is often described as consisting of 'eddies'. roughly circular motions with many different diameters corresponding to different flow scales.

2.6.1 The energy cascade

The eddies of different sizes interact in a non-linear fashion, and on average the large eddies are transformed into smaller ones. This process is called the energy cascade (first proposed by Richardson [2]). Vortex stretching, local interaction hypothesis, and return to isotropy are some of the key concepts generally used to explain the turbulent energy cascade.

Vortex stretching is a mechanism where eddies are distorted and stretched by the strain field of other eddies and their vorticity intensifies. Peter Bradshaw has proposed a vortex tree model to illustrate this concept [21]. A good discussion is presented by Tennekes and Lumley [21].

It is widely believed that the interactions most effective at transferring energy to smaller flow scales occur between eddies of similar sizes. Thus eddies are most effectively stretched by the strain field of slightly larger eddies leading to energy transfer to higher wavenumbers on average. However, for eddies very different in size, the smaller eddies are advected by the larger ones without much stretching action taking place. This advection leads to a change of phase of the small eddies but does not affect their vorticity or energy, while the effect on the large eddies is a small, viscous-like drain of energy. Primarily, it is the latter effect which is modeled by the eddy viscosity. For the present study of RNG eddy viscosity. the important questions are: i) just how local is the local energy transfer. or how rapidly does the rate of energy transfer change as the ratio of interacting wavenumbers increases; and ii) what is the asymptotic behavior of the energy transfer as the wavenumber ratio becomes very large or very small?

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The vortex streching mechanism tends not to transmit directional preferences so that smaller scales are more isotropic than large ones. This effect is called 'return to isotropy'.

For high Reynolds numbers, the spectrum of turbulent energy consists of three distinct regions as indicated in figure 2.1, the production range, the inertial region, and the dissipation range. The peak of the energy spectrum is located in the production region.

The production range. The largest eddies obtain energy from large scale pressure gradients or from the shear stress created by boundary motion relative to the fluid. The range of eddy sizes created in this way is called the production range. The eddies will tend to be aligned with the boundaries and so will have a directional preference. These scales of the turbulent flow in the production range are non-homogeneous, non-isotropic, and their energy content is flow dependent and thus non-universal. In the RNG method, the effects of the production range are replaced by a stirring force.

The inertial range. If the Reynolds number is sufficiently large, experiments have confirmed that over a range of eddy diameters known as the inertial range, the cascade approximately conserves kinetic energy as the molecular-viscous dissipation and the effects of the production range can be neglected. In the inertial range, the directional preferences of the large eddies have disappeared due to the many stages of interactions, and the turbulent motions which belong to these scales are homogeneous and isotropic. The rate of energy transfer from the wavenumbers lower than k to wavenumbers larger than k, $\Pi(k)$, is constant with wavenumber, so that $\Pi(k)$ = ε. According to Kolmogorov's hypothesis, the energy spectrum is dependent only on the energy transfer rate and on the wavenumber in this range. From dimensional arguments, the form of the energy spectrum is:

$$E(k) = C_k \varepsilon^{2/3} k^{-5/3} \qquad\qquad 2.6.1$$

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where C_k is the Kolmogorov constant, usually taken to be between 1.4 and 1.5. McComb [13]. Equation 2.6.1 is subject to minor corrections due to intermittency effects beyond the scope of the present study, Frisch [16].

An eddy of wavenumber magnitude k, has associated with it a characteristic local time, τ_k dependent only on k and E(k) and proportional to:

$$\tau_k \propto [k^3 E(k)]^{-1/2} \approx D^{-1/2} \varepsilon^{-1/3} k^{-2/3} \qquad 2.6.2$$

where D is the coefficient of the characteristic frequency of the eddy. According to a theory due to Kraichnan, the following relation holds:

$$D/C_k^2 = 0.1904 \qquad 2.6.3$$

The τ_k may be interpreted as a typical turnover time of an eddy at wavenumber k. It should be noted however that while 2.6.1 has an unambiguous meaning, the concept of the characteristic time is less well defined in the context of turbulence. and has been the subject of different interpretations and assumptions in the literature. Leslie [22]. The meaning of τ_k and its probable value will turn out to be important for the eddy viscosity investigation in the current work.

The upper boundary of the inertial range may be taken as the wavenumber Λ_0. given by:

$$\Lambda_0 = \gamma \left(\frac{\varepsilon}{v_0^3} \right)^{1/4} \qquad 2.6.4$$

where γ is usually assigned a value of about 0.1, Yakhot and Orzsag [3].

The inertial range of wavenumbers gets wider with increasing Reynolds number. and for sufficiently large Reynolds numbers, this range contains most of the turbulent energy. For the purpose of the current study, it is significant that the behavior of the inertial range of scales is universal and independent of the details of the

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production region. This justifies replacing the effects of boundaries and initial conditions by a stirring force and still recovering inertial range statistics.

The dissipation range. For wavenumbers larger than Λ_0, viscous dissipation dominates over the nonlinear interactions and the energy spectrum decreases at an approximately exponential rate. This third stage of the energy cascade extends from Λ_0 to Λ_m (Λ_m is the highest wavenumber present in the flow) and is called the dissipation range. For high Reynolds number flows where the inertial range is wide. the amount of energy contained in the dissipation range is relatively negligible.

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Figure 2.1 The turbulent kinetic energy spectrum.
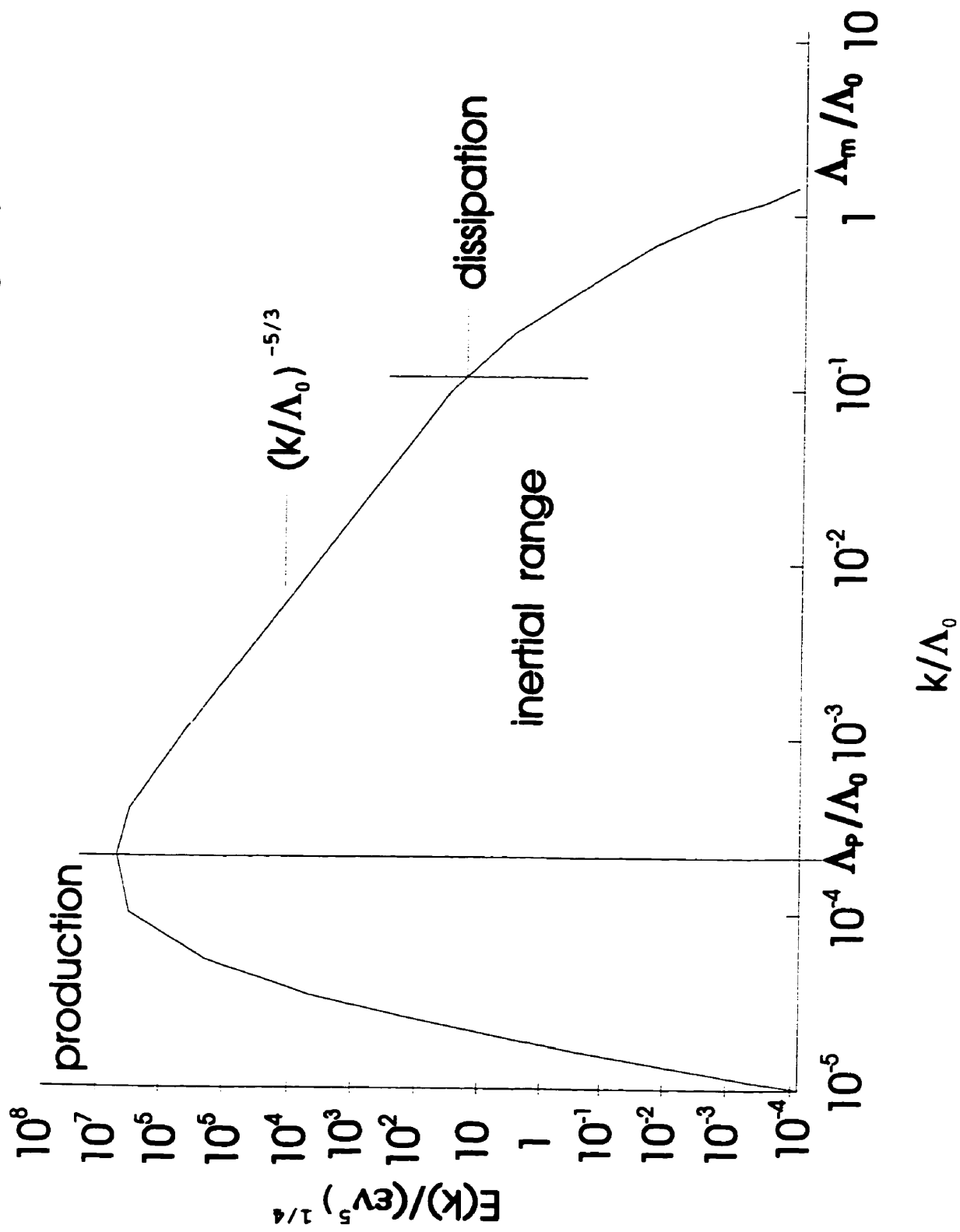
2.7 Large Eddy flow simulations

The objective of large eddy simulations is to correctly predict the behavior of the large scales of the flow, while scales smaller than the grid mesh. Δx. are filtered out. The formalism of the LES approach has been discussed in section 2.4. dealing with the filtering description of turbulence. The effects of the filtered scales on the resolved scales are represented by the subgrid model which approximates $T_{\alpha\beta}$, equation 2.4-4. This section introduces the methods of formulating the subgrid models in wavenumber space.

To set the stage for discussion of the large eddy simulations. the maximum resolved wavenumber magnitude, $\Lambda_c = \pi/\Delta x$, is selected. This is also called the cutoff wavenumber. The majority of LES models assume that the cutoff boundary between the resolved and filtered scales occurs in the inertial range of turbulent flow scales and this is also true for the RNG method.

The flow field $u(x,t)$ is set equal to $u(x,t)^< + u(x,t)^>$ where:

$$u(x,t) = u(x,t)^< + u(x,t)^>$$
2.7-1

$$u(x,t)^< = \sum_{|k|=0}^{\Lambda_c} u_\alpha(k,t)\exp\{ ik{\bullet}x \}$$
2.7-2

and

$$u(x,t)^> = \sum_{|k|>\Lambda_c}^{\Lambda_0} u_\alpha(k,t)\exp\{ ik{\bullet}x \}$$
2.7-3

for a finite-size flow domain with periodic boundaries. The flow variable. $u(x,t)^<$ constitutes the 'large eddies', while the $u(x,t)^>$ are the 'subgrid scales' and their effects must be modeled due to the non-linear coupling between these two variables. The decomposition above is called spectral splitting and may be viewed as a filtering operation using a sharp Fourier cutoff as discussed in section 2.4, so that $u(x,t)^<$ and $u(x,t)^>$ correspond to $\bar{u}$ and u'.

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The majority of the subgrid models are simple eddy viscosity closures but in recent years many new models have been proposed, Lesieur [19]. The most popular eddy viscosity model was proposed by Smagorinsky in 1963:

$$v(\Delta x) = (C_s \Delta x)^2 (2 \overline{S}_{\alpha\beta} \overline{S}_{\alpha\beta})^{1/2}$$

2.7-4

where:

$$\overline{S}_{\alpha\beta} = \frac{1}{2} \left\{ \frac{\partial \overline{u}_\alpha}{\partial x_\beta} + \frac{\partial \overline{u}_\beta}{\partial x_\alpha} \right\}$$

2.7-5

The Smagorinsky constant C_s is adjusted so that the ensemble averaged. subgrid kinetic energy dissipation is identical to ε. An approximate value for C_s is:

$$C_s \approx \frac{1}{\pi} \left(\frac{3C_k}{2} \right)^{-3/4}$$

2.7-6

Given $C_k \approx 1.4$ (Stull [2] discusses measurements of C_k for turbulence in the atmosphere made by Champagne et al 1977) relation 2.7-6 yields $C_s \approx 0.18$. However. the value $C_s = 0.1$ has been found to work better in practice. The Smagorinsky constant value of 0.19 has been obtained by YO [3] using the RNG method.

2.7.1 The concept and applications of spectral eddy viscosity

It is shown in equation 2-3.2 that a major role of the eddy viscosity. v_t is to provide the correct rate of energy drain from the resolved scales. In Fourier space. the spectral eddy viscosity, $v(k,\Lambda_c)$, replaces the nonlinear energy drain from a given. resolved mode ($k \leq \Lambda_c$), by all the subgrid modes $k \geq \Lambda_c$. Kraichnan [23]. This subgrid model is the research subject of the present work. This section introduces the concept of the spectral eddy viscosity and summarizes its successful forms and applications.

The spectral kinetic energy equation is:

$$\left(\frac{\partial}{\partial t} + 2v_0 k^2 \right) E(k,t) = \int_0^{\Lambda_0} \int_{\Delta k} dp \, dq \, S_E(k,p,q) + P(k) \qquad 2.7\text{-}7$$

where $S_E(k,p,q)$ is the non-linear, spectral energy transfer integrand. Δk is a notation for the constraint that k, p, and q form the sides of a triangle, and the magnitudes, p and q, range from 0 to Λ_0, with Λ_0 representing the highest wavenumber present in the flow. Also, $P(k)$ represents the external power input to the k mode.

For the purpose of large eddy simulations, the modes $\Lambda_c \le k \le \Lambda_0$ are the subgrid modes while $0 \le k \le \Lambda_c$ are the resolved modes. Using this spectral splitting approach, the limits of integration in equation 2.7-7 are changed to include only the resolved wavenumbers and the spectral eddy viscosity, $v(k,\Lambda_c)$ is introduced to compensate the energy transfer:

$$\left(\frac{\partial}{\partial t} + 2v_0 k^2 + 2v(k,\Lambda_c)k^2 \right) E(k,t) = \int_0^{\Lambda_c} \int_{\Delta k} dp \, dq \, S_E(k,p,q) + P(k) \qquad 2.7\text{-}8$$

The eddy viscosity $v(k,\Lambda_c)$ is obtained by isolating that portion of the integral in the right hand side of 2.7-7 where one or both of p and q is greater than k_c and dividing this term by $-2k^2 E(k,t)$. However, the integrand $S_E(k,p,q)$ is made up of the triple moments of the velocity and hence is unknown. Various theories approximate the form of $S_E(k,p,q)$ and of $v(k,\Lambda_c)$. Assuming that Λ_c is in the inertial range, the Eddy Damped Quasi-Normal Markovian (E.D.Q.N.M.) approximation due to Orszag [24] gives:

$$v(k,\Lambda_c) = 0.441 \; C_k^{-3/2} \left[\frac{E(\Lambda_c)}{\Lambda_c} \right]^{1/2} \tilde{v}(\frac{k}{\Lambda_c}) \qquad 2.7\text{-}9$$

where $E(\Lambda_c)$ is the kinetic energy spectrum at the cutoff Λ_c, and $\tilde{v}(\frac{k}{\Lambda_c})$ is a nondimensional eddy viscosity. It was noted in section 2.3 that the turbulent viscosity is proportional to a characteristic length scale and a velocity. ($v_t = v * \ell_m$). In the spectral domain, the length scale $\Lambda_c^{-1} \sim \Delta x$, and the characteristic velocity is $[\Lambda_c E(\Lambda_c)]^{1/2}$. The constants and the form of $\tilde{v}(\frac{k}{\Lambda_c})$ vary depending on the theory. A consensus in the research community is that $\tilde{v}(\frac{k}{\Lambda_c})$ is constant and equal to

1 for $k/\Lambda_c < \approx 0.3$, but increases for higher values of k and exhibits a cusp near $k/\Lambda_c = 1$, Kraichnan [23]. However, there is no consensus as to where the cusp starts, how steep it is, and what its peak value is. The cusp is taken as evidence of the local character of the energy transfer. For isotropic turbulence, the existence of the cusp (but not its exact form) has been confirmed by Lesieur & Rogallo and others (Lesieur & Metais [13]). Representative plots of two different theoretical forms of $\tilde{v}(\frac{k}{\Lambda_c})$ are shown in figure 2.2 where it is seen that the theory due to McComb [13] does not yield a cusp.

Figure 2.2 Comparison of subgrid eddy viscosities for isotropic turbulence, according to Kraichnan and McComb.
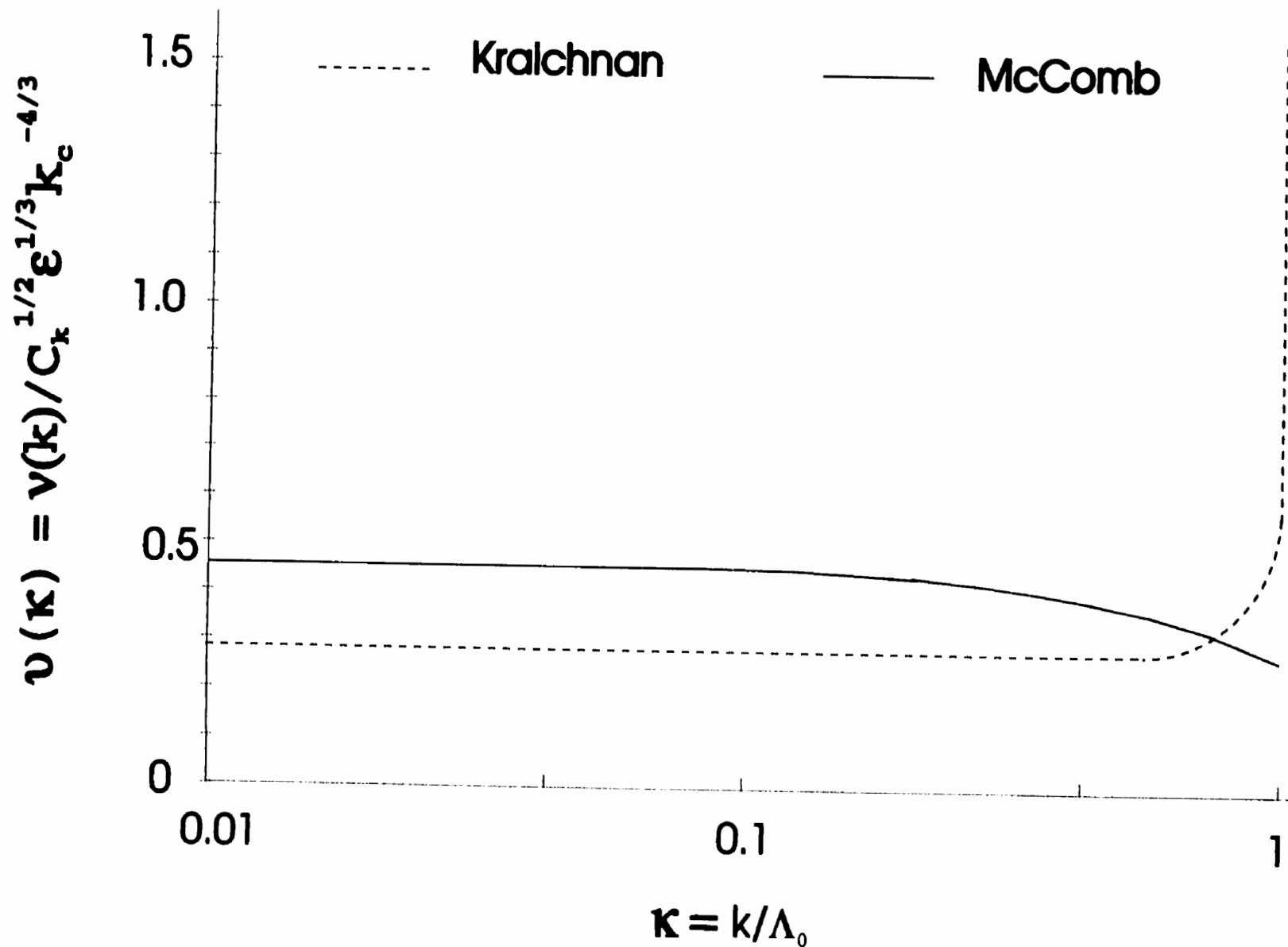
For many engineering problems, the geometry considered is such that transforming to spectral space is difficult. This will be the case when the physical extent of the domain is insufficiently large relatively to Λc^{-1} to assume isotropy of small scales and the boundaries cannot be assumed to be periodic. In cases where the solution domain is the physical space (x,t), the spectral eddy viscosity, $v(k,\Lambda_c)$, is not useful since individual values of k are not available. The remedy is to remove the cusp by averaging over k and then applying the criteria that the subgrid-scale kinetic energy dissipation be equal to ϵ, Leslie & Quarini [25]. The overall result is:

$$v(\Lambda_c) = \frac{2}{3} C_k^{-3/2} \left[\frac{E(\Lambda_c)}{\Lambda_c} \right]^{1/2}$$

2.7-10

This result is close to the Smagorinsky's model and to the YO RNG eddy viscosity model which will be derived in chapter 4.

2.7.2 Additional LES subgrid models

Many additional large eddy simulation models have been developed, both in wavenumber space, and in physical space. These models will not be discussed in detail here since they do not directly impact the RNG eddy viscosity on which this investigation is focused. The majority of the LES models are of the eddy viscosity type. However an important limitation of the eddy viscosity approach has been exposed by the DNS data obtained by Clark, McMillan & Ferziger, and by the experimental data of Liu et al as discussed by Lesieur & Metais [13]. Implicit in the eddy viscosity closure is the assumption of a one to one correlation between the subgrid scale stress and the large scale strain rate tensors. However, the findings by the above listed researchers shows very little correlation between the two tensors. This misrepresentation of the flow physics by the eddy viscosity closure provided motivation for the development of the scale similarity model, Bardina et al [26], and the dynamic model of Germano [27]. The implication for eddy viscosity models is that the resolved, near-grid scales will not be computed with the correct phase. This

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phase error will then propagate to larger scales, a phenomenon known as 'error backscatter', Lesieur [14]. Thus, the LES results will provide large scale flow statistics such as energy, maximum expected velocity magnitude, average mixing rates, etc. However, information related to phase such as the position of vortices will not be reliable. It is noted that this unpredictability problem affects all LES formulations regardless of which subgrid model is used, Lesieur & Metais [13]. The objective is to find a subgrid model that maximizes the decorrelation time between the 'real flow' and the LES simulation. These limitations of the eddy viscosity approach are beyond the scope of the present study.

3. Renormalization Group methods and turbulence modeling

The goal of this chapter is to outline the concepts of the Renormalization Group method as applied to turbulence. The presentation is qualitative and the mathematical details of the RNG are deferred to chapters 4, 5 and 6.

The concept of the Renormalization Group Method may be introduced with the example of the tree diagram shown in figure 3.1.a. The tree structure consists of layers of branches with the branches of each layer differing from the branches of the previous layer by only a scaling factor. Such a structure is called self-similar. Imagine now that random forces are acting on all the branches of the tree. Assume that the amplitude and frequency of the random forces are proportional to the size of the branches in each layer. Suppose that it is desired to determine the effective force acting on the trunk of the tree. One way to proceed is to first average the forces acting on the smallest branches (the first layer), then eliminate that layer from the diagram (figure 3.1.b) and add the averaged force to all the remaining branches. One then proceeds to average the random force component for the second layer (figure 3.1.c), add the increment to all the layers below and so on, until all the branches are removed and the effective force on the trunk is obtained. This is an example of the iterative scale removal procedure of the RNG method.

The above example illustrates important features and problems of the RNG method, primarily concerning the statistical averaging of only one layer at a time, a procedure called 'partial averaging'. Adding the average force from the first layer to the trunk of the tree is correct but adding this force to the adjacent, second layer in the same way is in error because each branch is affected by only two branches of the first layer. The forces transmitted between layers widely separated in scale can be treated accurately in average terms but large errors result if adjacent layers are treated in this manner. The question is whether such errors accumulate and affect the value of the resultant force on the tree trunk.

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Figure 3.1 Self-similar tree structure.
Iterative removal of branch layers.

a   b   c
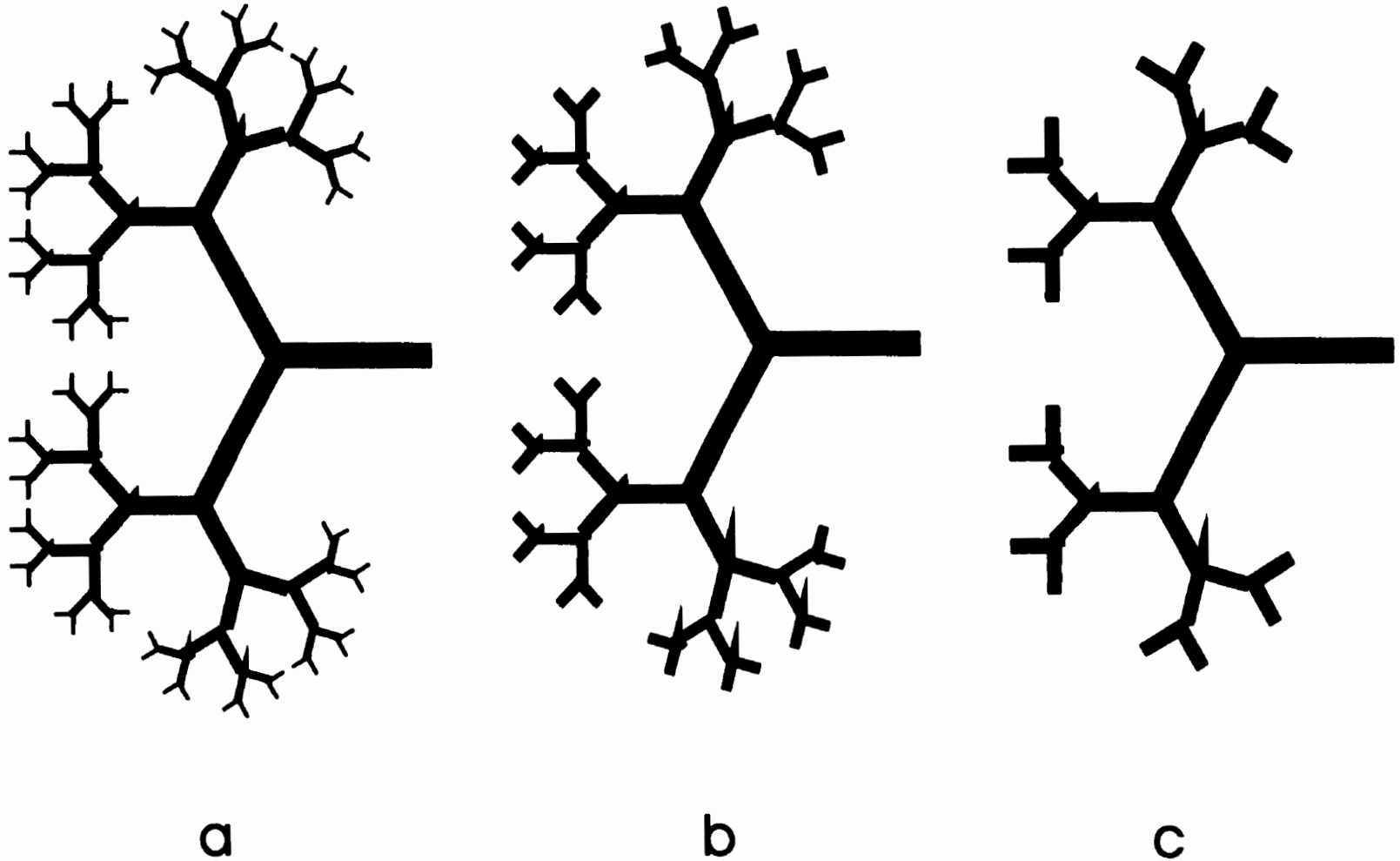
3.1 Renormalization Group (RNG) methods in turbulence

Objective: The study of turbulence yields scaling laws (such as $\epsilon^{2/3}k^{-5/3}$ for the energy spectrum) and numerical coefficients (such as the Kolmogorov constant C_k for the energy spectrum). The scaling laws may be obtained by a correct selection of the relevant parameters (like ϵ and k for the above example) and simple dimensional analysis. However, the numerical coefficients cannot be obtained in this manner and one must resort to experimental measurements or to a comprehensive analytical theory for which the RNG is a candidate.

The first objective of the RNG analysis is to derive the various coefficients of the turbulent flow such as the Kolmogorov constant, the skeweness coefficient. the turbulent Prandtl number etc. These turbulence constants are readily measured experimentally within a reasonable accuracy and will serve to verify the RNG results so that the method may be considered at least partially validated. The RNG values must be obtained without any empirically adjusted coefficients.

If the RNG predictions of the fundamental constants of turbulence are validated. there is justification to use this method for turbulence modeling. For the large eddy simulation of turbulence, the aim is to derive the Smagorinsky eddy viscosity constant or in a more advanced form a wavenumber function for the eddy viscosity. the k-ϵ model equations coefficients and so on.

Overview: The RNG procedure applied to fluid turbulence is similar to the example of the tree with random forces. First, the RNG involves the scaling away of a small range of the shortest wavelengths of the flow. This procedure is called spectral splitting. The average effect of the discarded rapid fluctuations on the longer wavelength coefficients is represented as an incremental change in the effective fluid viscosity in the momentum equation. The modified momentum equation is now redefined on a reduced range of wavenumbers and the procedure is iterated until the largest wavenumbers in the momentum equation fall within the range of the computational resolution.

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After the removal of a range of flow scales, the RNG method utilizes scaling transformations (hence the term 'renormalization'), to demonstrate that the equation coefficients approach a so called *fixed point*. At the fixed point, the renormalized viscosity becomes a constant. The fixed point offers a short cut to the final form of the eddy viscosity without going through a large number of iterations.

A number of issues and concepts associated with the RNG approach are discussed below.

The RNG turbulence model utilizes a stirring force. The derivations of RNG turbulence models require statistical symmetry and thus statistically homogeneous, isotropic, stationary turbulence. ZV [10] have also considered the case of freely decaying turbulence. It is to be noted that while the derivation uses idealized assumptions, the results are applied to practical flows. For stationary flows, the RNG method introduces a random stirring force as a source term in the Navier-Stokes equations. This force serves as a 'zero order' model of the turbulent fluctuations when the nonlinear terms in the Navier-Stokes equations are set to zero. Higher order approximations are constructed by 'nesting' convolution integrals of the forces to mimick the original non-linear terms. The stirring force has a Gaussian distribution of amplitude and is usually specified in terms of its correlation in Fourier space. This correlation of the forces is selected to reproduce inertial range flow statistics. Therefore, the stirring force is only a mathematical expedient to reproduce a stationary, homogeneous, unbounded flow field and to yield the correct first and second moment statistics. However, some researchers sought a physical interpretation of this force, generating debate in the RNG literature. Smith and Woodruff [7] interpret the force as the renormalized, particular class of non-linear interactions responsible for the inertial cascade rather than an actual external force. A different interpretation is offered by Lam [6], who asserted that the force cannot be scale invariant - thus challenging the inertial range analogy.

The idea that the statistical characteristics of real turbulence may be reproduced by a properly selected small scale force and without the knowledge of the boundaries or initial conditions of the flow has been labeled by YO as the

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'correspondence principle'. This term is used as an analogy to the well known correspondence principle of quantum physics where the statistical laws of quantum physics (microscopic level) yield the results of classical physics (macroscopic level).

The model problem vs. the turbulence problem. The current author is of the opinion that added insight may be obtained by alternating between two points of view. The 'model problem' is the Navier-Stokes equations with an external stirring force input. The statistical parameters of the force are varied and the resulting approximate solutions of the Navier-Stokes equations are examined. The 'turbulence problem' has the same mathematical form, but here the stirring force has the interpretation of a renormalized energy cascade. If one maintains a clear distinction between the two interpretations throughout the analysis, some of the ambiguity encountered in the literature may be avoided.

The RNG scale elimination is confined to the inertial range. The iterative spectral splitting and averaging of the RNG is confined to the inertial range. Λ_L - Λ_0. The RNG method ignores the dissipation range, (Λ_m - Λ_0 in Figure 2.1), and considers the turbulence to start at the upper end of the inertial range. Λ_0. (equation 2.6-3). It will be assumed that the Reynolds number is large so that the inertial range is wide and it contains most of the turbulent kinetic energy of the fluid. Under those conditions, the energy contained in the dissipation range may be ignored. For very large Reynolds numbers, some theorists neglect the production range as well, so that:

$$E = \int_0^{\Lambda_m} E(k)dk \approx \int_{\Lambda_L}^{\Lambda_0} E(k)dk \qquad 3.1\text{-}1$$

Elimination of a spherical shell. Since the flow field is considered to be homogeneous and isotropic, the Fourier space domain of the inertial range is a hollow sphere of inner radius Λ_L and outer radius Λ_0 (the inertial range extends equally in all directions of the wavenumber vector). Attention is focused on the outer shell of

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the sphere, where the thickness of the shell can be either finite or infinitesimally thin. If the method is iterated using shells of finite thickness, $\Lambda(l)\Delta l$, where $\Lambda(l)$ = $\Lambda_0 e^{-l}$, and Δl is a discrete increment of the scaling variable, the resulting RNG is called 'recursive', ZVH [8]. An example of such a method will be discussed in some detail in chapter 5. The YO method considers infinitesimally thin shells and converts the recursion into a differential equation for the eddy viscosity. The YO procedure will be presented in chapter 4. The effects of the Fourier velocity components in the shell on the momentum equation for wavenumbers smaller than the shell radius are replaced by an increment of the viscosity, Δv. Then the shell is discarded. The RNG procedure of Forster, Nelson and Stephen [28] (herein referred to as FNS) and YO [3] estimates the statistical effects of the small and fast eddies (high wavenumbers and frequency) on the large and slow ones so that the viscosity increment Δv is valid only for the Fourier velocity $u(k,\omega)$ where $|k| \longrightarrow 0$ and $\omega \longrightarrow 0$. ZVH [8] and ZV [10] claim that their recursive RNG yields an eddy viscosity valid for $0 \leq |k| \leq \Lambda(l)$ but their procedure and results have become a point of dispute. This issue will be examined in this study.

New non-linear terms are discarded. After the elimination of the wavenumber shell, new non-linear terms appear in the momentum equation, namely the triple and quadruple velocity products. In the version of RNG due to FNS [28], and adapted by YO [3] these new products were discarded to recover approximate momentum equations which appear the same as the initial Navier-Stokes equations. This allows for application of a recursive procedure, where the next shell may be removed with the same form of the results as before. Another version of RNG, originated by Rose [28] and expanded by ZVH [8] and ZV [10] retains and re-expands the triple velocity products on subsequent shells. The quadruple velocity products are almost universally discarded in the published literature, but the basis for doing so is not well understood except in the limit of very small wavenumbers. Still other variations of RNG as applied to turbulence make extensive use of field-theoretic methods [31], [16], and are beyond the scope of this study.

The Navier-Stokes equations may now be rescaled. Prior to removing the second shell, the original version of the RNG procedure rescaled the wavenumber domain to

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the original size and all the other variables were rescaled to suit. The purpose of rescaling was to demonstrate asymptotic limits for the various terms as a large number of shells was removed. In particular, the molecular viscosity becomes negligible and the rescaled coefficients of the momentum equations approach constant values. This condition is called the *fixed point* of the renormalized Navier-Stokes equations which now give self-similar solutions (related by a scaling factor) at any wavenumber. This condition is consistent with the characteristics of the inertial range of turbulence. The RNG terminology is derived from the rescaling process.

The unscaled eddy viscosity is of interest in turbulence modeling. As shells of wavenumbers are eliminated, the rescaled viscosity approaches a constant value, while the *unscaled* eddy viscosity increases. It is the latter quantity which is used to construct the subgrid turbulence model. For this reason, most recent versions of the RNG method applied to turbulence dispense with the rescaling process while retaining the terminology even though it strictly does not apply.

3.2 A brief history of RNG methods in turbulence

The RNG methodology has its origins in a wide range of scientific research. Part of the RNG mathematical framework is borrowed from earlier Renormalized Perturbation Theories of turbulence (RPT) due to Kraichnan [23], Edwards [13], and others. A comprehensive overview is given by McComb [13]. These sources gave rise to the use of stirring forces and the perturbation series. Other RNG tools such as spectral splitting, partial averaging, rescaling and fixed point analysis are taken over from quantum physics and from the study of critical phenomena. The works cited here are those most relevant to the current investigation.

The 'recursive' RNG vs the ϵ-RNG. Among the initial applications of RNG to turbulence is the work of Rose in 1976 [29] who applied the method to model the diffusion of a passive scalar in a randomly prescribed frozen velocity field. Shortly after, FNS [28] used the RNG method to study the large-distance and long-time behavior of velocity correlations generated by the Navier-Stokes equations for a

fluid stirred by a random force. These two works have originated two somewhat different schools of thought on RNG in turbulence, with some researchers labeling the method originated by Rose as the 'recursive' RNG and the analysis of FNS and their successors as the ϵ-RNG [8]. The two methods are similar in many respects but the 'recursive' RNG deals with finite shells of wavenumbers and numerically iterates the recursive algorithm until the desired range of scales is eliminated. The ϵ-RNG takes the limit of infinitesimally thin wavenumber shell so the recursion is converted into a differential equation which is then integrated. As the name implies, the parameter ϵ (to be defined in chapter 4) is of central importance in the ϵ-RNG theory. Initially a small value of ϵ is assumed to obtain convergence of a power series. However, ϵ is later set equal to 4 in order to recover the Kolmogorov energy spectrum, presumably invalidating the series convergence result.

The authors ZVH [8] and ZV [10] have criticized the ϵ-RNG based on the two contradictory ϵ values and have proposed the 'recursive' RNG as a viable alternative. However, ZV do not offer an alternative method to show series convergence. Instead, they justify truncating the series based on the analogous procedure for RNG applied to critical phenomena. For example, when the RNG was applied to the Kadanoff model of magnetism in metals near the Curie point, it was found that excellent agreement with experiment was obtained from a model using second order terms. Also, when higher order terms were included, the model results deviated from the experimental findings. [31]. However, while critical phenomena and inertial range turbulence have superficially similar governing equations, these are very different physical problems and success in one area does not guarantee satisfactory results with the other. [32].

It should be noted that since the ϵ-RNG originated by FNS was limited to very large scales, it really excluded the turbulence problem. Nonetheless their work was used to provide the foundations for the practical application of RNG to turbulence modeling. In 1979 DeDominicis and Martin [33] (herein referred to as DM) showed that a proper specification of the stirring force in the ϵ-RNG model will yield the Kolmogorov energy spectrum. Fournier and Frisch [35], [36] published several papers in the early 1980's extending the analysis of FNS and DM. In 1986, Yakhot and Orszag [3] built on the work of FNS and DM and took the ϵ-RNG method into the realm of

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practical application by deriving many of the hereto adjustable turbulent model parameters. The resulting turbulence models were the Smogarinsky eddy viscosity coefficient for large eddy simulations, and the RNG k-ε model.

The YO analysis has been somewhat controversial in the research community mainly because of the inconsistent ε value problem. The success of the YO k-ε model in simulating many industrial flows has renewed the interest of the research community in the theory during the late 80's and early 90's. Smith and Reynolds [5] have duplicated the YO analysis, identifing and correcting some algebraic errors. Subsequent corrections and extensions of the theory have been published by Dannevik. Yakhot, and Orszag, [36], and Yakhot and Smith [37]. Lam [6] analyzed the YO method by dimensional analysis and challenged some of its fundamental assumptions. Other insights and comments on the ε-RNG turbulence models have been made by Kraichnan [38]. McComb [13], and others.

Independently from the work of FNS and YO, researchers Zhou. Vahala and Hossain. (ZVH) have extended the work of Rose [29], in a series of publications [8]. [9]. and [10], to also develop a large eddy simulation eddy viscosity model which has not been implemented in practice. These researches claim that their analysis is free from the contradictions which cast doubt on the validity of the YO development. ZVH also claim that the eddy viscosity they obtained is valid for all resolved wavenumbers including those close to the cutoff between the resolved and unresolved scales. This means that the 'recursive' RNG is allegedly able to capture the 'local' interactions. energy transfer between wavenumbers of similar magnitude and straddling the cutoff boundary. Smith et al [39], and Carati [11] have disputed this claim and purport to show that only 'non-local' interactions may be handled with RNG methods. One of the aims of this study is to provide new evidence for one side of this argument if not to resolve it. Accordingly, the formulations and results of YO, ZVH, and Carati are reviewed in detail in chapters 4, 5, and 6.

4. The YO Application of RNG Theory to the Navier-Stokes Equations

The analysis given here follows that of Yakhot and Orszag [3], Smith and Reynolds [5], Yakhot and Smith [38], and Dannevik, Yakhot and Orszag [37] with some clarifications and explanations added. A brief discussion of the physical units of the various terms has been added following the later work of Lam [6]. Only the parts of the analysis which deal with derivation of the RNG relation between the stirring forces and the dissipation rate, the Kolmogorov constant, and the Smagorinsky eddy viscosity are presented here. YO went further to derive the RNG versions of the turbulent Prandtl number, the Batchelor constant, the skewness factor, and the constants for the k-ε model.

4.1 Mathematical framework

The objective is to determine the solution of the Navier Stokes equations 2.1-2 with a stirring force source term $f(x,t)$ and subject to the continuity equation 2.1-1. These equations are repeated here for convenience:

$$\frac{\partial u_\alpha}{\partial x_\alpha} = 0 \qquad\qquad 2.1\text{-}1$$

$$\frac{\partial u_\alpha}{\partial t} + u_\beta \frac{\partial u_\alpha}{\partial x_\beta} = -\frac{1}{\rho}\frac{\partial p}{\partial x_\alpha} + \upsilon_0 \frac{\partial^2 u_\alpha}{\partial x_\beta^2} + f_\alpha \qquad\qquad 2.1\text{-}2$$

The flow domain is assumed to be infinite in extent. The flow variables are assumed to be statistically homogeneous and stationary so that no boundary or initial conditions are specified. The Fourier space correlation of the stirring force. $f_\alpha(x,t)$ will be specified to yield a solution for the statistics of the inertial range of turbulence. It is noted that $f(x,t)$ is a force per unit mass with units of acceleration L/t^2.

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For most of the YO RNG analysis, time is Fourier transformed to frequency. ω (with the exception of the work of Dannevik et al [37]), so that one deals with $u(k.\omega)$. A tradition of the Renormalized Group Theory carried over from critical phenomena is to treat the number of space dimensions as a variable, d, where $d = 3$ is the case of practical interest. Formal solution of the momentum equation, written in terms of the Green's function operator $G_0(k,\omega)$, is given here with the addition of the bookkeeping parameter λ, (equal to unity) in front of the nonlinear term:

$$u_\alpha(k,\omega) = G_0(k,\omega)\left\{ \lambda M_{\alpha\beta\gamma}(k)\frac{1}{(2\pi)^{d+1}}\int d^dj\int d\Omega \ u_\gamma(k-j,\omega-\Omega)u_\beta(j,\omega) + f_\alpha(k,\omega) \right\} \qquad 4.1\text{-}1$$

where

$$G_0(k,\omega) = \frac{1}{(-i\omega + v_0k^2)} \qquad 4.1\text{-}2$$

The $M_{\alpha\beta\gamma}(k)$ has been defined in section 2.5 as:

$$M_{\alpha\beta\gamma}(k) = \frac{1}{2i}\left\{ k_\beta D_{\alpha\gamma}(k) + k_\gamma D_{\alpha\beta}(k) \right\} \qquad 2.5\text{-}6$$

with

$$D_{\alpha\beta}(k) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \qquad 2.5\text{-}7$$

It is noted that the Green's function operator, $G_0(k,\omega)$, would appear as $\int_{-\infty}^{t} G_0(k,t,\tau) \, d\tau$ in (k,t) domain (see chapter 5) or as $\int_{-\infty,\infty}^{\infty,t} \int G_0(|x-x'|,t,\tau) \, d^3x \, d\tau$ in (x,t) domain, allowing the usual symmetry w.r.t x and x'.

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The Fourier space correlation of the stirring forces is:

$$\langle f_\alpha(k,\omega) f_\beta(k',\omega') \rangle = 2W(k)(2\pi)^{d+1} D_{\alpha\beta}(k)\delta(k + k')\delta(\omega + \omega') \qquad 4.1\text{-}3$$

The amplitude of the force is assumed to be Gaussian with zero mean and uncorrelated in time. There is, however, a spatial correlation $W(|x\text{-}x'|)$ which is represented in Fourier space as $W(k)$. The Dirac delta functions $\delta(k + k')$ and $\delta(\omega + \omega')$ ensure a homogeneous and stationary force field, and $D_{\alpha\beta}(k)$ ensures isotropy and makes the force divergence free. Since the units of the force correlation in physical space are L^2/t^4, the units of $W(k)$ are L^5/t^3.

The wavenumber dependence of $W(k)$ is a power law:

$$W(k) = \begin{cases} 0 & \text{for } 0 < k < \Lambda_L \\[2ex] Wk^{-y} & \text{for } \Lambda_L \le k \le \Lambda_0 \end{cases} \qquad 4.1\text{-}4$$

where Λ_L is a minimum wavenumber required to avoid singular behavior of $W(k)$ and Λ_0 is the maximum wavenumber at which the force acts.

In the course of the analysis, the parameter $\epsilon = 4 + y - d$ will emerge as a variable which governs the solution of the RNG procedure. In terms of ϵ:

$$W(k) = Wk^{4-d-\epsilon} \qquad 4.1\text{-}5$$

and setting $d = 3$,

$$W(k) = Wk^{1-\epsilon} \qquad 4.1\text{-}6$$

It is noted that for dimensional consistency, W must have the units $L^{6-\epsilon}/t^3$, Lam [6].

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4.2 Overview of the RNG procedure

The mathematical steps of the RNG method are summarized below:

1) At the start of the procedure, the wavenumber domain is split into an outer shell; $\Lambda_0 e^{-\Delta \ell} \leq k < \Lambda_0$, and an inner sphere of low wavenumbers, $0 < k < \Lambda_0 e^{-\Delta \ell}$.

2) Next, the velocity is spectrally split:

$$u(k,\omega) = u^<(k,\omega) + u^>(k,\omega) \qquad \qquad 4.2\text{-}1$$

with $u^>(k,\omega)$ defined on the outer shell of wavenumbers, and $u^<(k,\omega)$ defined on the interior. Analogous splitting applies to the stirring force.

3) Two coupled momentum equations are then written for $u^<(k,\omega)$, and $u^>(k,\omega)$.

4) The high wavenumber velocity is expanded as a series in powers of an arbitrary parameter λ:

$$u^>(k,\omega) = u^{>0}(k,\omega) + \lambda u^{>1}(k,\omega) + \lambda^2 u^{>2}(k,\omega) \ldots \qquad \qquad 4.2\text{-}2$$

It should be noted that this series expansion applies to the velocity and implicitly also to the pressure, but not to the stirring force.

5) In order to find expressions for $u^{>0}(k,\omega)$, $u^{>1}(k,\omega)$, etc., the series expansion for $u^>(k,\omega)$ is substituted into the $u^>(k,\omega)$ momentum equation. Matching powers of λ yield terms in the series.

6) The series for $u^>(k,\omega)$ is substituted into the momentum equation for $u^<(k,\omega)$. The resulting series expansion for $u^<(k,\omega)$ is truncated to retain terms of $O(\lambda^2)$.

7) Ensemble averaging is applied to the high wavenumber forces in the equation for $u^<(k,\omega)$. This yields a viscosity correction term and a triple velocity product.

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$\int u^< \int u^< u^<$. The triple product is discarded in the YO method but is important in the methods of ZVH and Carati. It is also crucial to the present investigation.

8) The viscosity correction term is integrated with respect to frequency using contour integration and calculus of residues. This term is also integrated with respect to wavenumber over the spherical shell.

9) The correction is added to the viscosity and the procedure may be repeated in a recursive manner. At an intermediate step in the process, the wavenumber shell is $\Lambda(\ell)e^{-\Delta\ell} \leq k < \Lambda(\ell)$, where $\Lambda(\ell) = \Lambda_0 e^{-\ell}$. Any parameter with a zero subscript becomes instead a function of ℓ such as $v_0 \longrightarrow v(\ell)$. The eddy viscosity recursion is:

$$v(\ell+\Delta\ell) = v(\ell) + \Delta v(\ell+\Delta\ell) \qquad \qquad 4.2\text{-}3$$

10) YO convert equation 4.2-3 into a differential equation for $v(\ell)$ which is then integrated with respect to ℓ from zero ($v(0) = v_0$, the molecular viscosity) to ℓ_c where $\Lambda(\ell_c) = \Lambda_c$, the maximum wavenumber resolved by the grid.

11) The eddy viscosity derived in this manner contains the arbitrary coefficient of the stirring force correlation, W (see equations 4.1-3 and 4.1-4). YO substitute W = 1.59 $\pi^2\varepsilon$, relating the power input to the energy dissipation to complete the derivation.

Each step in the RNG process will be discussed in some detail in the following sections.

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4.3 The spectral splitting

The first step in the RNG procedure is to spectrally split the velocity and stirring force:

The Fourier velocity coefficients of the 'high' and 'low' wavenumbers are:

$$u_\alpha{}^<(k,\omega) = \begin{cases} u_\alpha(k,\omega) & \text{for } 0 < k < \Lambda_0 e^{-\Delta\ell} \\ 0 & \text{for } \Lambda_0 e^{-\Delta\ell} \le k \end{cases}$$

and

$$u_\alpha{}^>(k,\omega) = \begin{cases} 0 & \text{for } 0 < k < \Lambda_0 e^{-\Delta\ell} \\ u_\alpha(k,\omega) & \text{for } \Lambda_0 e^{-\Delta\ell} \le k < \Lambda_0 \end{cases}$$

An analogous definition applies for the stirring forces, $f^<(k,\omega)$ and $f^>(k,\omega)$. In addition, some authors [13] spectrally split the projection operator $M(k)$ into $M^<(k)$ and $M^>(k)$. Since the functional forms of $M^<(k)$ and $M^>(k)$ are the same, the purpose of this notation is to keep track of the domain of k when $M(k)$ appears in convolution integrals. The Green's function $G(k,\omega)$ also has the same form for all k and is usually not spectrally split in the literature.

For the purpose of comparison with the recursive RNG and for the discussion of higher order series expansion terms, the thickness of the eliminated shell will be needed.

$$\text{shell thickness} = \Lambda_0(1 - e^{-\Delta\ell}) \approx \Lambda_0 \Delta\ell \quad \text{when } \Delta\ell << 1 \qquad 4.3\text{-}2$$

The functions $u^<(k,\omega)$ and $u^>(k,\omega)$ are referred to as the 'high' and 'low' Fourier components of the velocity. These high and low Fourier coefficients of velocity and stirring force are substituted into equation 4.1-1, to generate two coupled momentum equations, 4.3-3 and 4.3-4:

$$u^<(k,\omega) = G_0(k,\omega) \left\{ \lambda M^<(k) \frac{1}{(2\pi)^{d+1}} \int d^d j \int d\Omega \left\{ u^<(k-j,\omega-\Omega)u^<(j,\Omega) + \right. \right.$$

$$\left. \left. 2u^<(k-j,\omega-\Omega)u^>(j,\Omega) + u^>(k-j,\omega-\Omega)u^>(j,\Omega) \right\} + f^<(k,\omega) \right\} \qquad 4.3\text{-}3$$

$$u^>(k,\omega) = G_0(k,\omega) \left\{ \lambda M^>(k) \frac{1}{(2\pi)^{d+1}} \int d^d j \int d\Omega \left\{ u^<(k-j,\omega-\Omega)u^<(j,\Omega) + \right. \right.$$

$$\left. \left. 2u^<(k-j,\omega-\Omega)u^>(j,\Omega) + u^>(k-j,\omega-\Omega)u^>(j,\Omega) \right\} + f^>(k,\omega) \right\} \qquad 4.3\text{-}4$$

In the above equations, the directional subscripts have been omitted. The wavenumber integration is carried out over the sphere of radius Λ_0 while the frequency integration domain is $\pm\infty$. The cross-terms $u^<(k-j,\omega-\Omega)u^>(j,\Omega) + u^>(k-j,\omega-\Omega)u^<(j,\Omega) = 2u^<(k-j,\omega-\Omega)u^>(j,\Omega)$ due to the symmetry of the integration domain.

4.4 The high wavenumber velocity series

The objective of the RNG procedure is to model the average effects of the high wavenumber velocity coefficients, $u^>$, on the low wavenumber coefficients, $u^<$ as an increment of the viscosity, then to eliminate the shell and iterate the process. The RNG method achieves this by representing $u^>$ as a function of $u^<$ and $f^>$, then averaging over the $f^>$ terms. The exact functional form of $u^>(u^<,f^>)$ is unknown since it requires the solution of 4.3-3 and 4.3-4. The RNG method approximates $u^>(u^<,f^>)$ as a series expansion in the artificial parameter λ:

$$u_\alpha^>(k') = u_\alpha^{>0}(k') + \lambda u_\alpha^{>1}(k') + \lambda^2 u_\alpha^{>2}(k') + \dots \qquad 4.4\text{-}1$$

where $k' = k,\omega$.

Since $\lambda = 1$, this is not a conventional perturbation series. It will be shown later that with the proper non-dimensionalization of equations 4.3-3 and 4.3-4. the

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effective expansion parameter $\bar{\lambda}$ (to be defined later) will be a small number under certain conditions.

The series expansion (4.4-1) is then substituted into equation (4.3-4) for the high wavenumber velocity. The notation is further simplified by using the abbreviated integral symbol $\int dj' = \dfrac{1}{(2\pi)^{d+1}} \int\limits_{j \leq \Lambda_0} d^dj \int d\Omega$. To obtain an explicit expression for $u^{>}$. the coefficients of λ^i are equated. The first two terms in the series are:

$$u^{>0}(k') = G_0(k')f^{>}(k')$$

4.4-2

$$u^{>1}(k') = G_0(k')M^{>}(k) \int dj' \left\{ u^{<}(k'-j')u^{<}(j') + \right.$$

$$\left. 2u^{<}(k'-j')u^{>0}(j') + u^{>0}(k'-j')u^{>0}(j') \right\}$$

4.4-3

Higher order terms may be obtained in an analogous manner but conventional versions of the RNG method utilize only the first two terms of the series. It is noted that $u^{>1}(k')$ may easily be written in terms of G_0, $f^{>}$, and $u^{<}$. This fact is exploited in section 4.6 since the contribution to the viscosity will be an average over $f^{>}f^{>}$.

The physical interpretation of the series 4.4-1 depends on the interpretation of the stirring force. It is difficult to provide a physical interpretation for the various terms of the series 4.4-1 because setting $\lambda = 0$ violates the principle of Galilean invariance and equation 4.4-2 no longer represents $\Sigma F = ma$. Equation 4.4-2 describes the individual decay of each Fourier mode due to viscosity in the absence of non-linear interactions. This is a diffusion equation for momentum with a forcing source term. The implications of Galilean invariance for the RNG have been discussed by Smith et al [30], and will be examined further later in this work.

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4.5 The low wavenumber velocity series

Next the high wavenumber series is substituted into the low wavenumber velocity equation in order to approximate the coupling between $u^<$ and $u^>$ as an interaction between $u^<$ and $f^>$ up to a given power of λ. This process yields the low wavenumber series in λ. Traditionally, in the RNG methods, terms up to λ^2 are retained:

$$u^<(k') = G_0(k')\lambda M^<(k)\int dj' u^<(k'-j')u^<(j') \; +$$

$$\underbrace{\hphantom{G_0(k')\lambda M^<(k)\int dj' u^<(k'-j')u^<(j')}}_{4.5\text{-}1a}$$

$$G_0(k')\lambda M^<(k)\int dj'2u^<(k'-j')\left(u^{0>}(j')+\lambda u^{1>}(j')+O(\lambda^2)\right) \; +$$

$$\underbrace{\hphantom{G_0(k')\lambda M^<(k)\int dj'2u^<(k'-j')(u^{0>}(j')+\lambda u^{1>}(j')+O(\lambda^2))}}_{4.5\text{-}1b}$$

$$G_0(k')\lambda M^<(k)\int dj'\left(u^{0>}(k'-j')+\lambda u^{1>}(k'-j')+O(\lambda^2)\right)\left(u^{0>}(j')+\lambda u^{1>}(j')+O(\lambda^2)\right) \; +$$

$$\underbrace{\hphantom{G_0(k')\lambda M^<(k)\int dj'(u^{0>}(k'-j')+\lambda u^{1>}(k'-j')+O(\lambda^2))(u^{0>}(j')+\lambda u^{1>}(j')+O(\lambda^2))}}_{4.5\text{-}1c}$$

$$\underbrace{G_0(k')f^<(k')}_{4.5\text{-}1d} \hspace{4cm} 4.5\text{-}1$$

4.6 Partial averaging of the high wavenumber stirring forces

Terms containing high wavenumber components of the stirring forces will now be averaged in order to obtain a statistical contribution to the eddy viscosity. Let $<..>$ be the symbol for the partial ensemble averaging operator which acts only on the high wavenumber components of the stirring forces. The variable $f^<$ is unaffected. A physical justification of this procedure is that high wavenumber fluctuations evolve much faster than the very large fluctuations at $k \longrightarrow 0$.

To the knowledge of the current author, all of the RNG literature uses the property:

$$\langle u^< u^> \rangle \rangle = u^< \langle u^> \rangle \rangle \qquad\qquad 4.6\text{-}1$$

and, in particular:

$$\langle u^< u^{>0} \rangle \rangle = u^< \langle G f^> \rangle \rangle = 0 \qquad\qquad 4.6\text{-}2$$

The key points to note about equation 4.6-1 are that i) $u^<$ (as well as $f^<$) is unaffected by the averaging operation, and ii) $\langle u^> \rangle \neq 0$ since the averaging affects only the high wavenumber stirring forces (as opposed to the high wavenumber velocities). The relation 4.6-1 is a standard property for the RNG partial averaging as published in the literature. However, this rule will be challenged in chapter 8 of this work for the following reason: The series expansion of $u^<(k,\omega)$ in powers of λ (4.5-1) contains $f^>$ components which will 'react' when averaged with the $f^>$ components of $u^>$ to any desired even moment of the forces. For example, relation 4.6-2 is only true to zero order in the λ expansion of $u^<$, when $u^<$ is represented by $G f^<$. This leads to the conclusion that the result of partial averaging depends on the order of λ expansion of $u^<$ and $u^>$.

The properties of the partial averaging operator $\langle .. \rangle \rangle$ will be discussed in chapter 8 of this study. At that stage, a modified RNG method will be proposed, where equation 4.6-2 is not used.

However, the work of YO uses relation 4.6-2 to carry out the partial averaging, as well as using the fact that the stirring forces are Gaussian variables with zero mean so that all the odd moments are zero. The results of the averaging are easier to demonstrate by first applying the procedure to the high wavenumber series 4.4-1 and then proceeding to average the low wavenumber series 4.5-1. Furthermore, according to 4.6-1, the sequence of averaging and substitution is interchangeable, an observation made by ZV [10] which is also challenged in the current investigation.

Averaging the first two terms of 4.4-1, gives:

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$$\langle u^>(k')\rangle\rangle = \langle u^{>0}(k')\rangle\rangle + \lambda\langle u^{>1}(k')\rangle\rangle + \lambda^2\langle u^{>2}(k')\rangle\rangle + \dots \qquad 4.6\text{-}3$$

with

$$\langle u^{>0}(k')\rangle\rangle = G_0(k')\langle f^>(k')\rangle\rangle = 0 \qquad 4.6\text{-}4$$

and

$$\langle u^{>1}(k,\omega)\rangle = G_0(k,\omega)M^>(k)\int dj'\ u^<(k\text{-}j,\omega\text{-}\Omega)u^<(j,\Omega) \qquad 4.6\text{-}5$$

where the last result has been obtained using the property 4.6-1 and

$$M^>(k)\int dj'\ \langle u^{>0}(k'\text{-}j')u^{>0}(j')\rangle\rangle = M^>(k)\int dj'G(k'\text{-}j')G(j')\langle f^>(k'\text{-}j')f^>(j')\rangle\rangle$$

$$= M^>(k)\int d^dj G(k'\text{-}j')G(j')W(j)^>\delta(k\text{-}j+j) = 0 \qquad 4.6\text{-}6$$

This is because $\delta(k\text{-}j+j) = \delta(k)$ and this Dirac Delta function does not equal zero only if $k = 0$. However, $M(k)$ is proportional to k and so $M(0) = 0$.

As noted by YO [3], the root mean square of terms of the form 4.6-6 does not vanish but constitutes an 'induced' force at the low wavenumber k due to two large wavenumbers, j and $k\text{-}j$. This induced force represents a backward cascade of energy (backscatter) from large wavenumbers to small wavenumbers. Carati [40] has incorporated the induced forces into an Iterative Filtering method, very similar to RNG.

Using the rules 4.6-1 to 4.6-5 to average equation 4.5-1 leaves term 4.5-1a unaltered, so that $\langle 4.5\text{-}1a\rangle\rangle = 4.4\text{-}5a$.

The average of term 4.5-1b takes the form:

$$\langle 4.5\text{-}1b\rangle\rangle = G_0(k')\lambda^2 M^<(k)\int dj' 2u^<(k'\text{-}j')M^>(j)\int dp' u^<(j'\text{-}p')u^<(p') \qquad 4.6\text{-}7$$

This triple velocity product $u^<u^<u^<$ is of central importance in this work. However, in the original YO analysis, the triple products are discarded so that the 'renormalized' equations can have the same form as the original Navier-Stokes equations.

Turning now to term 4.5-1c, the multiplication is performed and only terms of second order or lower in λ are retained. These terms are:

$$\lambda M^<(k)\int dj' \langle u^{o>}(k'\text{-}j')u^{o>}(j')\rangle, \quad \lambda^2 M^<(k)\int dj' \langle u^{l>}(k'\text{-}j')u^{o>}(j')\rangle, \text{ and}$$

$$\lambda^2 M^<(k)\int dj' \langle u^{o>}(k'\text{-}j')u^{l>}(j')\rangle. \text{ The average of the first term is zero. (equation}$$

4.6-6), and its root mean square is the induced force. The second and third term are combined by switching the wavenumber labels of (k-j) and (j). Noting that $G_0(\text{-}j')G_0(j') = |G_0(j')|^2$, the term $\langle 4.5\text{-}1c\rangle\rangle$ equals:

$$\langle 4.5\text{-}1c\rangle\rangle = u^<(k')8G_0(k')\lambda^2 M^<(k)\int dj' G_0(k'\text{-}j')M^>(k\text{-}j)|G_0(j')|^2 D(j)W^>(j)$$

$$= -u^<(k')\Delta v k^2 \qquad 4.6\text{-}8$$

Here $\Delta v = \dfrac{1}{k^2} 8G_0(k')\lambda^2 M^<(k)\int dj' G_0(k'\text{-}j')M^>(k\text{-}j)|G_0(j')|^2 D(j)W^>(j)$ is a correction to the viscosity owing to the averaged out shell of wavenumbers. The low wavenumber velocities can now be written as:

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$$(-i\omega + (\upsilon_0 + \Delta v)k^2)u^<(k') = \lambda M^<(k) \int dj' \ u^<(k'-j')u^<(j') +$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{<4.6-9a>}$$

$$\lambda^2 M^<(k)\int dj' 2u^<(k'-j')G_0(j')M^>(j)\int dp' u^<(j'-p')u^<(p') \ + \ f^< + \ O(\lambda^3) \qquad\qquad 4.6-9$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{<4.6-9b>}$$

Equation 4.6-9 is the momentum equation without any high wavenumber velocity components since these have been replaced by an average linear term, Δv. The triple velocity products, <4.6-9b> are discarded in the YO procedure discussed here. but are retained in other versions of the RNG procedure to be described later in this study. If the term 4.6-9b is dropped, the form of the resulting momentum equation is the same as the original Navier-Stokes equations.

4.7 Integration of frequency and wavenumber

An explicit expression for Δv is obtained by first substituting into 4.6-8 the explicit forms of the Green's functions as $G_0(j') = \dfrac{1}{-i\Omega + j^2 v}$ and $G_0(k'-j') = \dfrac{1}{i(\omega-\Omega) + |k-j|^2 v}$ and then performing the convolution integral over the frequency domain Ω:

$$u_\alpha^<(k,\omega)k^2\Delta v = u_\rho^<(k,\omega) \times$$

$$\left\{ \lambda^2 \frac{8W\pi}{v_0} M^<_{\alpha\beta\gamma}(k) \int\limits_0^{2\pi} d\phi \int\limits_0^\pi \sin\theta \ d\theta \int\limits_{\Lambda e^{-\Delta\ell}}^{\Lambda_0} j^2 \ dj \frac{M^>_{\gamma\sigma\rho}(k-j)D_{\beta\sigma}(j)j^{-\gamma-2}}{-i\omega + v_0 j^2 + v|k-j|^2} + O(\lambda^3) \right\} \qquad 4.7-1$$

The convolution integral over Ω has been evaluated over the limits $-\infty$ to ∞ using calculus of residues. Physical limits on the frequency are of the order of the

inverse of the Kolmogorov time scale; $(v_0\varepsilon)^{1/2}$, so that, approximately. $|\Omega| \le 2\pi(\varepsilon/v_0)^{1/2}$.

The subscript notation has been restored in equation 4.7-1 since it will serve to show that $u_\alpha{}^<(k,\omega)$ is picked out of the $u_\rho{}^<(k,\omega)$ in the process of angular integration.

Note that the k^2 factor will be extracted from the right hand side of 4.7-1. since $M^<_{\alpha\beta\gamma}(k)$ is proportional to k and the other k factor is obtained from the angular integration of $M^>_{\gamma\sigma\rho}(k-j)D_{\beta\sigma}(j)$.

The evaluation of the wavenumber integral is facilitated by changing the variable $j \longrightarrow j + k/2$. Prior to integration, the above expression is expanded in a power series of the small parameter $\dfrac{k}{\Lambda_0}$, and terms containing $\left\{\dfrac{k}{\Lambda_0}\right\}^2$ or higher powers are dropped. Also, the $i\omega$ term is dropped, implying that $\dfrac{\omega}{v_0 j^2} \ll 1$. Geometrically. the wavenumber integration domain is the overlap of two spherical shells of thickness $\Lambda\Delta\ell$ (assuming a small $\Delta\ell$ for a thin shell) with centers offset by a distance k. as shown in figure 4.1. The YO calculation assumes complete overlap of the shells. This implies that the magnitude of the low wavenumber is negligible relative to the thickness of the eliminated shell ($\dfrac{k}{\Lambda_0\Delta\ell} \ll 1$ or alternatively $\dfrac{k}{\Lambda_0} \ll \Delta\ell$ Smith et al [39]). The integration is carried out in spherical coordinates. first with respect to angles and then with respect to the magnitude. The angular integration associated with the vector j leads to a Kronecker delta which picks out the α component of the velocity vector $u_\rho{}^<(k,\omega)$. The details of this procedure are well explained by Leslie [22]. The final result is given as:

$$\Delta v(\Delta\ell) = A_d\frac{\exp(\varepsilon\,\Delta\ell) - 1}{\varepsilon}\left\{\frac{\lambda^2 W}{v_0{}^2\Lambda_0\varepsilon}\right\}$$

4.7-2

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$u^{>0}(k-J)u^{>1}(J),\ u^{>1}(k-J)u^{>0}(J)$

Δv contribution

K-J   shell

J   shell

J

K

$u^{<}(J)u^{>}(k-J)$ $u^{>}(J)u^{<}(k-J)$
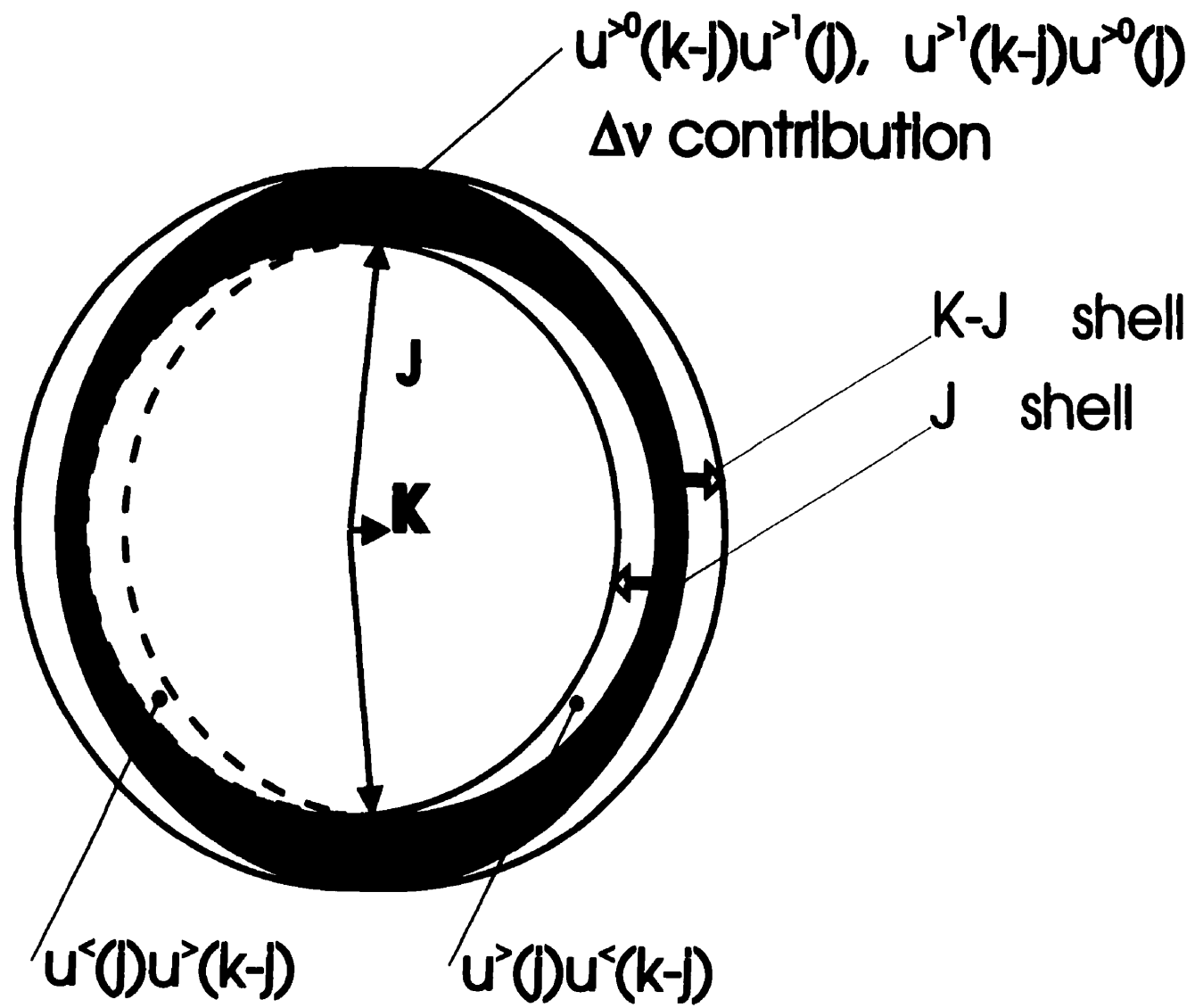
Figure 4.1 The wavenumber integration domain for k < shell thickness.

where it is recalled that W is the amplitude of the stirring force correlation (equation 4.1-4):

$$A_d = \overline{A_d}\frac{S_d}{(2\pi)^d}; \qquad \overline{A_d} = \frac{1}{2}\frac{d^2 - d - \in}{d(d + 2)}; \qquad \in = 4 + y - d \qquad\qquad 4.7\text{-}3$$

and S_d is the area of the d - dimensional unit sphere.

Reynolds and Smith [5] have shown that to consistently retain terms of the same power of $\in$ (that is $\in^1$) it is necessary to expand $\overline{A_d}$ in powers of $\in$:

$$\overline{A_d} = \overline{A_d}^0 + \overline{A_d}^1 \in = \frac{d^2 - d}{2d(d + 2)} + \frac{-1}{2d(d + 2)}\in$$

and use the zero order term, $\overline{A_d}^0$ in place of $\overline{A_d}$ (and A_d^0 in place of A_d) in the eddy viscosity calculations, again assuming that $\in$ is small. This correction has been incorporated into the RNG literature, [6].

$$v_1 = v_0 + A_d^0 v_0 \frac{\exp(\in\Delta\ell) - 1}{\in}\left(\overline{\lambda_0}^2 + O(\overline{\lambda_0}^4) \right) \qquad\qquad 4.7\text{-}4$$

where v_1 is the combined molecular and eddy viscosity obtained after removing the band $\Lambda_0\exp[-\Delta\ell] < k < \Lambda_0$ of wavenumbers. The parameter $\overline{\lambda_0}$ is given as:

$$\overline{\lambda_0}^2 = \frac{\lambda^2 W}{v_0^3 \Lambda_0 \in} \qquad\qquad 4.7\text{-}5$$

The parameter $\overline{\lambda_0}$ has been identified by FNS [28] as the effective expansion parameter of the non-dimensionalized Navier-Stokes equations.

Examining equation 4.7-4, the term $O(\overline{\lambda_0}^4)$ comes from the contribution of the fourth moment of the stirring forces, or the so called 'induced' force.

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It is noted that the eddy viscosity correction is known to $O(\overline{\lambda}_0{}^4)$ but the low wavenumber momentum equation has been truncated at $O(\overline{\lambda}_0{}^2)$. The improvement in accuracy (assuming that $\overline{\lambda}_0$ is smaller than one) is due to the fact that the terms of order $O(\overline{\lambda}_0{}^3)$ go to zero. Some terms of order $O(\overline{\lambda}_0{}^3)$ are odd moments of the Gaussian forces and vanish as a result of averaging. Other $O(\overline{\lambda}_0{}^3)$ terms go to zero as $\frac{k}{\Lambda_0} \longrightarrow 0$ due to Galilean invariance requirements, FNS [28].

To show that $\overline{\lambda}_0$ indeed plays the role of an effective Reynolds number, the characteristic velocity scale is selected as:

$$v_0 = \frac{W^{1/2} \; \Lambda_0{}^{1-\epsilon/2}}{v_0{}^{1/2}} \qquad\qquad 4.7\text{-}6$$

The form of v_0 has been modified by the current author so that it is dimensionally correct (L/t) for arbitrary ϵ. This modification takes into account that the units of W are $L^{6-\epsilon}/t^3$ (Lam [6]). It is noted that since the units of W vary with ϵ, W must be a function of ϵ, Lam [6], but its functional form is unknown. Published literature use $v_0 = \dfrac{W^{1/2} \; \Lambda_0{}^{-1}}{v_0{}^{1/2}}$, valid only for $\epsilon = 4$. The length scale is $l_0 = \Lambda_0{}^{-1}$ and the time scale is $\Lambda_0{}^{-2}/v_0$. These parameters are characteristic of the eliminated wavenumber shell. Non-dimensionalizing the Navier-Stokes equations with respect to these values gives a form of the Reynolds number $\dfrac{v_0 l_0}{v_0} = \dfrac{W^{1/2} \; \Lambda_0{}^{-\epsilon/2}}{v_0{}^{3/2}} = \overline{\lambda}_0(\epsilon)$. It is desired that $\overline{\lambda}_0(\epsilon)$ is less than one to justify the series expansion. Clearly, this may be achieved if Λ_0 is treated as an independent parameter and chosen sufficiently large. However, this choice of Λ_0 may fall in the dissipation range where the use of $W \sim \epsilon$ is no longer appropriate. For further discussion see Carati [40].

Lam [6] also proposed that W should be a function of Λ. This interpretation is problematic (Smith and Woodruff [7]) since W is intended to correspond to ε in the inertial range and hence be independent of wavenumber. A possible scenario is that:

$$\frac{\partial W(\Lambda, \in)}{\partial \Lambda}\bigg|_{\in = 4} = 0$$

After the range of shells Λ_0 to $\Lambda_0 e^{-\ell}$ are eliminated, the parameters v_0, z_0, l_0 . and $\overline{\lambda}_0(\in)$, are replaced by $v(\ell)$, $z(\ell)$, $l(\ell)$, and $\overline{\lambda}(\ell\in)$. It will be shown in the next section that for sufficiently large ℓ, $\overline{\lambda}(\ell\in)$ is proportional to $\in^{1/2}$ if W is treated as a constant (its functional dependence on $\in$ is ingored, Lam [6]). Then, the series expansion in powers of $\overline{\lambda}(\ell\in)$ is justified for small $\in$. However, it will be demonstrated shortly that the spectrum of the inertial range requires that $\in = 4$. The YO hypothesis is that the results of calculations performed for small $\in$ may be extended to the case where $\in = 4$. Some authors such as Theodorovicz [41], have suggested that a form of analytical continuation may be used to justify this hypothesis but to the knowledge of the current author no such attempt has been successful to date. Smith and Reynolds estimated the range of $\overline{\lambda}(\ell)$. At $\ell = 0$ relating Λ_0 to v_0 through the Kolmogorov relation; $\Lambda_0 \sim 0.2(\varepsilon/v_0)^{1/4}$ and using $W \sim \varepsilon$. $\overline{\lambda}_0(4) \sim 25$. At $\ell \longrightarrow \infty$, $\overline{\lambda}(\ell\in) \sim 11.5$ to the lowest order of the $\in$ expansion.

4.8 Recursion relations

The objective is to obtain a recursion algorithm by successively removing wavenumber shells of Fourier coefficients using the same steps as in sections 4.3 - 4.7. This iterative process will yield a sequence of eddy viscosity corrections. each of the same form but not the same value as the previous. After removing the first shell of wavenumbers corresponding to $\Lambda_0 e^{(-\Delta\ell)} \leq k \leq \Lambda_0$, the low wavenumber momentum equations are:

$$(-i\omega + v_1 k^2)u_\alpha^<(k') = M_{\alpha\beta\gamma}^<(k)\lambda \int d^d j \int d\Omega u_\gamma^<(k'-j')u_\beta^<(j') + f_\alpha^<(k')$$

$$+ \lambda^2 M_{\alpha\beta\gamma}^<(k)\int dj' 2u_\beta^<(k'-j')G_0(j')M_{\gamma\rho\sigma}^>(j)\int dp' u_\rho^<(j'-p')u_\sigma^<(p') + O(\lambda^3) \qquad 4.8\text{-}1$$

$$\underbrace{\hspace{10cm}}_{4.8\text{-}1b}$$

Note that equation 4.8-1 is the same as equation 4.6-9 $v_1 = v_0 + \Delta v_1$. The YO method achieves the recursion by discarding the triple velocity product. 4.8-1b. so that the momentum equations 4.8-1 have the same form as the original Navier-Stokes equations. The justification for neglecting the triple velocity product is subject to the condition that $\frac{k}{\Lambda_0} \ll \Delta\ell$ so that the integration domain of that term goes to zero.

Recalling relation 4.7-4;

$$v_1 = v_0 + A_d{}^0 v_0 \left(\frac{\exp(\in \Delta\ell) - 1}{\in} \right)\left(\bar{\lambda}_0{}^2 + O(\bar{\lambda}_0{}^4) \right) \qquad 4.7\text{-}4$$

the YO procedure first transforms this equation into a general recursion relation. valid after removing a range of wavenumbers, Λ_0 to $\Lambda_0 e^{-\ell}$ and the resulting increment of the eddy viscosity is $\Delta v(\ell+\Delta\ell)$. Setting $v_0 = v(\ell)$, $v_1 = v(\ell+\Delta\ell)$. and $\bar{\lambda}_0{}^2 = \bar{\lambda}^2(\ell)$ in equation 4.7-4 gives:

$$v(\ell+\Delta\ell) = v(\ell) + A_d{}^0v(\ell)\,\frac{\exp(\epsilon\Delta\ell) - 1}{\epsilon}\left\{\ \overline{\lambda}^2(\ell) + O(\overline{\lambda}^4)\ \right\}$$

4.8-2

where:

$$\overline{\lambda}^2(\ell) = \frac{\lambda^2 W}{v(\ell)^3 \Lambda(\ell)\epsilon}$$

4.8-3

To transform 4.8-2 into a differential equation, both sides are divided by $\Delta\ell$ and the limit $\Delta\ell \longrightarrow 0$ is taken:

$$\frac{dv(\ell)}{d\ell} = \lim_{\Delta\ell \longrightarrow 0}\frac{\Delta v(\ell)}{\Delta\ell} = \lim_{\Delta\ell \longrightarrow 0} A_d{}^0\frac{\exp(\epsilon\Delta\ell) - 1}{\Delta\ell\epsilon}\left\{\ \overline{\lambda}^2(\ell) + O(\overline{\lambda}^4)\ \right\}$$

4.8-4

Expanding the $\exp(\epsilon\Delta\ell)$ in a series of $\Delta\ell$ to the first order, the ϵ and $\Delta\ell$ factors cancel and the resulting differential equation is:

$$\frac{dv(\ell)}{d\ell} = A_d{}^0v(\ell)\left\{\ \frac{\lambda^2 W}{v(\ell)^3\Lambda(\ell)\epsilon} + O(\overline{\lambda}^4)\ \right\}$$

4.8-5

where $\Lambda(\ell) = \Lambda_0 e^{-\ell}$. Now the parameter $\overline{\lambda}$ is assumed to be small, so the $O(\overline{\lambda}^4)$ is neglected and equation 4.8-5 may be easily integrated. The solution is obtained using the boundary condition, $v(\ell=0) = v_0$, and integrating up to $\ell = \ell_c$. For the purpose of evaluating an eddy viscosity model for large eddy simulation, ℓ_c controls the 'cutoff' wavenumber $\Lambda_c = \Lambda_0 e^{-\ell_c}$ which is the maximum wavenumber resolved by the computational grid.

$$v(\ell) = \left\{\ 3A_d{}^0\,\frac{\lambda^2 W}{\Lambda_0\epsilon}\,\frac{\exp[\epsilon\ell] - 1}{\epsilon} + v_0{}^3\ \right\}^{1/3} + O(\overline{\lambda}^4)$$

4.8-6

The expression 4.8-6 may be substituted into 4.8-5 in order to obtain an explicit relation for $\overline{\lambda}$:

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$$\bar{\lambda}(\ell) = \bar{\lambda}_0 \exp[\tfrac{\in}{2}\ell][1 + \frac{3A_d^0\ W}{\Lambda_0^{\in}\ v_0^3}\ \frac{\exp[\in\ell] - 1}{\in}]^{-1/2} + O(\bar{\lambda}^4)$$

4.8-7

For $\in > 0$, and large ℓ equation 4.8-7 yields:

$$\bar{\lambda}^2 \longrightarrow \frac{\in}{3A_d^0} + O(\in^2)$$

4.8-8

The expression 4.8-8 is crucial for the YO RNG method for two reasons. First, as ℓ becomes large, $\bar{\lambda}$ becomes independent of ℓ - a condition called the 'fixed point'. Second, if $\in$ is close to zero, the truncation of the high wavenumber series is justified, again for large values of ℓ. At the fixed point, the eddy viscosity is approximated by equation 4.8-9:

$$v(\ell) \sim [\ \frac{3A_d^0 W}{\in}\ ]^{1/3}(\Lambda_0 e^{-\ell})^{-\in/3} = [\ \frac{3A_d^0 W}{\in}\ ]^{1/3}\Lambda_c^{-\in/3} \equiv v(0,\Lambda_c)$$

4.8-9

Once ℓ is large enough for relations 4.8-8 and 4.8-9 to be valid, any further increases of ℓ increase $v(\ell)$ according to $e^{\in/4\ell}$. The notation $v(0,\Lambda_c)$ has been used by Smith and Reynolds [5] in the context of RNG and by Kraichnan [38] and others for different versions of spectral eddy viscosity. The general form, $v(k,\Lambda_c)$ has been introduced in equation 2.7-8, (the t subscript has been dropped). For the YO version of the RNG method, $k/\Lambda_c \longrightarrow 0$ so that the method obtains the limit $v(k,\Lambda_c) \longrightarrow v(0,\Lambda_c)$. Note that here $k/\Lambda_c \longrightarrow 0$ means $k/\Lambda_c \ll 1$, but k is still in the inertial range of the turbulent flow.

It is a requirement of the RNG method that the fixed point relations apply over a large fraction of the total range of eliminated wavenumbers. Let $\ell = \ell_f$, where ℓ_f is a sufficiently large value of ℓ for $v(\ell)$ and $\bar{\lambda}(\ell)$ to be close to their fixed point values. In the wavenumber range Λ_0 to $\Lambda_0 e^{-\ell_f}$, $v(\ell)$ and $\bar{\lambda}(\ell)$ vary between their initial values v_0 and $\bar{\lambda}_0$ and their fixed point values and are not useful for turbulence modeling. In the wavenumber range $\Lambda_0 e^{-\ell_f}$ to $\Lambda_0 e^{-\ell}$, $\bar{\lambda}(\ell) \propto \in^{1/2}$, the power series expansion in $\bar{\lambda}(\ell)$ is justified for small $\in$, and the fixed point values of $v(\ell_f)$ and $\bar{\lambda}(\ell_f)$ are accurate for small $\in$. The overall validity of the eddy viscosity

model requires that the range $\Lambda_0 e^{-f}$ to $\Lambda_0 e^{-\ell}$ is much larger than the range Λ_0 to $\Lambda_0 e^{-f}$ so that the fixed point values may be said to apply over most of the eliminated wavenumbers. It may be concluded that the RNG procedure is applicable to high Reynolds number turbulence where the inertial range is wide.

4.9 The inertial range parameters

So far the eddy viscosity, $v(0,\Lambda_c)$ has been derived for a small value of ϵ ($\epsilon = 4 + y - d$) and an arbitrary W, the coefficient of amplitude for the correlation of the stirring forces. The objective now is to make a connection to the physics of turbulent flows in the Kolmogorov inertial range. The statistics of the inertial range will provide constraints on the values of ϵ and W so that the resulting $v(0,\Lambda_c)$ will hopefully be applicable to real turbulence.

It has been noted that the form of $v(k,\Lambda_c)$ has been derived for $\frac{k}{\Lambda_c} \ll 1$. However, Dannevik, Yakhot and Orszag [36] (herein called DYO), claim that as $\epsilon \longrightarrow 0$, the eddy viscosity result is correct even as $\frac{k}{\Lambda_c} \longrightarrow 1$. The validity of this claim will be examined in chapter 6, however it will be taken as correct for now so one may replace Λ_c by k in equation 4.8-9 to write:

$$v(k) \sim [\ \frac{3A_d{}^0 W}{\epsilon}\ ]^{1/3} k^{-\epsilon/3} \quad \text{with } \Lambda_L < k \leq \Lambda_c \qquad 4.9\text{-}1$$

where Λ_L is the lowest wavenumber of the inertial range. Relation 4.9-1 was used by YO [3] to obtain the energy spectrum:

$$E(k) = \frac{1}{2}[\frac{2WS_d}{(2\pi)^d}]^{2/3}(\frac{2\overline{A_d{}^0}}{2\epsilon})^{-1/3} k^{1-2\epsilon/3} \qquad 4.9\text{-}2$$

Inspection of 4.9-2 indicates that ϵ must equal to 4 in order to recover the Kolmogorov energy spectrum $E(k) = C_K \epsilon^{2/3} k^{-5/3}$. Therefore, YO set $\epsilon = 4$, ignoring that

the convergence criteria for the high wavenumber series as well as the limit $\frac{k}{\Lambda_c} \longrightarrow 1$ require $\in$ to be close to zero. YO make the ad hoc claim that their theory extends the results for $\in \longrightarrow 0$ to the case of $\in$ = 4. Their justification is based on the successful derivation of some experimentally known parameters as the end result of the nonrigorous procedure. However, with $\in$ = 4, DYO proceeded to derive a relation betweeen the amplitude of the stirring force variance, W, and the power transfer rate $\Pi(k)$, equal to ε in the inertial range.

To achieve this goal, DYO [36] solved the Navier-Stokes equations, 4.1-2 to second order in the $\in$ expansion to recover the Eddy-Damped-Quasi-Normal-Markovian (EDQNM) equations introduced by Orszag [24] in 1970. These equations express the mean dissipation rate, ε in terms of the energy spectrum and the effective viscosity in the Kolmogorov inertial range;

$$\varepsilon = \varepsilon(\ E(k),v_t(k)\ )$$

4.9-3

Combining 4.9-1, 4.9-2, and 4.9-3, and setting $\in$ = 4. DYO numerically integrated the EDQNM equations to obtain the relation:

$$C_{mg}\ \varepsilon = \frac{2WS_d}{(2\pi)^d}$$

4.9-4

where YO established that $C_{mg} \approx 1.59$, in agreement with a previous result by Kraichnan [23]. Equation 4.9-4 is the most important relation in the YO RNG because it establishes the relationship between the arbitrary RNG parameter W and the real flow parameter ε. Substituting 4.9-4 into 4.9-1 and 4.9-2 gives:

$$E(k) = 1.61\varepsilon^{2/3}k^{-5/3}$$

4.9-5

$$v(k) = 0.49\varepsilon^{1/3}k^{-4/3}$$

4.9-6

The coefficient 1.61 in equation 4.9-5 is the RNG value of the Kolmogorov constant, C_k. Since the accepted range for C_k from experimental measurements is 1.4 - 1.7, YO concluded that their result is in agreement with experiments.

Equation 4.9-6 may be rewritten in a form suitable for large eddy simulations. For this purpose, k is again replaced with Λ_c in equation 4.9-6. YO let $\tilde{\Delta}$ denote the computational grid spacing while Δ denotes the width of a filter, where $\Delta = 2\tilde{\Delta} = 2\pi/\Lambda_c$. The eddy viscosity, $v(\Lambda_c)$ is now written as $v(\Delta)$ to make a connection with the x-t space. Also, the eddy viscosity must satisfy the equivalence of the dissipation rate given by equation 2.3-2:

$$\varepsilon = \frac{v_0}{2} < \left\{ \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} \right\}^2 > = \frac{v(\Delta)}{2} < \left\{ \frac{\partial u_\alpha^<}{\partial x_\beta} + \frac{\partial u_\beta^<}{\partial x_\alpha} \right\}^2 > \qquad 2.3-2$$

where the $<..>$ notation denotes ensemble average as usual and $u^<(x,t)$ is the resolved, large eddy component of the velocity consisting of Fourier components $u(k,t)$ such that $|k| \leq \Lambda_c$. Substituting the above relations into equation 4.9-7 yields:

$$v(\Delta) = 0.49 \left(\frac{v(\Delta)}{2}\right)^{1/3} \left\{ \frac{\partial u_\alpha^<}{\partial x_\beta} + \frac{\partial u_\beta^<}{\partial x_\alpha} \right\}^{2/3} (2\pi)^{-4/3} \Delta^{4/3} \qquad 4.9-7$$

Rearranging finally yields:

$$v(\Delta) = c_s \Delta^2 \left| \frac{\partial u_\alpha^<}{\partial x_\beta} + \frac{\partial u_\beta^<}{\partial x_\alpha} \right| \qquad 4.9-8$$

where $c_s = 0.0061$ as calculated by the present author (YO [3] report 0.0062).

Equation 4.9-8 is the well known Smagorinsky eddy viscosity formula which has been widely used in large eddy simulations. Early users debated which value of c_s gave the best results, with Deardorff (1971) claiming that $c_s \approx 0.005$ worked best. Moin and Kim pointed out that the computed LES turbulent intensity were insensitive to variations of the constant in 4.9-8 by 40% (Lesieur et al [19]). Therefore YO

concluded that their RNG value of the RNG derived Smagorinsky constant is within the 'experimental' range as established by early LES computations.

4.10 Discussion

The YO version of the RNG application to turbulence has a number of controversial steps. The main issues are summarized below:

1) Interpretation of the stirring forces.

2) Series expansion in powers of $\overline{\lambda}$, where $\overline{\lambda} > 1$.

3) Functional dependence of W on $\in$ and possibly on Λ as well.

4) Partial averaging of the stirring forces based on wavenumber magnitude with a sharp boundary between the averaged and resolved variables.

5) Wavenumber integration domain is assumed to be a spherical shell.

6) Triple velocity products are discarded.

7) Derived eddy viscosity is valid only for $k/\Lambda_c \longrightarrow 0$. However it is used at the next stage for $u^{>0}$.

Item 1, the various interpretations of the stirring forces will now be briefly discussed. The series expansion in powers of $\overline{\lambda}$ (item 2) will be discussed in chapter 10. Item 3, the comprehensive role of $\in$ in the RNG model is beyond the scope of this study although it is closely related to the series expansion. For the purposes of the current investigation $\in$ is simply fixed at 4. A possible future study is the model problem (where the stirring force is interpreted as a real external force acting on the fluid) analyzed for the case where $\in$ varies from near zero to near 4 in order to determine if such a problem is physically realizable and what is the nature of the

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non-linear energy transfer. The functional dependence of W on $\in$ can also be investigated. The partial averaging operator will be discussed in chapters 8. and 10. Items 5 and 6 are related and they will be addressed later in this section. Item 7 will be addressed in chapter 9.

The interpretation of the stirring forces should be in the context of the 'correspondence principle' introduced by YO [3], largely misunderstood in subsequent literature according to Smith and Woodruff [7]. To quote YO [3]:

'A turbulent fluid characterized in the inertial range by scaling laws can be described in this inertial range by a corresponding Navier-Stokes equation in which a random force generates velocity fluctuations that obey the scaling of the inertial range of the original unforced system. The dynamical equation with the random force is the basis for the systematic elimination of small scales and calculation of the renormalized transport coefficients.'

The difference of opinion among various researchers concerns the nature of this stirring force. The present author interprets the above statement literally to mean that the inertial range statistics of turbulent flow may be reproduced by the 'model problem' of a Navier-Stokes equation driven by a properly specified external stirring force. In this interpretation, the force is real and separate from the effects of the energy cascade. The difficulty with the model problem driven by an external stirring force is that since the force acts on all wavenumbers of interest there can be no inertial range energy cascade for the model problem because such a cascade by definition excludes external input. The energy spectrum may still have the required $k^{-5/3}$ slope.

Smith and Woodruff [7] interpret the force as a renormalized effect of the energy cascade. The problem with this interpretation is that since the complete non-linear convolution integral is still included in the Navier-Stokes equation. there is a duplication in the representation of some non-linear interactions. However. this problem may later be alleviated by the truncation of the high wavenumber series at λ^2. a procedure which discards higher order non-linear terms. A similar interpretation is offered by Lam [6], who however relaxes the fundamental YO requirement that the force correlation W be scale invariant.

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These different interpretations of the stirring force may perhaps be reconciled by alternating one's viewpoint between the literal model problem with an external stirring force and the equivalent turbulent problem with a renormalized force. The interpretation of RNG results will be less ambiguous if one clearly specifies the context.

An important restriction of the YO results is that they are valid only for $k \sim 0$ or more precisely for $k/(\Lambda\Delta\ell) \sim 0$. The resulting eddy viscosity is rather analogous to the low frequency 'beats' caused by two superposed linear waves at nearly equal frequency. Even though the shell thickness, $\Lambda\Delta\ell$ is thin (it tends to zero in the differential limit), the restriction $k/(\Lambda\Delta\ell) \ll 1$ means that only the interactions which are nearly opposite in phase contribute to the eddy viscosity. It is noted. however, that the RNG calculations are strictly valid only for k values in the inertial range due to the isotropic form of the Navier-Stokes equations in Fourier space. Therefore, the inertial range must be very wide, so that $k/(\Lambda\Delta\ell) \ll 1$ even as $\Delta\ell \longrightarrow 0$. The restrictions on the RNG procedure should be interpretted as relative to the limits of the inertial range which are assumed to be very wide. Chapters 8 and 9 of this study will examine the extension of YO results to the case where $\dfrac{k}{\Lambda\Delta\ell} \geq 1$.

YO obtain their recursion relation by neglecting the triple velocity products. 4.8-1b. Neglecting the subscripts, term 4.8-1b is rewritten below:

$$\lambda^2 M^<(k)\int dj'2u^<(k'-j')G_0(j')M^>(j)\int dp'u^<(j'-p')u^<(p')$$

YO show that if the triple products were retained in the renormalized equation of motion, these terms would scale as $e^{-\ell(d-y)}$, where $e^{-\ell}$ is the fraction of eliminated scales, d is the number of space dimensions, and y is the negative power law exponent of the stirring force correlation. Therefore, taking $d = 3$, and letting ℓ become large, the triple products will diminish in magnitude if $y < 3$. However. the physics of Kolmogorov's turbulence requires $\varepsilon = 4$ so that $y = 3$ and the triple products are

Region C

Region A

Region C

Region B

z

θ

K

J

z

θ

K

J
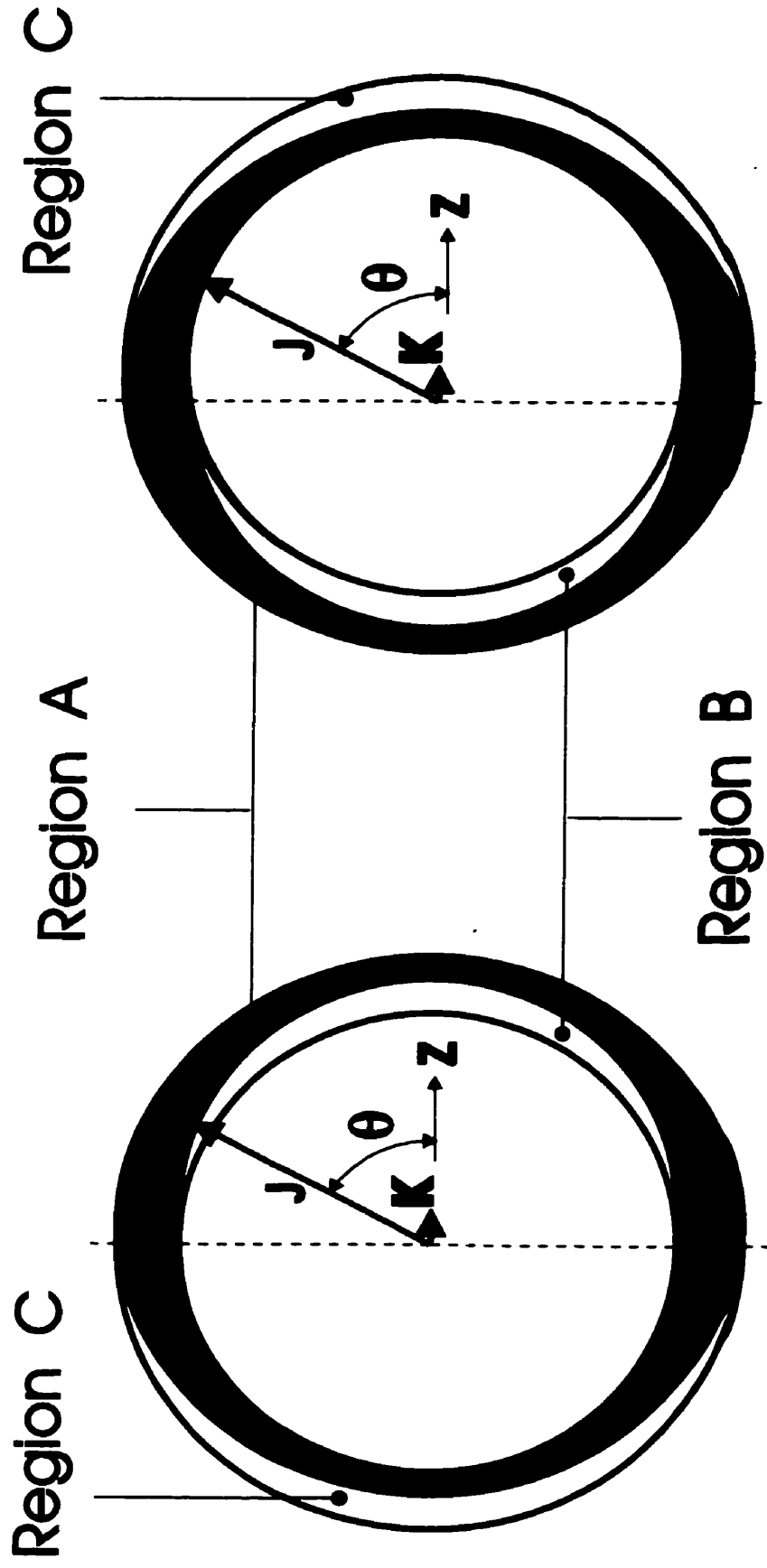
a. Shell $\Lambda e^{-s} < |J| < \Lambda$

b. Shell $\Lambda e^{-s} < |k-J| < \Lambda$

Figure 4.2 Integration domains for Δv and the triple products for $k/(\Lambda \delta L) < 1$.

not affected by rescaling. Therefore, YO state that discarding the triple products is not justified but express the hope that the contributions from these terms are not too large and lead only to logarithmic corrections to their theory. However, the magnitude of the triple products vs. the magnitude of the viscosity correction due to a single eliminated shell may be assessed by examining their domain of integration in wavenumber space. The domains of integration of some RNG terms are indicated in figure 4.2 as described in the following discussion.

It is noted that terms such as $\int u^{<}u^{>}$, or $\int u^{>1}u^{>0}$ in the RNG analysis come in pairs obtained by interchanging the j and k-j variables. Due to the symmetry of the integration domain indicated in figure 4.2, the two terms of each pair are equal so that $\int u^{>1}(k\text{-}j)u^{>0}(j)$ + $\int u^{>1}(j)u^{>0}(k\text{-}j)$ is usually expressed as $2\int u^{>1}(k\text{-}j)u^{>0}(j)$. The integration domain of the YO viscosity correction, Δv, consists of region A labelled in figure 4.2. It should be noted that for each shell, region A is symmetric with respect to the angle θ. Region B is the domain of terms of the form $\int u^{<}u^{>}$ and to $O(\lambda^2)$, these consist of $\lambda \int u^{<}u^{>0}$ and $\lambda^2 \int u^{<}u^{>1}$. The contribution of $\lambda \int u^{<}u^{>0}$ to the eddy viscosity (a new contribution of the present work) will be taken into account in chapters 8 and 9. As demonstrated in section 4.6, the term $\lambda^2 \int u^{<}u^{>1}$ gives rise to the triple velocity products (equation 4.6-9). It will be shown in chapter 5 that region C represents the domain of the re-expanded triple products from previous shells - not used in the YO analysis. The magnitudes of the respective integration volumes for regions A, B, and C, will now be examined.

Since the flow is isotropic, the vector k may be taken to be parallel to the z axis without loss of generality. The intersection angle between two circles both of radius Λ centered at origin and $\{0,0,k\}$, respectively, is $\theta_3 = \arccos(\frac{k}{2\Lambda})$, and similarly, $\theta_2 = \arccos(\frac{k}{2\Lambda e^{-\Delta\ell}})$ is the intersection angle for circles of radii $\Lambda e^{-\Delta\ell}$. The equation for the magnitude of the j vector from the origin to a point on the circle of radius Λ and centered at $\{0,0,k\}$ is:

$$J(k,\theta,\Lambda) = k\cos\theta + \Lambda\sqrt{1 - \left(\frac{k}{\Lambda}\right)^2(1 - \cos^2\theta)} \qquad 4.10\text{-}1$$

For thin shells, $\theta_3 \approx \theta_2$ and the integration volumes of regions A, B, and C, in figure 4.2 may be evaluated as:

$$\text{region A} = 2\pi\int_0^{\theta_3} \text{Sin}\theta \ d\theta \int_{J(k,\theta,\Lambda e^{-\Delta\ell})}^{\Lambda} j^2 \ dj + 2\pi\int_{\theta_3}^{\pi} \text{Sin}\theta \ d\theta \int_{\Lambda e^{-\Delta\ell}}^{J(k,\theta,\Lambda)} j^2 \ dj \qquad 4.10\text{-}2$$

$$\text{region B} = 2\pi\int_0^{\theta_3} \text{Sin}\theta \ d\theta \int_{\Lambda e^{-\Delta\ell}}^{J(k,\theta,\Lambda e^{-\Delta\ell})} j^2 \ dj + 2\pi\int_0^{\theta_3} \text{Sin}\theta \ d\theta \int_{J(k,\theta,\Lambda e^{-\Delta\ell})}^{\Lambda e^{-\Delta\ell}} j^2 \ dj \qquad 4.10\text{-}3$$

$$\text{region C} = 2\pi\int_0^{\theta_3} \text{Sin}\theta \ d\theta \int_{\Lambda}^{J(k,\theta,\Lambda)} j^2 \ dj + 2\pi\int_{\theta_3}^{\pi} \text{Sin}\theta \ d\theta \int_{J(k,\theta,\Lambda)}^{\Lambda} j^2 \ dj \qquad 4.10\text{-}4$$

Since the magnitude limit is a function of angle as given by 4.10-1, the magnitude integral must be performed first and the θ integration follows. Integrals 4.10-2 to 4.10-4 can be evaluated analytically in three dimensions. To $O(\Delta\Lambda)$ and with $k \leq \Delta\Lambda$, the volume of region A is $4\pi\Lambda^2(\Lambda\Delta\ell - 0.5k)$, while the volumes of regions B and C are both $2\pi\Lambda^2 k$. It is noted that YO approximate 4.10-1 as $4\pi\Lambda^2\Delta\Lambda$, this approximation is equivalent to an expansion in powers of $\left(\frac{k}{\Delta\Lambda}\right)$ to $\left(\frac{k}{\Delta\Lambda}\right)^0$. If $\left(\frac{\Delta\Lambda}{\Lambda}\right) < 1$, $\left(\frac{k}{\Delta\Lambda}\right) > \left(\frac{k}{\Lambda}\right)$ and since the Δv integrand is expanded to $\left(\frac{k}{\Lambda}\right)^1$ so should be the integral domain and thus the YO procedure is inconsistent in this aspect. However, as verified by the present author, removing this inconsistency does not affect the final result, since the added terms cancel over the integral domain.

The integration volume ratio of the triple terms (region C) to the eddy viscosity (region A) is $\frac{k}{2(\Lambda\Delta l - 0.5k)}$, proportional to $\frac{k}{2\Lambda\Delta l}$ as $\frac{k}{\Lambda\Delta l} \longrightarrow 0$. Since the integrand of the triple products is finite in this region, the triple products may be neglected in this limit. The same argument applies to region B. For values of k not quite so small, where $\frac{k}{\Lambda\Delta l} \leq 1$, the YO calculation is inaccurate.

The general conclusion from the above discussion is that while some of the YO analytical steps may appear oversimplified, their results are robust. More exact definition of the integration domain does not change the answers.

It is instructive to review the relevant parameters in the RNG procedure. The Navier-Stokes equations have been non-dimensionalized in section 4.7 in order to recover $\overline{\lambda}$ as the effective expansion parameter, a form of the Reynolds number. It is noted that at the start of the procedure, there are three relevant dimensional parameters in RNG, W [$L^{6-\in}/t^3$], Λ [L^{-1}], and v_0. In addition, there are several dimensionless parameters; $\in$, Δl and $\frac{k}{\Lambda}$. In the YO version of RNG, two limits are taken in sequence; first $\frac{k}{\Delta l \Lambda} \longrightarrow 0$, second $\Delta l \longrightarrow 0$, and ZV [9] claim that this sequence cannot be interchanged. After a wide range of eliminated scales, when $v_0^3 \ll$ $\frac{3A_d^0}{\in}W\Lambda(l)^{-\in}$, inertial effects dominate viscous effects in the eliminated range of flow fluctuations so v_0 may be neglected. The remaining parameters {W.Λ.$\in$} are used to construct the values of characteristic velocity, length scale, and eddy viscosity (Lam [6]). The effective expansion parameter $\overline{\lambda}$ is proportional to $\in^{1/2}$ and so the series expansion of the high wavenumber Fourier coefficients is justified for small $\in$. However, $\in$ = 4 is required to recover the Kolmogorov energy spectrum.

YO assert that their analysis is valid even if $\in$ = 4, when the high wavenumber velocity series does not appear to converge. Authors such as Theodorovicz [41] and Lam [6] interpret the extrapolation of results from $\in \longrightarrow 0$ to $\in$ = 4 as a form of analytic continuation but a rigorous proof has never been given. YO justify their claim by two means of validation; the derivation of several fundamental constants of

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turbulence in good agreement with experiment and good simulation results with the RNG turbulence models. In chapter 10, the current study will address the question whether the limit of infinitesimally thin shell causes the high wavenumber series to converge or whether the approximated solution for $u^>$ has at least improved accuracy for this case. Also, in view of the non-rigorous analytical steps taken by YO, the question arises: what are the criteria for the RNG analysis of turbulence to yield meaningful results? How does one tell if a new term generated by the analysis has physical meaning and whether the LES results will be more accurate if this term is incorporated in the model? The present work will attempt to give a partial answer to this question.

5. The recursive RNG method

This chapter outlines the recursive RNG method as pioneered by Rose [29], and extended by Zhou, Vahala and Hossain, [8], hereafter referred to as ZVH. Rose worked with a model of a passive scalar convected by a known velocity field and derived an expression for the renormalized eddy diffusivity. ZVH applied similar methods to the model of stationary, isotropic turbulence, and to free decaying turbulence in order to obtain a renormalized eddy viscosity for each case. ZVH refer to their stationary turbulence model as 'forced' turbulence since energy is supplied at a steady rate by a stirring force. Here, the scope is restricted to the forced turbulence case.

The model equations at the start of the procedure are the same as those used by YO but the method used by ZVH is different from the $\in$-RNG in several important respects. One important difference is that the eddy viscosity contribution is calculated numerically for a finite size shell, and the total eddy viscosity is accumulated iteratively rather than integrated from an approximate differential equation. Also, the triple velocity products are retained in the momentum equation and contribute to the eddy viscosity. A third important difference between the work of YO and the eddy viscosity model of ZVH is that the latter model is allegedly valid for all $0 \le k \le \Lambda_c$ while the YO model is valid only for $k/\Lambda_c \longrightarrow 0$. In chapter 6 of the current study, the YO eddy viscosity is extended to finite wavenumbers by reexpanding the triple velocity products. However the implementation and the results are different from ZVH. The methodology of the present work has common points with both YO and ZVH and will be compared to both.

5.1 The recursive RNG method - mathematical model

The presentation given here will follow the publication of Zhou, Vahala, and Hossain [8] and also a later work by Zhou and Vahala (ZV) [11] for the derivation of an eddy viscosity for forced turbulence. Their notation is modified to match the other chapters of this work. Comparisons with the work of YO will be made along the way. Small differences in numerical factors will be pointed out since these may affect the final RNG results for numerical values of the turbulence constants.

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Emphasis is placed on the assumptions made by ZVH or their methodology which are at odds with the procedure adopted in this investigation.

In contrast to YO, ZVH worked in the time domain so that the spatially Fourier transformed momentum equation is recalled from chapter 2 as:

$$\left(\frac{\partial}{\partial t} + \upsilon_0 k^2\right)u_\alpha(k,t) = M_{\alpha\beta\gamma}(k)\int d^3j \ u_\gamma(k\text{-}j,t)u_\beta(j,t) + f_\alpha(k,t) \qquad 2.5\text{-}5$$

with

$$M_{\alpha\beta\gamma}(k) = \frac{1}{2i}\left\{ k_\beta D_{\alpha\gamma}(k) + k_\gamma D_{\alpha\beta}(k) \right\} \qquad 2.5\text{-}6$$

and

$$D_{\alpha\beta}(k) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \qquad 2.5\text{-}7$$

Note that ZVH omit the $1/(2\pi)^3$ factors from the Fourier integrals, in contrast to YO. The stirring force has a zero mean and ZVH restrict it to a Gaussian distribution. ZVH specify the Fourier space correlation of the stirring forces as:

$$<f_\alpha(k,t)f_\beta(k',t')> = W(k)D_{\alpha\beta}(k)\delta(k + k')\delta(t - t') \qquad 5.1\text{-}1$$

Comparison of equation 5.1-1 with equation 4.1-3 of YO shows that a factor of 2 has been omitted by ZVH as well as the $(2\pi)^3$ factor. The specification of the functional dependence on (t-t') as $\delta(t - t')$ in equation 5.1-1 implies a white noise temporal correlation of the force.

The wavenumber dependence of $W(k)$ is a power law exactly the same as for YO equation 4.1-4:

$$W(k) = \begin{cases} 0 & \text{for } 0 < k < \Lambda_L \\ Wk^{-y} & \text{for } \Lambda_L \leq k \leq \Lambda_0 \end{cases} \qquad 4.1\text{-}4$$

where Λ_L is a minimum wavenumber required to avoid singular behavior of $W(k)$ and Λ_0 is the maximum wavenumber at which the force acts.

The recursive RNG partitions the subgrid region $\Lambda_c < k < \Lambda_0$ into N shells. each with thickness $k(1-h)$ where k is the outer radius of the shell being considered and h is held constant, $0 < h < 1$:

$$\Lambda_c \equiv k_N \equiv h^N\Lambda_0 < ...< k_i \equiv h^i\Lambda_0 <...< k_1 = h\Lambda_0 < \Lambda_0 \qquad 5.1\text{-}2$$

ZVH use the usual notation to denote variables within a shell with a superscript $>$ while the variables below the shell have the superscript $<$. The resulting equations are analogous to 4.3-3 and 4.3-4 except that they are in the time domain.

$$(\frac{\partial}{\partial t} + \upsilon_0 k^2)u^<(k,t) = \lambda M^<(k)\int d^3j \ \Big\{ u^<(k\text{-}j,t)u^<(j,t) + $$

$$2u^<(k\text{-}j,t)u^>(j,t) + u^>(k\text{-}j,t)u^>(j,t) \Big\} + f^<(k,t) \qquad 5.1\text{-}3$$

$$(\frac{\partial}{\partial t} + \upsilon_0 k^2)u^>(k,t) = \lambda M^>(k)\int d^3j \ \Big\{ u^<(k\text{-}j,t)u^<(j,t) + $$

$$2u^<(k\text{-}j,t)u^>(j,t) + u^>(k\text{-}j,t)u^>(j,t) \Big\} + f^>(k,t) \qquad 5.1\text{-}4$$

The directional indexes have been suppressed for clarity and the artificial parameter λ has been inserted. If λ is set to zero, equations 5.1-3 and 5.1-4 become linear and may be solved using the Green's function:

$$G_0(k,t,\tau) = e^{-\upsilon_0 k^2(t-\tau)} \qquad 5.1\text{-}5$$

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The Green's function 5.1-5 is analogous to that used by YO (equation 4.1-2) and may be used to write a formal solution for $u^>(k,t)$:

$$u^>(k,t) = \int_{-\infty}^{t} d\tau \left\{ G_0(k,t,\tau) \; \lambda M^>(k) \int d^3j \left\{ u^<(k-j,\tau)u^<(j,\tau) + \right.\right.$$

$$\left.\left. 2u^<(k-j,\tau)u^>(j,\tau) + u^>(k-j,\tau)u^>(j,\tau) \right\} + f^>(k,\tau) \right\}$$ 5.1-6

5.2 The high wavenumber velocity series

Following the same procedure as YO, (equation 4.4-1), the subgrid velocity $u^>(k,t)$ is expanded in powers of λ:

$$u_\alpha^>(k,t) = u_\alpha^{>0}(k,t) + \lambda u_\alpha^{>1}(k,t) + \lambda^2 u_\alpha^{>2}(k,t) + ...$$ 5.2-1

Substituting equation 5.2-1 into 5.1-6 and matching the powers of λ, the first two terms of this power series are:

$$u^{>0}(k,t) = \int_{-\infty}^{t} G_0(k,t,\tau)f^>(k,\tau)d\tau$$ 5.2-2

Similarly, the expression for $u^{>1}(k,t)$ may be written as:

$$u^{>1}(k,t) = M^>(k) \int_{-\infty}^{t} d\tau \; G_0(k,t,\tau) \times$$

$$\int d^3j \left\{ u^<(k-j,\tau)u^<(j,\tau) + 2u^<(k-j,\tau)u^{>0}(j,\tau) + u^{>0}(k-j,\tau)u^{>0}(j,\tau) \right\}$$ 5.2-3

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5.3 The substitution and partial averaging

The conventional sequence of operations in the ϵ-RNG is to first substitute the high wavenumber series 5.2-1 up to order λ^2 into the resolvable scales equation 5.1-3 and then to perform ensemble averaging over the subgrid forces. ZVH follow the same sequence in their first publication [8] but in their later work ZV [11] claim to have demonstrated that the results of RNG are independent of the order of substitution and averaging. This issue is discussed further in chapter 8, where it is shown that the significance of the sequence of substitution and partial averaging is determined by the properties of the partially averaged product $<u^<u^>>$. Here the presentation follows the convention of performing the substitution first, carrying it out to $O(\lambda^2)$ for the overall expression, and followed by the partial averaging.

Restoring the directional subscripts and substituting the high wavenumber series (equation 5.2-1) into the momentum equation for low wavenumbers (equation 5.1-3) yields:

$$\left(\frac{\partial}{\partial t} + \upsilon_0 k^2\right) u_\alpha^<(k,t) = \lambda M_{\alpha\beta\gamma}^<(k)\int d^3j \; u_\beta^<(k\text{-}j,t) u_\gamma^<(j,t) \; +$$

$$\underbrace{\hspace{5cm}}_{\text{5.3-1a}}$$

$$\lambda M_{\alpha\beta\gamma}^<(k)\int d^3j \; 2u_\beta^<(k\text{-}j,t)\left(u_\gamma^{>0}(j,t) + \lambda u_\gamma^{>1}(j,t) \right) \; +$$

$$\underbrace{\hspace{5cm}}_{\text{5.3-1b}}$$

$$+ \; \lambda M_{\alpha\beta\gamma}^<(k)\int d^3j \left(u_\beta^{>0}(k\text{-}j,t) + \lambda u_\beta^{>1}(k\text{-}j,t)\right)\left(u_\gamma^{>0}(j,t) + \lambda u_\gamma^{>1}(j,t)\right) + f_\alpha^<(k,t) \qquad \text{5.3-1}$$

$$\underbrace{\hspace{5cm}}_{\text{5.3-1c}}$$

Turning now to the ensemble averaging of the fine scale forces, term 5.3-1a is unaffected. The average of the term 5.3-1b is evalulated below:

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$$\langle5.3\text{-}1b\rangle\rangle = \lambda M_{\alpha\beta\gamma}{}^{<}(k)\int d^3j\langle2u_{\beta}{}^{<}(k\text{-}j,t)(\ u_{\gamma}{}^{>0}(j,t) + \lambda u_{\gamma}{}^{>1}(j,t))\rangle\rangle =$$

$$2\lambda^2 M_{\alpha\beta\gamma}{}^{<}(k)\int d^3j\ u_{\beta}{}^{<}(k\text{-}j,t)\int d\tau\ G_0(j,t,\tau)\ M_{\gamma\rho\sigma}{}^{>}(j)\int u_{\rho}{}^{<}(j\text{-}p,\tau)u_{\sigma}{}^{<}(p,\tau)d^3p \qquad 5.3\text{-}2$$

The result in 5.3-2 has been derived using the same principles as utilized in chapter 4. The details of this derivation are given in reference [8]. Equation 5.3-2 gives the 'triple velocity products' which play a central role in the work of ZVH as well as for the current study. Averaging term 5.3-1c gives:

$$\langle5.3\text{-}1c\rangle\rangle =$$

$$\lambda M_{\alpha\beta\gamma}{}^{<}(k)\int d^3j\langle(\ u_{\beta}{}^{>0}(k\text{-}j,t) + \lambda u_{\beta}{}^{>1}(k\text{-}j,t)\ )(\ u_{\gamma}{}^{>0}(j,t) + \lambda u_{\gamma}{}^{>1}(j,t)\ )\rangle\rangle \qquad 5.3\text{-}3$$

Expanding the products and temporarily suppressing the subscripts gives:

$$\lambda M^{<}(k)\int d^3j\langle u^{>0}(k\text{-}j,t)u^{>0}(j,t)\rangle\rangle + 2\lambda^2 M^{<}(k)\int d^3j\langle u^{>1}(k\text{-}j,t)u^{>0}(j,t)\rangle\rangle +$$

$$\underbrace{\hspace{4cm}}_{5.3\text{-}3a} \qquad \underbrace{\hspace{4cm}}_{5.3\text{-}3b}$$

$$\lambda^3 M^{<}(k)\int d^3j\langle u^{>1}(k\text{-}j,t)u^{>1}(j,t)\rangle\rangle$$

$$\underbrace{\hspace{4cm}}_{5.3\text{-}3c}$$

Term 5.3-3a contains the product $\langle u^{>0}(k\text{-}j,t)u^{>0}(j,t)\rangle\rangle$ which yields $\delta(k)$ when averaged. Since $\delta(k)M(k) = 0$, the term 5.3-3a has a zero average but a non-zero rms value - it is the induced force indentified by YO. The term 5.3-3c is of $O(\lambda^3)$ and is dropped in the conventional $O(\lambda^2)$ RNG closure of ZVH. It is noted that the $O(\lambda^2)$ closure is artificial since $\lambda = 1$. If the partial averaging is carried out on term 5.3-3c, the result renormalizes $M^{<}(k)$, a so-called 'vertex renormalization'. FNS [28] have shown that such terms vanish when the ratio of k/j << 1 as a consequence of

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Galilean invariance. Since the work of YO satisfies $k/j \ll 1$ they are justified in dropping term 5.3-3c. However, ZVH claim that their analysis is valid even as $k/j \longrightarrow 1$ and so their discarding the vertex renormalization terms is based on the $O(\lambda^2)$ closure only and hence is problematic.

Partial averaging the remaining term 5.3-3b yields:

$$\langle\langle 5.3\text{-}3b\rangle\rangle = 2\lambda M_{\alpha\beta\gamma}^{<}(k)\int d^3j \langle u_\beta^{>0}(k\text{-}j,t)\lambda u_\gamma^{>1}(j,t)\rangle\rangle =$$

$$4\lambda^2 M_{\alpha\beta\gamma}^{<}(k)D_{\alpha\rho}(k)\int d^3j W_0 |k\text{-}j|^{-\gamma} D_{\beta\sigma}(k\text{-}j) \times$$

$$M_{\gamma\rho\sigma}^{>}(j) \int_{-\infty}^{t} G_0(j,t,\tau)u_\rho^{<}(k,\tau)d\tau \int_{-\infty}^{t} G_0^2(|k\text{-}j|,t,\tau')d\tau'$$

$$\equiv - \int_{-\infty}^{t} d\tau\ \eta_0(k,t,\tau)\ u_\alpha^{<}(k,\tau) \qquad\qquad 5.3\text{-}4$$

where $\eta_0(k,t,\tau)$ is a 'non-local' generalized eddy-damping function defined by:

$$\eta_0(k,t,\tau) \equiv$$

$$4\lambda M^{<}(k)D_{\alpha\rho}(k)\int d^3j\ W_0 |k\text{-}j|^{-\gamma}\ G_0(j,t,\tau)\ D_{\beta\sigma}(k\text{-}j)M_{\gamma\rho\sigma}^{>}(j) \int_{-\infty}^{t} G_0^2(|k\text{-}j|,t,\tau')d\tau' \qquad 5.3\text{-}5$$

It is noted that the term $D_{\alpha\rho}(k)$ is introduced artificially by ZVH in order to extract $u_\alpha^{<}(k,\tau)$ from the three terms $u_\rho^{<}(k,\tau)$. This term is convenient for the notation in the intermediate expression 5.3-5 but is not necessary, since $u_\alpha^{<}(k,\tau)$ is naturally obtained from $u_\rho^{<}(k,\tau)$ once the angular j wavenumber integration is carried out.

The term 'non-local' refers to the time convolution integrals also called 'memory integrals' or 'history integrals'. Prior to the application of the RNG procedure, the

momentum equations in Fourier space are written for a given instant in time. These equations are in effect Markovian, with the rate of change of $u(k,t)$ determined only by the state of the fluid at time t. After the application of the RNG method. the new momentum equations contain history effects, so that the rate of change of $u(k,t)$ is affected by values of $u(k,\tau)$ via the function $\eta_0(k,t,\tau)$. The representation of physical reality by such a model is questionable since it is well known that variables containing two different time arguments do not possess Galilean invariance. However, the complete convolution integral could have Galilean invariance.

It should also be noted that the introduction of the time convolution integral into the momentum equations in some ways violates the general intent of RNG methods. as does the retention of the triple products, [39]. The general principle of the RNG is to absorb the effects of small scale variables into enhanced versions of the coefficients in a given equation without changing the form of the equation. However. in the ZVH procedure, the form of the equation changes only at the elimination of the first shell and remains the same thereafter.

5.4 Re-expanding the triple products and the recursion relations

After eliminating the first shell which is now labelled shell number 0. $u^<(k,t)$ is replaced by $u(k,t)$ and the momentum equations take the form:

$$(\frac{\partial}{\partial t} + \upsilon_0 k^2)u_\alpha(k,t) + \int_{-\infty}^{t} \eta_0(j,t,\tau)u_\alpha(k,\tau)\ d\tau =$$

$$\lambda M_{\alpha\beta\gamma}(k)\int d^3j\ u_\gamma(k\text{-}j,t)u_\beta(j,t) + f_\alpha(k,t) +$$

$$2\lambda^2 M_{\alpha\beta\gamma}(k)\int d^3j u_\beta(k\text{-}j,t) \int_{-\infty}^{t} d\tau\ G_0(j,t,\tau)M_{\gamma\rho\sigma}{}^{>0}(j)\int u_\rho(j\text{-}p,\tau)u_\sigma(p,\tau)d^3p \qquad 5.4\text{-}1$$

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It is noted that the first shell label appears as $M_{\gamma p \sigma}{}^{>0}(j)$ in order to indicate that $|j|$ or j is constrained to the first shell.

Following ZVH [8], equations 5.4-1 will still be referred to as the Navier - Stokes equations even with the new triple velocity product. Now the ZVH procedure departs from that of YO since the latter method was to discard the triple product while ZVH retain it for the next RNG step. This next step consists once again of spectral splitting, expanding the high wavenumber velocity in powers of λ. and substituting the series into the equation for the low wavenumbers. The equations shall be simplified in appearance by dropping directional indexes with the objective of showing the main effects of the triple terms on the renormalization procedure. The spectrally split equations appear as:

$$(\frac{\partial}{\partial t} + \upsilon_0 k^2)u^<(k,t) + \int_{-\infty}^{t} \eta_0(j,t,\tau)u^<(k,\tau) \, d\tau =$$

$$\lambda M^<(k)\int d^3j \left(u^<(k-j,t) + u^>(k-j,t)\right)\left(u^<(j,t) + u^>(j,t)\right) +$$

$$2\lambda^2 M^<(k)\int d^3j \left(u^<(k-j,t) + u^>(k-j,t)\right) \times$$

$$\int_{-\infty}^{t} d\tau G_0(j,t,\tau)M^{>0}(j)\int\left(u^<(j-p,\tau) + u^>(j-p,\tau)\right)\left(u^<(p,\tau) + u^>(p,\tau)\right)d^3p + f^<(k,t) \qquad 5.4-2$$

with an analogous equation for $u^>(k,t)$. The high wavenumber velocity is once again expanded in a power series of λ:

$$u_\alpha{}^>(k,t) = u_\alpha{}^{>0}(k,t) + \lambda u_\alpha{}^{>1}(k,t) + \lambda^2 u_\alpha{}^{>2}(k,t) + \ldots$$

As had been shown in section 4.4, expressions for different terms in the series may be obtained by substituting the series into both sides of the high wavenumber momentum equation and matching equal powers of λ. The first term is given by:

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$$\left(\frac{\partial}{\partial t} + v_0 k^2\right)u^{>0}(k,t) + \int_{-\infty}^{t} d\tau\ \eta_0(k,t,\tau)u^{>0}(k,\tau) = f^{>}(k,t) \qquad 5.4\text{-}3$$

This equation is linear and may be solved using a Green's function, $G_1(k,t,\tau)$, defined through equation 5.4-4:

$$\left(\frac{\partial}{\partial t} + v_0 k^2\right)G_1(k,t,\tau) + \int_{\tau}^{t} ds\ \eta_0(k,s,\tau)G_1(k,s,\tau) = \delta(t-\tau) \qquad 5.4\text{-}4$$

where the lower limit of the integral is changed to ensure causality, $G_1(k,s,\tau) = 0$ if $\tau > s$ (the cause must precede the effect). The formal solution for $u_\alpha^{0>}(k,t)$ may now be written as:

$$u^{>0}(k,t) = \int_{-\infty}^{t} G_1(k,t,\tau)f^{>}(k,t)d\tau \qquad 5.4\text{-}5$$

It is noted that ZVH do not present an explicit form for $G_1(k,t,\tau)$ but leave its definition implicit in equation 5.4-4. The reason is that ZVH use these Green's functions only to formulate a general mathematical framework. For the purpose of deriving an actual eddy viscosity model, ZVH introduce simplifications through multiple time scale approximations.

The expression for $u^{>1}(k,t)$ may be written as:

$$u^{>1}(k,t) = M^{>}(k) \int_{-\infty}^{t} d\tau\ G_1(k,t,\tau) \times$$

$$\int d^3 j \Big\{ u^{<}(k-j,\tau)u^{<}(j,\tau) + 2u^{<}(k-j,\tau)u^{>0}(j,\tau) + u^{>0}(k-j,\tau)u^{>0}(j,\tau) \Big\} \qquad 5.4\text{-}6$$

After substituting $u^>(k,t) = u^{>0}(k,t) + \lambda u^{>1}(k,t)..$ into the low wavenumber equation 5.4-2 and ensemble averaging of the fine scale forces, there are two $O(\lambda^2)$ contributions to the new non-local eddy-damping function. The first contribution is exactly analogous to 5.3-5 and ZVH label it $\eta_1{}^D(k,t,\tau)$:

$$\eta_1{}^D(k,t,\tau) = -4\lambda_0{}^2 M(k) \int d^3j \ M^{>1}(j) G_1(j,t,\tau) \ D(k) \times$$

$$\int_{-\infty}^{t} d\tau' \ G_1{}^2(|k-j|,t,\tau') D(k-j) W_0 |k-j|^{-y} \qquad\qquad 5.4\text{-}7$$

The second contribution, due to the re-expanded triple term is labeled $\eta_1{}^T(k,t,\tau)$:

$$\eta_1{}^T(k,t,\tau) = -4\lambda_0{}^2 M_{\alpha\beta\gamma}(k) \int d^3j \ M^{>0}(j) G_0(j,t,\tau) \ D(k) \times$$

$$\int_{-\infty}^{t} d\tau' \ G_1{}^2(|k-j|,t,\tau') D(k-j) W_0 |k-j|^{-y} \qquad\qquad 5.4\text{-}8$$

It is noted that the difference between $\eta_1{}^D$ and $\eta_1{}^T$ are the factors $M^{>1}(j)G_1(j,t,\tau)$ in the former and $M^{>0}(j)G_0(j,t,\tau)$ in the latter. The integration domain of $\eta_1{}^D$ is the overlap of $|j|^{>1}$ and $|k-j|^{>1}$ (ie. the same shell radius) while the domain of $\eta_1{}^T$ is the overlap of $|j|^{>0}$ and $|k-j|^{>1}$ illustrated in figure 5.2. Also, $\eta_1{}^T$ contains G_0 which contains v_0. This point is another deviation from the intent of RNG methodology which is to upgrade a given coefficient at each step of scale elimination. For this reason the triple velocity term cannot be properly renormalized. Both of these points are important to the method used in this investigation and will be discussed further.

Thus, the final form of the momentum equation after elimination of the second shell is:

$$\left(\frac{\partial}{\partial t} + v_0 k^2\right) u(\mathbf{k},t) + \sum_{i=0}^{1} \int_{-\infty}^{t} d\tau\, \eta_i(\mathbf{k},t,\tau)\, u(\mathbf{k},\tau) =$$

$$f(\mathbf{k},t) + \lambda M(\mathbf{k}) \int d^3 j\, u(\mathbf{k}-\mathbf{j},t) u(\mathbf{j},t) +$$

$$+ \sum_{i=0}^{1} 2\, \lambda^2 M(\mathbf{k}) \iint d^3 j\, d^3 j'\, M^{>i}(\mathbf{j}) \int_{-\infty}^{t} d\tau\, G_i(\mathbf{k},t,\tau)\, u(\mathbf{k}-\mathbf{j},t) u(\mathbf{j}',\tau) u^<(\mathbf{j}-\mathbf{j}',\tau) \qquad 5.4\text{-}9$$

where

$$\eta_i(\mathbf{k},t,\tau) = \eta_i^T(\mathbf{k},t,\tau) + \eta_i^D(\mathbf{k},t,\tau) \qquad 5.4\text{-}10$$

and $\eta_0^T(\mathbf{k},t,\tau) = 0$.

The above result may be generalized to the removal of the $(n+1)$ subgrid shell:

$$\left(\frac{\partial}{\partial t} + v_0 k^2\right) u_\alpha(\mathbf{k},t) + \int_{-\infty}^{t} d\tau \sum_{i=0}^{n} \eta_i(\mathbf{k},t,\tau)\, u_\alpha(\mathbf{k},\tau) =$$

$$f_\alpha(\mathbf{k},t) + \lambda M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3 j\, u_\gamma(\mathbf{k}-\mathbf{j},t) u_\beta(\mathbf{j},t)$$

$$+ 2\sum_{i=0}^{n} \lambda^2 M_{\alpha\beta\gamma}(\mathbf{k}) \iint d^3 j\, d^3 j'\, M_{\beta\beta'\gamma'}^{>i}(\mathbf{j}) \int_{-\infty}^{t} d\tau\, G_i(\mathbf{k},t,\tau) \times$$

$$\left\{ u_\gamma(\mathbf{k}-\mathbf{j},t) u_{\beta'}(\mathbf{j}',\tau) u_{\gamma'}(\mathbf{j}-\mathbf{j}',\tau) \right\} \qquad 5.4\text{-}11$$

where the directional subscripts have been restored.

The functions $\eta_i(\mathbf{k},t,\tau) = \eta_i^D(\mathbf{k},t,\tau) + \eta_i^T(\mathbf{k},t,\tau)$ are given by the relations:

$$\eta_i^T(\mathbf{k},t,\tau) = -4\lambda_0^2 M_{\alpha\beta\gamma}(\mathbf{k}) \sum_{i'=0}^{i-1} \int d^3 j\, M_{\beta\beta'\gamma'}^{>i'}(\mathbf{j}) G_i(\mathbf{j},t,\tau)\, D_{\alpha\gamma'}(\mathbf{k}) \times$$

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$$\int_{-\infty}^{t} d\tau' \ G_i^2(|\mathbf{k\text{-}j}|,t,\tau')D_{\beta'\gamma}(\mathbf{k\text{-}j})W_0|\mathbf{k\text{-}j}|^{-y}$$

5.4-12

and

$$\eta_i^D(\mathbf{k},t,\tau) = -4\lambda_0^2 M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j M_{\beta\beta'\gamma^{>i}}(\mathbf{j})G_i(\mathbf{j},t,\tau) \ D_{\alpha\gamma}(\mathbf{k}) \ \times$$

$$\int_{-\infty}^{t} d\tau' \ G_i^2(|\mathbf{k\text{-}j}|,t,\tau')D_{\beta'\gamma}(\mathbf{k\text{-}j})W_0|\mathbf{k\text{-}j}|^{-y}$$

5.4-13

The Green's function $G_i(\mathbf{k},t,\tau)$ is given by ZVH as a solution to the equation:

$$(\frac{\partial}{\partial t} + v_0k^2)G_i(\mathbf{k},t,\tau) + \int_{\tau}^{t} ds \ \eta_{i\text{-}1}(\mathbf{k},s,\tau)G_i(\mathbf{k},s,\tau) = \delta(t\text{-}\tau)$$

5.4-14

5.5 Simplified, renormalized momentum equations

The above equations are too complex as a model of turbulence to be useful in practice. ZVH simplify them through a multitime scale analysis to recover an eddy viscosity turbulence model in place of the non-local eddy-damping function. The simplifying approximations of ZVH have some analogy with the YO procedure of setting $\omega \longrightarrow 0$ where ω is the Fourier transform of time for $u(\mathbf{k},\omega)$ in the eddy viscosity expression 4.6-8. However, it is found in the current investigation that as k/Λ increases, the size of terms involving ω increases and the non-local behavior of the 'eddy viscosity' becomes more important. For this reason, the ZVH approximations will be presented here in detail and compared to the equivalent results of this investigation in chapter 9.

ZVH assume that $\eta_i(k,t,\tau)$ evolves on a much faster time scale than $u_\alpha(k,\tau)$ and so the latter term is taken outside the integral, with τ replaced by t. Also. $u_\alpha(k,t)$ is taken outside the summation over the shell contributions. This step implies that $k < \Lambda_n$.

$$\int_{-\infty}^{t} d\tau \sum_{i=0}^{n} \eta_i(k,t,\tau)\, u_\alpha(k,\tau) \approx u_\alpha(k,t) \int_{-\infty}^{t} d\tau \sum_{i=0}^{n} \eta_i(k,t,\tau) \qquad 5.5\text{-}1$$

This approximation reduces the eddy-damping function to an eddy viscosity coefficient:

$$v_{n+1}(k)k^2 = v_0 k^2 + \int_{-\infty}^{t} d\tau \sum_{i=0}^{n} \eta_i(k,t,\tau) \qquad 5.5\text{-}2$$

Note that near the cutoff the time constants of $u^<(k,\tau)$ and $u^>(k,\tau)$ are of the same order, so moving the $u^<(k,\tau)$ outside the temporal integral and then calculating the eddy viscosity near the cutoff is problematic.

Once the approximation 5.5-1 is made, the temporal integrals in equation 5.5-2 may be evaluated exactly. Taking the example of $\eta_0(k,t,\tau)$, the exponential Green's functions of the form 5.1-5 are readily evaluated to yield:

$$\int_{-\infty}^{t} d\tau\ \eta_0(k,t,\tau) =$$

$$\int_{-\infty}^{t} d\tau G_0(j,t,\tau)4\lambda M_{\alpha\beta\gamma}^<(k)D_{\alpha\rho}(k)\int d^3j\ W_0 |k\text{-}j|^{-y}\ D_{\beta\sigma}(k\text{-}j)M_{\gamma\rho\sigma}(j)\int_{-\infty}^{t} d\tau' G_0^2(|k\text{-}j|,t,\tau')$$

$$= 2\lambda M_{\alpha\beta\gamma}^<(k)D_{\alpha\rho}(k)\int d^3j\ W_0 |k\text{-}j|^{-y}D_{\beta\sigma}(k\text{-}j)M_{\gamma\rho\sigma}(j)\frac{1}{v_0 |k\text{-}j|^2 v_0 j^2} \equiv \delta v_0(k) \qquad 5.5\text{-}3$$

Figure 5.1 Shell integration domains for finite k in the outer region.

Region A, η^D

Region C, η^T

Region B

Region C, η^T
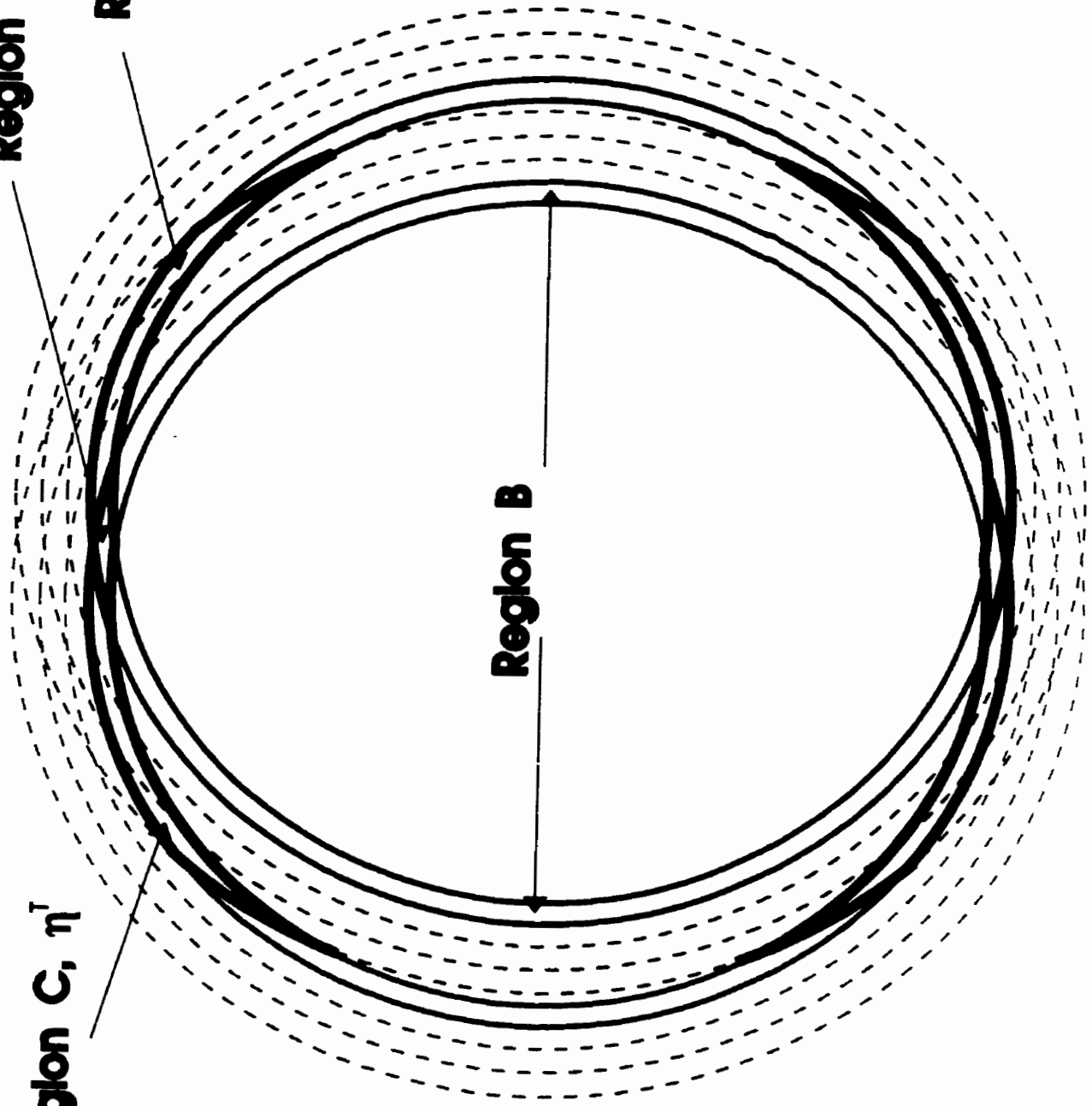
Figure 5.2 Shell integration domains for finite k in the self-similar region.
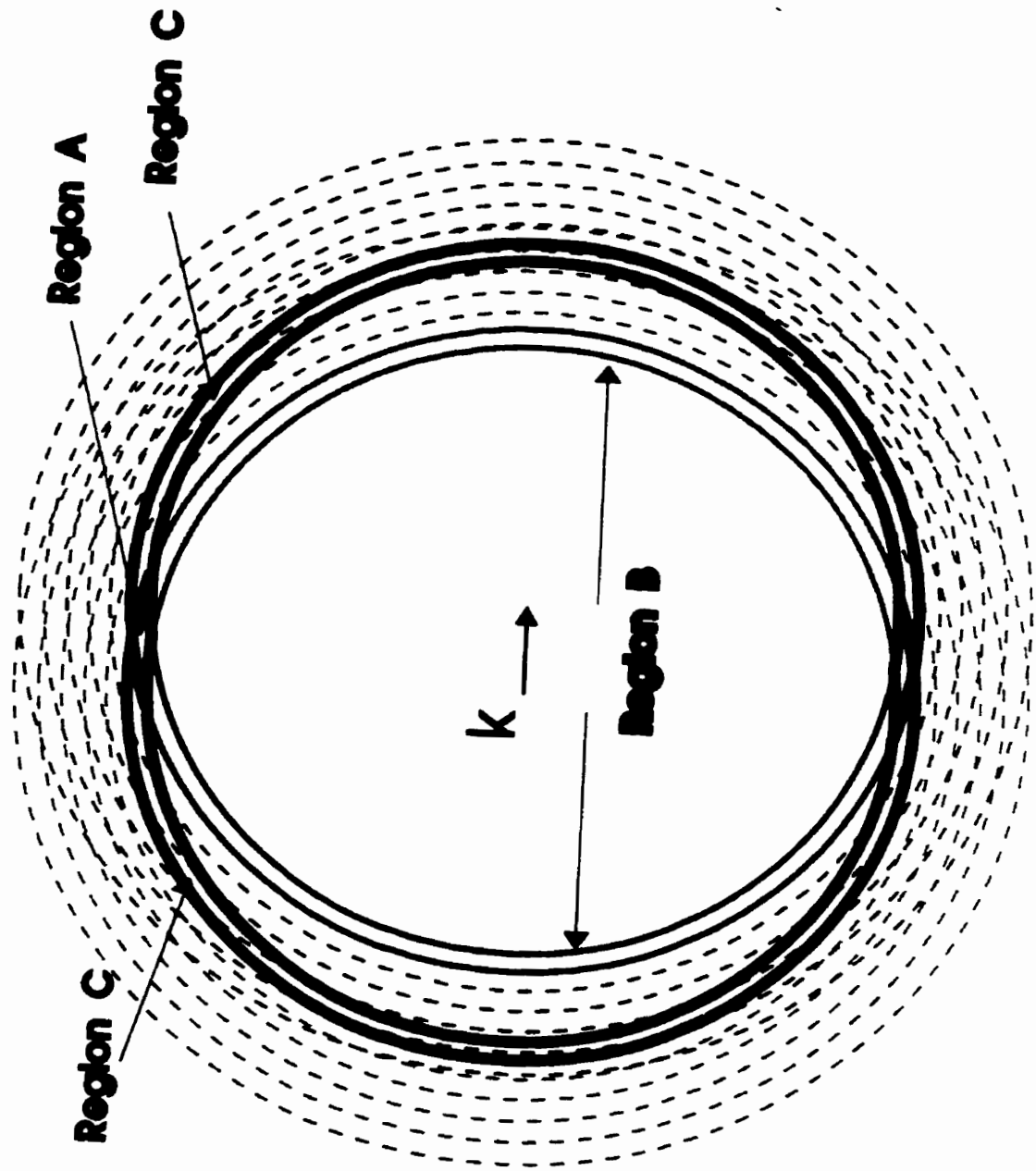
Using approximations 5.5-1 and the result 5.5-3, and proceeding to the removal of the (n+1) shell, ZVH obtain the recursion relation:

$$V_{n+1}(k) = V_n(k) + \delta V_n(k) \qquad 5.5\text{-}4$$

where:

$$\delta V_n(k) =$$

$$\frac{1}{k^2}\sum_{i=0}^{n} 2\lambda M_{\alpha\beta\gamma}(k)D_{\alpha\rho}(k)\int d^3j \; W_0 \, |k\text{-}j|^{-\gamma} D_{\beta\sigma}(k\text{-}j)M_{\gamma\rho\sigma}(j)\frac{1}{v_n(|k\text{-}j|)|k\text{-}j|^2 v_i(j)j^2} \qquad 5.5\text{-}5$$

with the integration limits $k_{n+1} < |k\text{-}j| < k_n$ and $k_{i+1} < j < k_i$, $i = 0.1, \ldots n$. It will be seen later in this investigation that the presence of previous values of the eddy viscosity, v_i, in the expression for $\delta V_n(k)$ poses major problems for the transformation of the recursion relation into a differential equation for v_n.

The integration domains in wavenumber space corresponding to equation 5.5-3 are indicated in figures 5.1 and 5.2. Region A represents the intersection of two spherical shells of radii Λ_n. In three dimensions it is a ring with the cross-section shown in figure 5.1. Region A has a volume of order $\Lambda(\Delta\Lambda)^2$ and corresponds to the YO intersection which is of order $\Lambda^2(\Delta\Lambda)$ when $\frac{k}{\Delta\Lambda} << 1$ but goes to $\Lambda(\Delta\Lambda)^2$ for $\frac{k}{\Delta\Lambda} >> 1$. Stirring forces at a wavenumber in the nth shell interact with each other through an intermediate wavenumber also in the nth shell. Physically, the intermediate wavenumber corresponds to a component of the Fourier velocity coefficient in the nth shell so that $M^{>n}(j)\int u^{<}u^{>0}$ contributes to $u^{>}$. It is noted that the variation of the parameter $\frac{k}{\Delta\Lambda}$ is the connecting link between the methods of YO and ZVH.

Region C represents the contributions of the triple velocity products to the eddy viscosity. Stirring forces at a wavenumber in the nth (current) shell interact with each other through a higher, intermediate wavenumber in a previous shell number n-i. This contradicts the general intent of the RNG method which is to eliminate the turbulent fluctuations in a given shell so that they play no further part in the problem and the equation coefficients are increased in compensation. However, the

partial averaging procedure affects only those components of $u^>$ which contain the stirring forces. The components of $u^>$ with the form $M^> \int u^< u^<$ are not affected by the partial averaging. These terms form the triple products, $\int u^< M^> \int u^< u^<$ when substituted into the equation for $u^<$.

For the range of wavenumbers from Λ_0 to Λ_0 - k, region C increases in size with the number of the eliminated shell. This is because the interacting shells are offset by a distance k and the overlap of shell at radius Λ with shells of larger radii is incomplete for $\Lambda > \Lambda_0$ - k. For this reason, the range Λ_0 to Λ_0 - k has been called the 'outer' region in this study. The outer region is not well suited for the RNG method since the calculation is not self-similar here as the portion of the shell contributing to Δv changes from shell to shell. ZVH do not discuss this region, since their numerical calculation adjusts the wavenumber limits for each discrete shell that is removed. However, for those versions of RNG which formulate a differential equation for $v(k,\Lambda_c)$, self-similarity is important. Figure 5.1 shows region B in the outer range of wavenumbers, while figure 5.2 shows the calculation in the self-similar range.

It is noted that region B is not taken into account by the method of ZVH. This is the overlap of a shell of radius Λ with shells of smaller radii. This volume contains the interactions of stirring forces in the shell number n through intermediate wavenumbers in smaller shells, n+i. Assuming that the turbulent velocity coefficients exists at all wavenumbers $0 \le k \le \Lambda$, the region C is always self-similar. The contribution of region C will be taken into account in this study by generalizing the properties of the partial averging operator. This is the subject of chapter 8.

5.6 Renormalized eddy viscosity and numerical results

ZVH perform the transformation:

$$k \longrightarrow k_{n+1} \tilde{k} \qquad\qquad 5.6\text{-}1$$

and define a renormalized eddy viscosity, $v_n^*(\tilde{k})$ by the relation:

$$v_n^*(\bar{k}) \equiv k_{n+1}^{(y+1)/3} v_n(k_{n+1}\bar{k}) \text{ for } \bar{k} \leq 1 \qquad 5.6\text{-}2$$

Therefore, the recursive relations 5.5-4 and 5.5-5 become:

$$v_{n+1}^*(\bar{k}) = h^{(y+1)/3}[v_n^*(h\bar{k}) + \delta v_n^*(h\bar{k})] \qquad 5.6\text{-}3$$

and

$$\delta v_n^*(\bar{k}) = \frac{1}{\bar{k}^2} \sum_{i=0}^{n} 2\lambda M(\bar{k}) D_{\alpha\rho}(\bar{k}) \int d^3\bar{j} \; W_0 \mid \bar{k}\text{-}\bar{j} \mid^{-y} D_{\beta\sigma}(\bar{k}\text{-}\bar{j}) M_{\gamma\rho\sigma}(\bar{j}) \frac{1}{v_n^*(\mid \bar{k}\text{-}\bar{j} \mid)\mid \bar{k}\text{-}\bar{j} \mid^2 v_i^*(\bar{j})\bar{j}^2} \qquad 5.6\text{-}4$$

with the integration limits $1 < \mid \bar{k}\text{-}\bar{j} \mid < \frac{1}{h}$ and $1 < h^i\bar{j} < \frac{1}{h}$, $i = 0,1, \ldots, n$.

ZVH numerically solved equations 5.6-3 and 5.6-4 to obtain fixed point values of v_n^* as a function of $\bar{k}$. The subgrid partition parameter was set at $h = 0.7$ and various values of y were tried, with $y = 3$ yielding the Kolmogorov energy spectrum. Figure 5.3 shows the result for $y = 3$ as compared to Kraichnan's test field model. ZVH claim that their eddy viscosity plot has a 'mild cusp' near $\bar{k} = 1$ but inspection of figure 5.3 shows that their eddy viscosity values first drop at $\bar{k}$ just below 0.2, then rise again close to $\bar{k} = 1$ but never exceed their flat region value for $\bar{k}$ close to 0. Criticisms of these results by Smith and Woodruff [7] allege that the computations of ZVH are very sensitive to the size of the partition parameter h and show no evidence of convergence as $h \longrightarrow 0$. The present author agrees with this objection regarding the ZVH result for eddy viscosity without the contribution of the triple products. As discussed above, such contribution is represented by region A of figures 5.1 and 5.2 and is of order $\Lambda(\Delta\Lambda)^2$ for a given shell. After the elimination of n shells, the resulting sum of such volumes will be of order $n\Lambda(\Delta\Lambda)^2$, and the range of eliminated wavenumbers is $n\Delta\Lambda = \Lambda_0 - \Lambda_c$. Therefore, the eliminated volume of wavenumbers is $(\Lambda_0 - \Lambda_c)\Lambda(\Delta\Lambda)$, which goes to zero as $\Delta\Lambda \longrightarrow 0$. In this limit, the ZVH recursion becomes a differential equation as discussed by Carati [11] in chapter 6.

Figure 5.3 Comparison of subgrid eddy viscosities for isotropic turbulence, according to Kraichnan and Zhou, Vahala, and Hossain (ZVH). Recursive shell thickness 0.3 used by ZVH
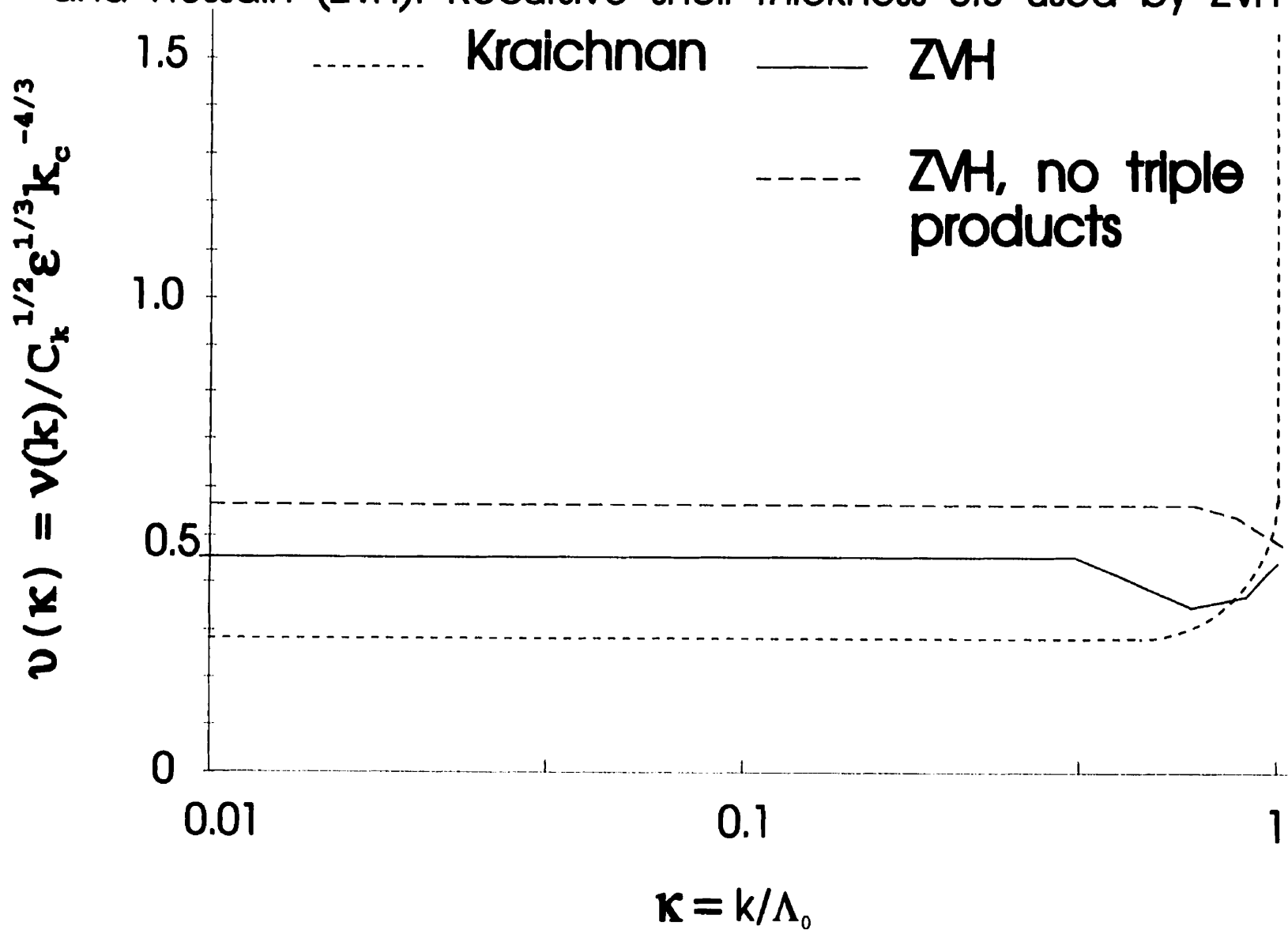
5.7 Summary

ZVH have presented a 'recursive' version of RNG which eliminated finite shells of wavenumbers and re-expanded the triple velocity products. First, a general derivation was presented involving an eddy damping function η(k,t) coupled to u(k,t) through temporal convolution integrals. While theoretically correct in the treatment of the temporal dependence of RNG, the resulting mathematical model is too complex to generate a workable turbulence model. Therefore, ZVH removed the time dependence. in effect retaining the zero order term in a Taylor series expansion of the time function. This approximation, suitable for wavenumbers far apart in the spectrum. is equivalent to disregarding the ω dependence of the eddy viscosity expression by YO. ZVH produced a numerically generated plot of the resulting eddy viscosity for h = 0.7. This could be called the 'thick shell' method since h = $\frac{\Delta\Lambda}{\Lambda}$ ~ 1. ZVH do not discuss the sensitivity of their results to the variation of h. This issue has been investigated by Carati [11] and will be presented in the next chapter. The augmented momentum equation for the resolved velocities contains triple velocity products which are difficult to simulate.

6. Local interactions in renormalization methods for Navier-Stokes equations

As described in the previous chapter, ZVH re-expanded the triple velocity products in their recursive version of RNG in order to include these in the resultant eddy viscosity, $v(k,\Lambda_c)$ and to capture the behavior of tehis quantity for wavenumbers near the cutoff, $k/\Lambda_c \longrightarrow 1$. The transfer of energy between wavenumbers close in magnitude is called 'local'. The application of RNG near $k/\Lambda_c = 1$ is problematic for a number of reasons, the first of which is that a small ratio of k/Λ_c is required to justify the partial averaging and the 'viscous-like' effect of small scales on large scales. Carati [11], Smith [30], and Smith and Woodruff [7] pointed out other problems with the recursive RNG method. Carati [11] repeated the numerical simulations of ZVH for various values of their scaling factor h, and found that the ZVH eddy viscosity results are sensitive to the value of h. Carati also carried out an approximate analytical calculation for $h \longrightarrow 1$ (shell thickness approaching 0) to show that an unphysical term of order k is obtained in addition to the eddy viscosity term. Smith [30] showed that the triple products do not have Galilean invariance. Zhou and Vahala [9] responded to the initial criticism of Carati. To date, the research community remains divided on the ability of RNG to represent local interactions.

This chapter reviews the work of Carati [11] in some detail since his approach is similar to that of the present work. The subsequent rebuttal of ZV [9] is summarized and discussed. The objective is to prepare the background for the procedure carried out in part 2 of this study.

6.1 The recursive RNG in the vanishing shell thickness limit

Carati's formulation is the same as ZVH, except that it is carried out in the frequency domain. The notation will be modified to match that used elsewhere in the thesis where similar quantities are represented. The Carati equation for the total eddy damping function is:

$$\eta(k,\Lambda_n - \Delta/\Lambda_n,\omega) = \eta(k,\Lambda_n,\omega) +$$

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$$\frac{2D_0\lambda^2{}_0}{d-1}\sum_{j=0}^{n}\frac{1}{(2\pi)^{d+1}}\int_{B_j^n} d^dj \ A(k,j)\,|k\text{-}j|^{-\gamma} \times$$

$$\int d\Omega \ [-i\Omega + v(|j|,\Lambda_j)]^{-1}[\Omega^2 + v^2(|k\text{-}j|,\Lambda_n)]^{-1} \qquad\qquad 6.1\text{-}1$$

where

$$A(k,j) = M_{imn}(k)M_{mrs}(j)D_{is}(k)D_{nr}(k\text{-}j) \qquad\qquad 6.1\text{-}2$$

and

$$M_{\alpha\beta\gamma}(k) = \frac{1}{2i}\left\{ k_\beta D_{\alpha\gamma}(k) + k_\gamma D_{\alpha\beta}(k) \right\} \qquad\qquad 2.5\text{-}6$$

$$D_{\alpha\beta}(k) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \qquad\qquad 2.5\text{-}7$$

Figure 6.1 shows that the integration domain c_j^n is bounded by the following inequalities: $\Lambda_n - \Delta/\Lambda_n < |k\text{-}j| < \Lambda_n$ and $\Lambda_j - \Delta/\Lambda_n < |j| < \Lambda_j$. Here Λ_n is the radius of the current shell and Λ_j is the radius of a previous shell.

Following YO, Carati carried out the frequency integration using calculus of residues. Also, assuming the shell thickness to be small relatively to each of Λ_a, Λ_j, and k, $j \approx \Lambda_j + O(\Delta/\Lambda_n)$, and $|k\text{-}j| \approx \Lambda_n + O(\Delta/\Lambda_n)$, Carati recovered the following expression:

$$h(k,\Lambda_a - \Delta\Lambda,\omega) = h(k,\Lambda_a,\omega) +$$

$$\frac{D_0\lambda^2{}_0}{d-1}\sum_{j=0}^{n}\frac{1}{(2\pi)^d}\frac{\Lambda_n^{-\gamma}}{u(\Lambda_n)[u(\Lambda_n) + u(\Lambda_j)]}\int_{c_j^n} d^dj \ A(k,j) \qquad\qquad 6.1\text{-}3$$

where:

$$h(k,\Lambda) = k^2 v(k,\Lambda) \text{ and } u(j) = h(|j|,|j|) \qquad\qquad 6.1\text{-}4$$

Figure 6.1 Shell integration domains for finite k in the self-similar region of |k-j| shell.
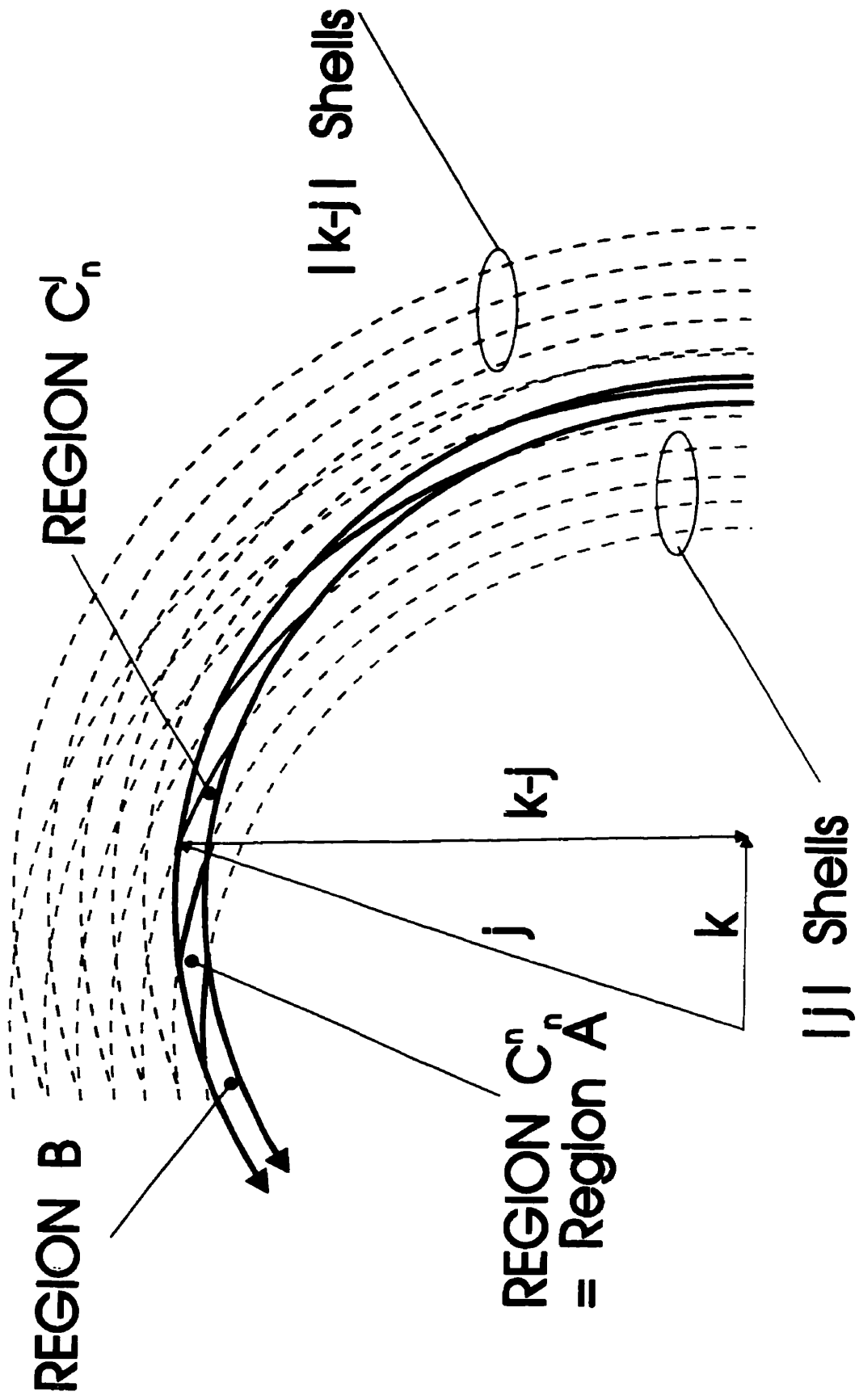
Here Carati introduced u(j) as the local viscous term. since it represents the energy transfer at the cutoff between wavenumbers of equal magnitude. It is noted that a portion of the integrand has been moved to the outside of the integral since it is approximated as constant over the small volume c_j^n. In connection with the previous notation of this study, the volume c_j^n, corresponding to the intersection of shell n with shell j, is a part of region C. Also, c_n^n = region A. Carati uses this approximation for the remaining integrand as well by first expressing the angle θ_{kj} between the vectors **k** and **j** in terms of Λ_n and Λ_j.

Letting $j_{\parallel}$ and $j\perp$, represent the longitudinal and the perpendicular components of the vector j with respect to **k**, the following approximations may be made:

$$j_{\parallel} = \frac{j^2 + k^2 - |k\text{-}j|^2}{2k} \approx \frac{\Lambda_j^2 + k^2 - \Lambda_n^2}{2k} + O(\Delta l)$$

6.1-5

where relation 6.5 was obtained from the cosine law: $j^2 + k^2 = |k\text{-}j|^2 - 2jk\cos\theta_{jk}$

$$j\perp^2 = j^2 - j_{\parallel}^2 \approx j^2 - j_{\parallel}^2(\Lambda_j,\Lambda_n,k) + O(\Delta l)$$

6.1-6

The above relations are illustrated in figure 6.1:

Carati uses the identity:

$$A(k,j) = j\perp^2 \left\{ \frac{(nd\text{-}1)k^2}{|k\text{-}j|^2} - \frac{2kj_{\parallel}}{j^2} \right\}$$

6.1-7

Now relations 6.5 and 6.6 may be substituted into 6.7, so the dependence of A(k,j) on the angle between vectors **j** and **k**, θ_{jk} is replaced by a dependence on Λ_j, Λ_n, and k so that $A(k,j) \approx A(\Lambda_j,\Lambda_n,k)$. These quantities are approximated as constant within the small integration domain B_j^n so that $A(\Lambda_j,\Lambda_n,k)$ is moved outside the integral in equation 6.3. This integral is now reduced to the evaluation of the volume V_j^n of the integration domain, B_j^n. Carati evaluates V_j^n in d-dimensions as:

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$$V_j^n = \int_{B_j^n} d^dj \approx 2S_{d-1}(\Lambda_n \Delta l)^2 \frac{\Lambda_j \Lambda_n}{k} \left[\Lambda_j^2 - \frac{(\Lambda_j^2 + k^2 - \Lambda_n^2)^2}{4k^2} \right]^{(d-2)/2}$$

6.1-8

if $\Lambda_j < \Lambda_n + k$. $V_j^n = 0$ if $\Lambda_j \geq \Lambda_n + k$

Here S_{d-1} is the area of a sphere of a unit radius in (d-1) dimensional space.
Taking d = 3,

$$V_j^n = \int_{C_j^n} d^3j \approx 2\pi(\Delta\Lambda)^2 \frac{\Lambda_j \Lambda_n}{k}$$

6.1-9

Carati now uses relations 6-5 to 6-8 to simplify equation 6.3. Also. taking Δl to be an infinitesimal quantity, the sum in 6.3 becomes an integral so the overall result is a differential - integral equation:

$$\frac{\partial h(k,\Lambda)}{\partial \Lambda} = - \frac{2W_0 \lambda^2 S_{d-1}}{(d-1)(2\pi)^d k} \frac{\Lambda^{-y}}{u(\Lambda)} \int_\Lambda^{\Lambda+k} d\Lambda' \frac{1}{u(\Lambda) + u(\Lambda')} \times$$

$$\left[\Lambda'^2 - \frac{(\Lambda'^2 + k^2 - \Lambda^2)^2}{4k^2} \right]^{(d-1)/2} \times \left[\frac{(d-1)k^2}{\Lambda^2} - \frac{(\Lambda'^2 + k^2 - \Lambda^2)^2}{\Lambda'^2} \right]$$

6.1-10

It should be noted that equation 6.1-10 is valid only in the self-similar region, $\Lambda \leq \Lambda_0 - k$, and not in the 'outer' region, as defined in chapter 5.

It is noted that Carati evaluated the wavenumber integral 6.1-3 by taking a small integration volume V_j^n, resulting from the intersection of two thin shells of thickness $\Delta l \Lambda$, Figure 6.1. The simplification available due to the small V_j^n avoids integration with respect to θ_{kj} and shifts the complications from 6.1-3 to 6.1-10. To solve the latter, two integrations are required, the first of which does have a complicated analytical solution and the second requires a numerical approximation. Further discussion will be found in chapter 10.

Carati proposed a similarity solution to 6.1-10 of the form:

$$h(k,\Lambda) = \Lambda^c \phi(k/\Lambda)$$

<div align="right">6.1-11</div>

Letting $k/\Lambda = 1$, the expression for the local viscous term is:

$$u(\Lambda) = \phi(1)\Lambda^c$$

<div align="right">6.1-12</div>

The objective is to find the exponent c and the function ϕ. Equation 6.10 is now rewritten using the dimensionless variables $r = k/\Lambda$ and $\xi = \Lambda'/\Lambda$:

$$\frac{\partial h(k,\Lambda)}{\partial \Lambda} = -M \ \Lambda^{d-y-2c+2} \ F(r)$$

<div align="right">6.1-13</div>

where $M = 2W_0\lambda^2/[(d-1)(2\pi)^d\phi^2(1)$ and

$$F(r) = \frac{S_{d-1}}{r} \int\limits_{1}^{1+r} d\xi \ \frac{1}{1 + \xi^c} \ \times$$

$$\left[\xi^2 - \frac{(\xi^2 + r^2 - 1)^2}{4r^2}\right]^{(d-1)/2} \left[(d-1)r^2 - \frac{(\xi^2 + r^2 - 1)^2}{\xi^2}\right]$$

<div align="right">6.1-14</div>

The above expression results from factoring out Λ from the right hand side of expression 6.1-10. The derivative on the left hand side of 6.1-10 may also be expressed in terms of the non-dimensional variables by using the chain rule of differential calculus:

$$\frac{\partial h(k,\Lambda)}{\partial \Lambda} = c\Lambda^{c-1}\phi(r) - \Lambda^{c-1}r\frac{d\phi(r)}{dr}$$

<div align="right">6.1-15</div>

Carati equates the right hand side of equation 6.1-15 to the right hand side of equation 6.1-14 to obtain:

$$r\frac{d\phi(r)}{dr} = c\phi(r) + M\Lambda^{d-y-3c+2} F(r)$$
<div align="right">6.1-16</div>

The solution to 6.1-16, $\phi(r)$, cannot depend on Λ by definition of a similarity solution. Therefore, the exponent of Λ in 6.1-16 must be zero yielding an equation for c:

$$c = (d-y+2)/3$$
<div align="right">6.1-17</div>

In three dimensions and with $y = 3$ as generally required by RNG methods, this gives $c = 2/3$. Substituting 6.1-17 into 6.1-16, a separable differential equation for $\phi(r)$ results with the solution:

$$\phi(r) = Mr^c \int_0^r d\rho \, \rho^{-c-1} F(\rho)$$
<div align="right">6.1-18</div>

where the integral apparently cannot be evaluated analytically. Thus, in order to evaluate 6.18, an expansion of $F(r)$ in powers of $r = \frac{k}{\Lambda}$ is necessary, [30]:

$$F(r) = \sum_{i=0}^{\infty} F_i r^i$$
<div align="right">6.1-19</div>

Carati [11] called this an *a posteriori* k/Λ expansion. Substituting 6.19 into 6.18, the resulting integral can be evaluated term by term to give:

$$\phi(r) = \frac{M}{\phi^2(1)} \sum_{i=0}^{\infty} r^i \frac{F}{i-c}$$
<div align="right">6.1-20</div>

It will be shown in the second part of this thesis that the above calculation consists of a symmetric and an non-symmetric part with respect to the eliminated wavenumber shell. The symmetrical portion allows only even powers of r (Wilson [41])

<div align="center">100</div>

and this is the basis of the conventional RNG procedure as imported from other branches of physics. Odd powers of r indicate a break of symmetry and the validity of the RNG method becomes questionable in this case.

Carati carried out the expansion 6.1-19 to third order, [30]. In three dimensions, the result is:

$$F(r) = \frac{r}{4} + \frac{4+c}{15}r^2 + \frac{26-3c}{48}r^3 + O(r^4)$$ 6.1-21

Expressing $c = (5-y)/3 = 2 - \epsilon/3$, and using 6.1-21 and 6.1-20 in 6.1-11, the result is:

$$h(k,\Lambda) = \frac{M}{\phi^2(1)} \Lambda^{2-\epsilon/3} \left[\frac{-3}{\epsilon-3} \frac{k}{4\Lambda} + \frac{3}{\epsilon} \frac{18-\epsilon}{45} \frac{k^2}{\Lambda^2} + \frac{3}{\epsilon+3} \frac{20+\epsilon}{48} \frac{k^3}{\Lambda^3} + O(\frac{k}{\Lambda})^4 \right]$$ 6.1-22

Carati observed that the first term of the series cannot be interpreted as a viscous linear term since it is proportional to k instead of k^2. The behavior of $h(k,\Lambda)$ at large scales, where $k/\Lambda \longrightarrow 0$ is:

$$h(0,\Lambda) \approx \frac{M}{\phi^2(1)} \Lambda^{2-\epsilon/3} \left[\frac{-3}{\epsilon-3} \frac{k}{4\Lambda} \right] = k^2 \frac{M}{\phi^2(1)} \Lambda^{1-\epsilon/3} \left[\frac{-3}{4(\epsilon-3)} k^{-1} \right]$$ 6.1-23

Thus, the resulting 'eddy viscosity' behaves as k^{-1} at large scales which is unphysical. Also, letting $\epsilon = 4$, as generally required in RNG theories, it is noted that the quantity in 6.1-23 is negative.

Checking the behavior of $h(k,\Lambda)$ at small scales where $k/\Lambda \longrightarrow 1$ and using k in place of Λ:

$$h(k,k) = \frac{M}{\phi^2(1)} k^{2-\epsilon/3} \left[\frac{-3}{\epsilon-3} \frac{1}{4} + \frac{3}{\epsilon} \frac{18-\epsilon}{45} + \frac{3}{\epsilon+3} \frac{20+\epsilon}{48} + O(\frac{k}{\Lambda})^4 \right] =$$

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$$k^2 \frac{M}{\phi^2(1)} \left[\frac{-3}{\epsilon-3} \frac{1}{4} + \frac{3}{\epsilon} \frac{18-\epsilon}{45} + \frac{3}{\epsilon+3} \frac{20+\epsilon}{48} + O(\frac{k}{\Lambda})^4 \right] k^{-\epsilon/3} \qquad 6.1\text{-}24$$

Substituting $\epsilon = 4$ into 6.1-24, the 'eddy viscosity' at the cutoff is:

$$v(k) \approx \frac{M}{\phi^2(1)} \left[\frac{-3}{4} + \frac{7}{30} + \frac{3}{14} + O(\frac{k}{\Lambda})^4 \right] k^{-4/3} \qquad 6.1\text{-}25$$

Therefore, up to $O(\frac{k}{\Lambda})^4$, expression 6.1-25 has a negative value of approximately $-0.302 \frac{M}{\phi^2(1)} k^{-4/3}$ which is unphysical as an 'eddy viscosity'. in effect indicating a reverse flow of energy from small scales to large scales.

Carati also reproduced the numerical calculations of ZVH for a range of shell thicknesses, using the ZVH notation of h to represent the consecutive shell radius ratio. His results are shown in figure 6.2, where the sensitivity of the result to the value of h is apparent. Also, for values of $h > 0.8$ (shell thickness of 0.2Λ) the 'eddy viscosity' becomes negative for much of the range of k/Λ in agreement with 6.23 and 6.25.

Figure 6.2 Renormalized ZVH eddy viscosity for various values of h.

$$\upsilon(\kappa) = \nu(\kappa)/C_{\mu}^{1/2}\epsilon^{1/3}\kappa_c^{-4/3}$$

$\kappa = k/\Lambda_0$

h = 0.7
h = 0.75
h = 0.8
h = 0.85

0.01
0.1
1

1.0
0.5
0.0
-0.5
-1.0
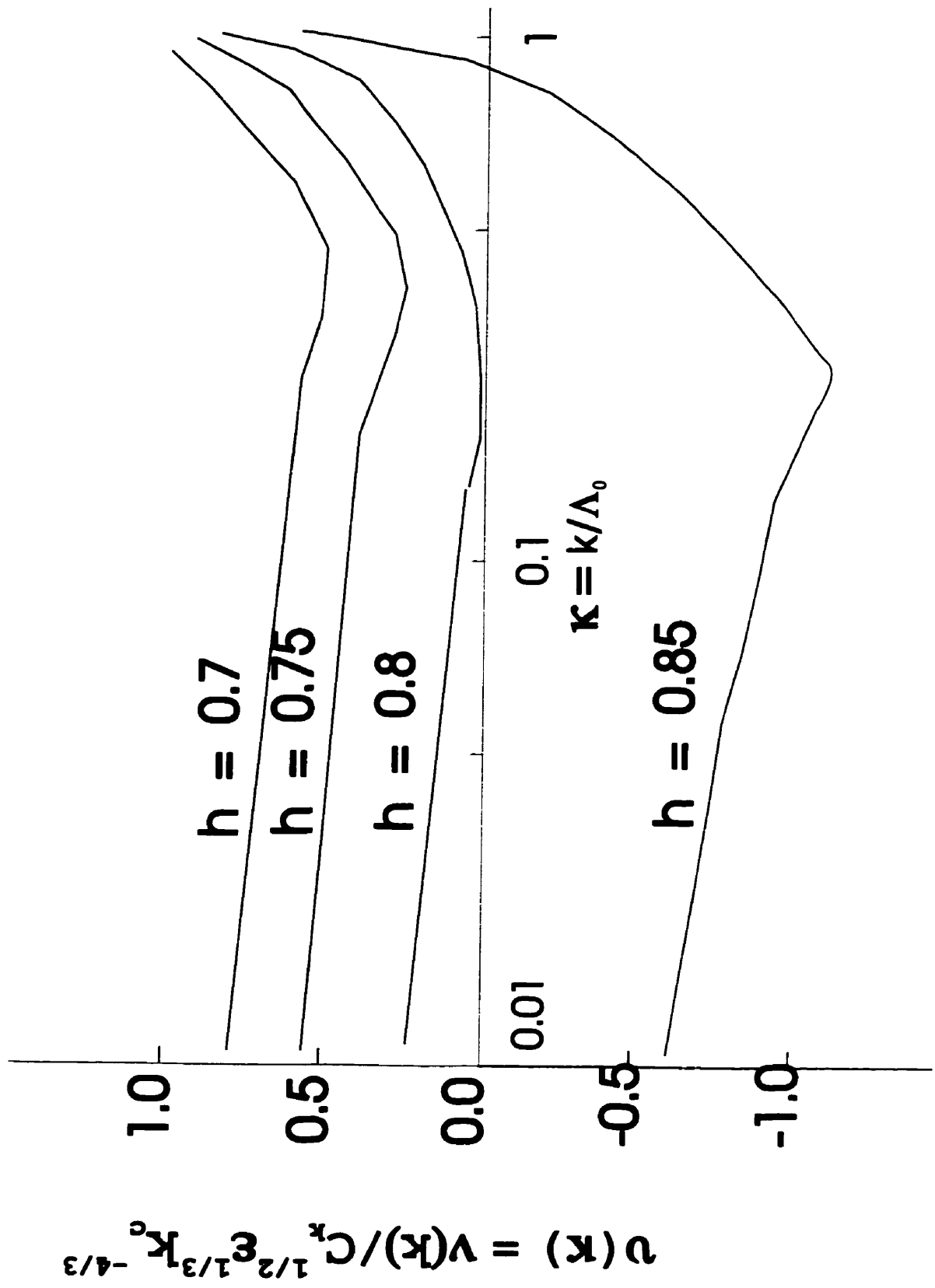
Based on the above findings, Carati concluded that the 'recursive' RNG which includes the re-expanded triple velocity products is unsuitable for evaluation of the 'eddy viscosity' and that the RNG methods are only suited to represent very 'non-local' interactions in turbulence (k/Λ << 1). This view is in agreement with the philosophy of YO $\in$-RNG.

In their rebuttal publication, Zhou and Vahala [9] acknowledged that their eddy viscosity model has a 'parametric dependence' on the shell thickness $\Delta\Lambda$ and responded to Carati's criticism concerning the k^{-1} behavior of their eddy viscosity as k $\longrightarrow$ 0. They claimed that in order to obtain meaningful results. the sequence of the two limits, $\Delta\Lambda \longrightarrow$ 0 and k $\longrightarrow$ 0 is not interchangeable.

Considering first the case of $\in$-RNG, the k $\longrightarrow$ 0 limit is taken first while $\Delta\Lambda$ remains finite as discussed in chapter 4. The ZV argument really applies to the case where $\frac{k}{\Delta\Lambda} \leq$ 1 rather than $\frac{k}{\Delta\Lambda} \longrightarrow$ 0. The outer, unshaded regions of Figure 6.3a and figure 6.3b contain the contributions to Δv_n from the triple products generated on shell n-1 and re-expanded on shell n. These are labelled as regions C_L and C_R. while the YO domain is the shaded region A. As discussed in section 4.10. the integration volume associated with the triple products is of order $k\Lambda^2$. ZV [10] use their free decaying turbulence formula for the discussion, but here their forced turbulence formula will be used instead to correspond to the rest of this study:

$$\Delta v_n(k) =$$

$$\frac{1}{k^2}\sum_{i=0}^{n}2\lambda M_{\alpha\beta\gamma}(k)D_{\alpha\rho}(k)\int d^d j \ W_0 |k\text{-}j|^{-y}D_{\beta\sigma}(k\text{-}j)M^{>i}{}_{\gamma\rho\sigma}(j)\frac{1}{v_n(|k\text{-}j|)|k\text{-}j|^2v_i(j)j^2} \qquad 5.5\text{-}5$$

$M_{\alpha\beta\gamma}(k)$ is of order k, all the integrand terms are of order $\Lambda^i + k^n$. where the exponents i and n depend on the term except for $D_{\beta\sigma}(k\text{-}j)$ which is of order unity. After including the $\frac{1}{k^2}$ factor, the whole expression prior to integration is of order k^{-1}.

$u^>(k-j)M(j)^{>n-1} \int u^< u^>$
Region C_R

$u^>(j)u^<(k-j)$
Region B_R

$u^>(k-j)u^<(j)$
Region B_L
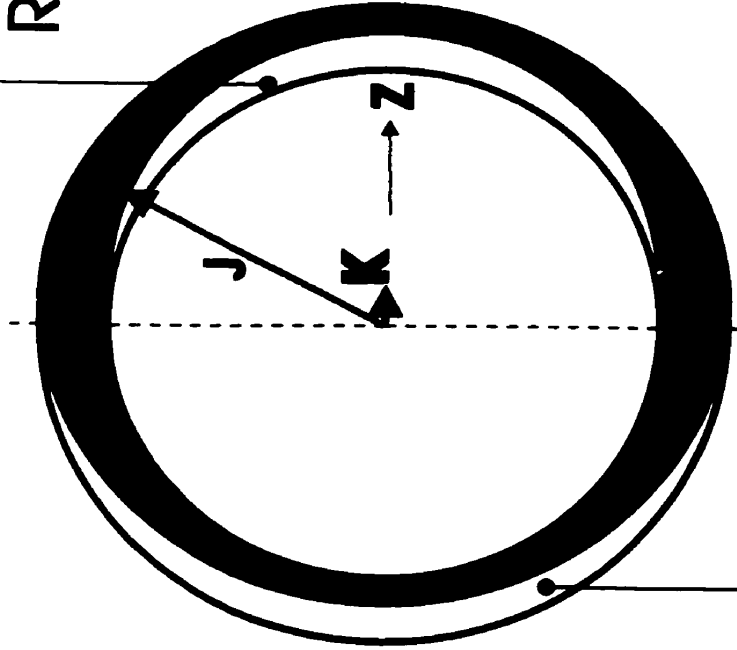
$u^>(j)M(k-j)^{>n-1} \int u^< u^>$
Region C_L

a. Shell $\Lambda e^{-\delta l} < |j| < \Lambda$
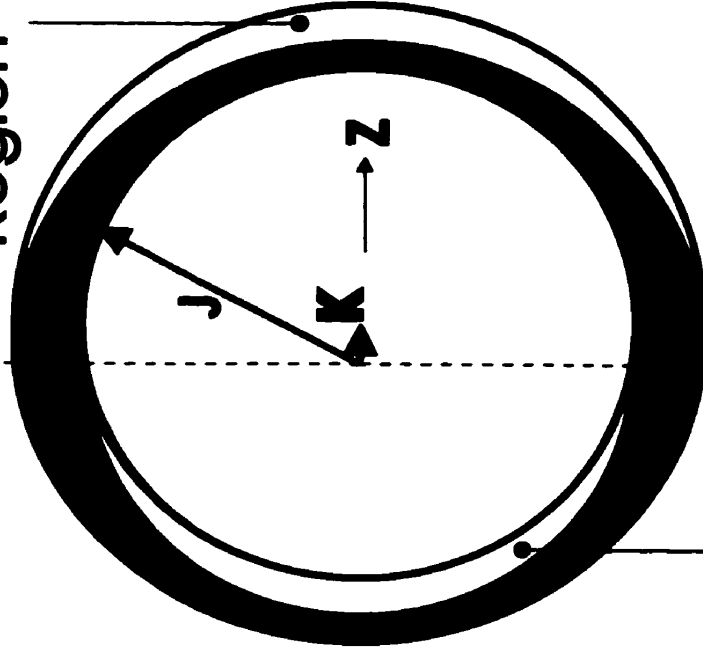
b. Shell $\Lambda e^{-\delta l} < |k-j| < \Lambda$

Figure 6.3 Integration domains for Δv and the triple products for $k/(\Lambda \delta L) < 1$.

Referring to figure 6.3, the ZVH calculation includes regions C and A. As discussed in section 4.10, the integration volume of region C is proportional to $k\Lambda^2$ cancelling with the k^{-1} factor in the integrand to create a constant value. Region A is symmetric with respect to θ and since the odd powers of k have odd powers of $\cos\theta$ as coefficients, they vanish after the angular integration. This eliminates the threat of k^{-1} singularity as $k \longrightarrow 0$ for $\frac{k}{\Delta\Lambda} \leq 1$. However, ZV do not address the question whether the contribution of regions C is negative for $\frac{k}{\Delta\Lambda} \leq 1$. This question will be considered in chapter 9. It should be noted that here ZVH in effect validate the YO method, since the contribution of region C will go to zero as $k \longrightarrow 0$. It is also noted that the ZVH calculation with h = 0.7. corresponding to $\Delta\Lambda$ = 0.3. corresponds to the YO procedure for small values of k although it is not as accurate.

Having disposed of the singularity problem for $\frac{k}{\Delta\Lambda} \leq 1$, ZV then considered the case where $\frac{k}{\Delta\Lambda} > 1$, corresponding to the Carati analysis. More precisely. ZV let $\Delta\Lambda \longrightarrow 0$. while k is 'small but fixed'. ZV derive the size of the integration volume of region A as $O(\Delta\Lambda^2)$. This may also be surmized from figure 6.1 since $A = C_n^n$. However. ZV only admit C_n^{n-1} as the additional domain of the triple products. Since the latter volume is also $O(\Delta\Lambda^2)$, ZV conclude that Δv. the correction to eddy viscosity is $O(\Delta\Lambda^2)$ and so an ordinary differential equation for $v(k,\Lambda_c)$ cannot be formed and that the $\Delta\Lambda \longrightarrow 0$ limit is singular for the recursive RNG, when k is finite.

It is the observation of the author of this study that ZV have considered only n = 2 with $\frac{k}{\Delta\Lambda} \gg 2$ which led them to the above conclusion. The correct application of the 'recursive' RNG is in the self-similar region. $\Lambda_0 - k > \Lambda$. as discussed in section 5.6 and indicated in figure 5.2. The ZV argument applies only in the begining of the 'outer' region, as shown in figure 5.1. The Carati analysis implicitly discards the 'outer' region which is permissible if $k/\Lambda \ll 1$.

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6.2 Galilean invariance of the triple products

Smith [30] demonstrated the lack of Galilean invariance of the triple velocity products and proposed that the unphysical results obtained by Carati may be due to this fact. It is recalled that the Navier-Stokes equations possess Galilean invariance since the equations are a statement of Newton's law, $F = ma$. Galilean invariance requires that the solution for a, or $\frac{Du(x,t)}{Dt}$ is invariant under a constant velocity shift of the reference frame. The YO method temporarily loses Galilean invariance at a given step of the procedure by introducing the λ expansion. In particular, the Stokes solution, $u^{>0}$, does not have Galilean invariance. Also, it is well known that the two-point, two-time correlations introduced by RNG lack Galilean invariance. However, the white noise correlation of the stirring forces eliminates the two-time property of the correlations, and after the λ series is absorbed by the eddy viscosity, the Galilean invariance property is restored at each step of the procedure. For the recursive RNG, the unphysical triple products are retained in the modified momentum equation so the Galilean invariance is not recovered until the very end of the procedure, when the remaining triple products are discarded. Smith [30] concludes that the limit $k/\Delta\Lambda \longrightarrow 0$ may be necessary to obtain physically meaningful results. Foster, Nelson, Stephen, FNS [28] have also stated that the symmetry of the wavenumber shell is implied by the Galilean invariance requirement. Kraichnan (Leslie [25] explains that lack of Galilean invariance to random shifts in the velocity of the reference frame causes spurious convection effects in the approximate solution. This spurious convection causes artificial decay of the energy spectrum due to 'phase mixing' - spurious cancelling of covariances of the velocity Fourier modes. Kraichnan recommends a Lagrangian rather than Eulerian formulation in order to preserve Galilean invariance.

6.3 The iterative filtering method

Following up on his previous work, Carati [40] has developed the iterative filtering method (IF) which is as yet unpublished but was made available to the present author through a personal communication. A detailed review of this work is

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beyond the scope of this thesis, however the major principles will be outlined here. The IF method is applicable to finite k wavenumbers. Like the RNG, the IF relies on the λ^2 closure and incrementally evaluates corrections to the eddy damping function. Unlike the RNG, the IF also evaluates corrections to the random stirring force. Since any random variable may be decomposed into an average plus a zero-mean fluctuation. the IF evaluates the $O(\lambda^2)$ averages on a given shell and retains fluctuations from the term $\lambda \int u^< u^{i>0}$ where $u^{i>0}$ is the response due to the stirring force on shell (i). It is noted that the average $\lambda \langle \int u^< u^{i>0} \rangle$ is taken equal to 0 like the RNG. Proceeding to the next shell, spectral splitting and expanding in powers of λ yields $\lambda \int u^{i+1>0} u^{i>0}$, a zero mean random variable which is now used to augment $f^{>i-1}$. This step is a major departure from the FNS and YO RNG methods.

Among the methods discussed so far, only the IF accounts for the $\lambda \int u^< u^{i>0}$ term which contains the stirring forces acting in region B of the shell (i). However, instead of integrating over region B directly (as will be done in the current study), the IF takes these forces into account as secondary induced forces on subsequent shells. An encouraging characteristic of the method is that there is a general symmetry of the integrands on each half of the integration contour C.

6.4 Discussion

In summary, Carati has correctly extended the ZVH procedure to the case of an infinitesimal shell. His results do not represent the physics of an eddy viscosity due to the k coefficient of the leading term and also due to the negative sign of this term. The leading term is considered 'not physical' because it is not of the same form as any of the terms present in the original Navier-Stokes equations and cannot be identified as the effect of viscosity, the non-linear advection or the forcing. Including such terms in the modified momentum equations is unlikely to produce the correct solution for the large scales because it changes the mathematical characteristics of the equations. Thus, the recursive RNG appears to fail in the limit of $\frac{\Delta\Lambda}{k} \longrightarrow 0$. From a numerical standpoint, the ZVH recursive equation is not consistent as a representation of the differential equation for the eddy viscosity.

In the opinion of the present author, the ZVH rebuttal argument does not address the cause of the failure of the recursive RNG.

7. Objectives and Outline of Part 2.

7.1 Introduction

In the context of modeling isotropic turbulence, the objective of the RNG method is to model the average effects of the small scales on the large scales where the small scales have a high spatial wavenumber or temporal frequency. During the RNG procedure, one imagines iterative spatial filtering, as shells of wavenumber radius Λ and thickness $\Delta\Lambda$ are removed. This aim is unambiguous when there is a wide separation of scales between the resolved velocities, $u(k,t)$ and the subgrid boundary Λ. $\frac{k}{\Lambda} \ll 1$. This constraint has been labelled 'non-local interactions' in the literature. In the YO analysis, the non-local requirement takes on the extreme form $\frac{k}{\Delta\Lambda} \ll 1$. Considering the convolution integral, $\int^{\Lambda} u^>(k\text{-}j)u^>(j)d^3j$, a physical interpretation of this constraint is that k is so much smaller than Λ that the momentum drain from $u(k)$ by wavenumbers of order Λ is the same in all directions on average. Also, both j and $k\text{-}j$ reach into the unresolved subgrid region in the shell so that the energy transferred from eddies at k via the eddy viscosity to the subgrid scales, may be transfered into both j and $k\text{-}j$. Averaging over a thin shell of the in. gration domain corresponds unambiguously to eliminating a range of small flow scales. Therefore, as had been discussed in chapter 4, the RNG method for turbulent flows has some success in modeling the non-local interactions of turbulence.

However, for $\frac{k}{\Delta\Lambda} \gg 1$, (and $\frac{k}{\Lambda} \leq 1$), the k vector triad includes only one vector in the current shell, say $|j| = \Lambda$. Interactions where $\frac{k}{\Lambda} \sim 1$ have been labelled 'local' in the literature. As the vector j travels around the shell, $|k\text{-}j|$ varies from supergrid scales (figure 7.1, region B) to the range of subgrid scales (figure 7.1, region C). In this case, the objective is that the RNG incrementally models the effects of wavenumber triad interactions which involve at least one subgrid wavenumber. In the conventional RNG method, these interactions take the form of the

figure 7.1

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triple products, $\int u^<(k\text{-}j)M^>(j)\int u^<u^<$, where the wavenumber j falls in the subgrid range. In this case, the elimination of a shell of Fourier coefficients cannot be interpreted as spatial filtering since higher wavenumbers remain in the equation. For a temporal filtering analogy, the term $M^>(j)\int u^<u^<$ may be taken as that portion of $u^>(j,\omega)$ which falls below the temporal filtering cutoff. In this context. re-expanding the triple products on subsequent shells is analogous to gradually lowering the low-pass limit to reduce the remaining portion of $u^>(j,\omega)$.

Another term which poses difficulty for the RNG is $\int u^>M^<(j)\int u^<u^>$. which originates from $\int u^<u^>$ and is typically set to zero. The Iterative Filtering procedure due to Carati deals with such terms as discussed below, while a different method is proposed in this study.

All of the RNG methods discussed up to this point involve integrations over regions C as indicated in figure 7.1. To the best of this author's knowledge. the effects of the stirring forces present in regions B are not taken into account in any published literature. The Iterative Filtering method (herein referred to as IF) proposed by Carati takes region B forces into account as secondary induced forces on subsequent shells. In this way, region B forces affect the eddy viscosity through the fourth moment:

$$\sum_{j=1}^{n} <f^i>f^{i+j}>f^i>f^{i+j}>>$$ 7.1-1

where the $|k\text{-}j|$ is divided into n shells of thickness $\Delta\Lambda$ to include all of region C. However, if the region B forces (forces which act through an intermediate wavenumber lower than the shell) were to be treated in the same way as the forces acting in regions A and C, these would affect the eddy viscosity through the second moment:

$$\sum_{j=1}^{m} <f^i>f^i>>$$ 7.1-2

where the $|k\text{-}j|$ is divided into m shells of thickness $\Delta\Lambda$ to include all of region B.

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It may be concluded that a number of difficulties emerge when attempting to use the RNG method to model 'local interactions'. Carati [11] and Smith [30] have shown that eddy viscosity calculations do not yield physical results. The Iterative Filtering method [40] is beyond the scope of this study. In the remaining part of this work, the RNG method will be modified to better represent the 'local interactions' of turbulence in Fourier space.

7.2 Objectives and outline

Refering to section 1.3, for the objectives of the present work, a review of the RNG methods for fluid turbulence has been presented in chapters 3 through 6. Additional RNG procedures have been applied to fluid flow, for example those based on the field theoretic approach, and also the IF method of Carati. These are beyond the scope of this study. The remaining objectives for part 2 are to examine if the RNG analytical tools are suitable to represent local interactions, and to attempt to obtain an eddy viscosity function $v(k,\Lambda)$, hopefully with a cusp behavior as reported in literature dealing with other theories. The investigation will proceed in the following stages:

1. The RNG method uses several steps which depend on the non-local approximation. These are the partial averaging, and the Markovian approximation of the time dependence. Also, the λ series expansion is affected by the non-local assumption. The impact of local interactions on the above approximations will be examined in chapter 8. First, the partial averaging procedure will be generalized allowing different properties to be specified. Modification of the partial averaging will allow including region B forces and also possible elimination of the triple velocity products. Second, some characteristics of the λ expansion series will be examined. Finally, an observation will be made about the Markovian approximation for local interactions.

2. Three different variations of partial averaging leads to three differential equations for the eddy viscosity. These equations will be derived in chapter 9.

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3. The numerical solutions for the eddy viscosity formulations will be presented in chapter 10. The results will be discussed and compared to the values proposed in literature. Chapter 11 will conclude the current investigation with recommendations for future work.

8. Modified partial averaging of the Navier-Stokes equations for the RNG method

8.1. Introduction

This chapter deals with several steps of the RNG procedure, the partial averaging of the Navier-Stokes equations in Fourier space, the λ series expansion, and the Markovian approximation.

8.2 The partial averaging operation

Typically, the partial averaging operation is defined as equivalent to the full ensemble average for those random variables which satisfy certain conditions. If the conditions for partial averaging are not satisfied, the random variable will be unaffected and will appear to be statistically sharp (invariant under the averaging). Such definition is unphysical for variables very close in temporal and spatial scales that fall on different sides of the defined demarcation line, i.e. for a wavenumber lower or higher than a given limit, or a temporal frequency lower or higher than a given limit.

A simple definition of the partial averaging operator is based on wavenumber magnitude alone. The Fourier coefficients with a magnitude lower than the demarcation value Λ are unaffected by the operation, those with a magnitude higher than Λ are ensemble averaged:

$$\langle u(k) \rangle^{\!>} = \begin{cases} u(k) & \text{for } 0 < k < \Lambda \\ \langle u(k) \rangle & \text{for } \Lambda \le k \le \Lambda_0 \end{cases} \qquad 8.2\text{-}1$$

where $\langle .. \rangle$ is the ensemble average and $\langle .. \rangle^{\!>}$ is the partial average. It is easy to show that such a definition is inadequate for turbulence models. Consider the Navier-Stokes equation for $u^<(k,t)$:

$$\left(\frac{\partial}{\partial t} + v_0 k^2\right) u^<(k,t) = \lambda M^<(k)\int u^< u^< + \lambda 2M^<(k)\int u^< u^> + \lambda M^<(k)\int u^> u^> + f^<(k) \qquad 8.2\text{-}2$$

$$\underbrace{\qquad\qquad}_{8.2\text{-}2a} \quad \underbrace{\qquad\qquad}_{8.2\text{-}2b} \quad \underbrace{\qquad\qquad}_{8.2\text{-}2c} \quad {}_{8.2\text{-}2d}$$

Applying the operation (8.2-1) to equation 8.2-2 causes terms 8.2-2b and 8.2-2c to vanish and clearly, the equation cannot remain valid. Therefore, the properties of the partial averaging operator must be modified.

Motivated by the above discussion, a general procedure for partial averaging of the Navier-Stokes equations in Fourier space is presented here. Let u_i be a component of the velocity vector $\mathbf{u}$. First, the following general property is proposed:

$$< <u_i u_j ... u_k>> > = < u_i u_j ... u_k > \qquad 8.2\text{-}3$$

where $< >>$ again denotes partial averaging.

Next, the Fourier velocity coefficient, $u_i(k,t)$ is decomposed as:

$$u_i(k,t) = \bar{u}_i(k,t) + u'_i(k,t) \qquad 8.2\text{-}4$$

Further properties of the hypothetical partial averaging are as follows:

$$< \bar{u}_i \bar{u}_j ... \bar{u}_k >> = \bar{u}_i \bar{u}_j ... \bar{u}_k + \lambda^m \text{ (remainder)} \qquad 8.2\text{-}5$$

$$< u'_i u'_j ... u'_i >> = < u'_i u'_j ... u'_i > + \lambda^n \text{ (remainder)} \qquad 8.2\text{-}6$$

where $m \geq 3$ for the first moment and $m \geq 2$ for the second moment. Also, $n \geq 3$ for the first moment and $n \geq 1$ for the second moment. Since RNG retains terms up to λ^2, higher order moments are beyond the scope of this work. The remainder terms have a zero average.

For the products of $\bar{u}$ and u':

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$$< \overline{u_i u_j ... u_k} \ \overline{u'_i u'_j ... u'_i} >> \ = \ \overline{u_i u_j ... u_k} < u'_i u'_j ... u'_i >> + \ \lambda^m \ (\text{remainder}) \qquad 8.2\text{-}7$$

Combining the results of equations 8.2-3 and 8.2-7 leads to the requirement:

$$< \overline{u_i u_j ... u_k} \ \overline{u'_i u'_j ... u'_i} > =$$
$$< \overline{u_i u_j ... u_k} < u'_i u'_j ... u'_i >> > + \ \lambda^m < (\text{remainder}) > \qquad 8.2\text{-}8$$

where the remainder term in equation 8.2-8 will have a non-zero average in general. If the remainder terms in equations 8.2-5 to 8.2-8 have the wavenumber and temporal frequency similar to those of $\overline{u}(k,t)$ then the properties of partial averaging are similar to filtering in space or in time.

Further discussion of the partial averaging will be limited to first and second moments. Recalling from section 2.5 that the spectral correlation tensor is defined as:

$$Q_{ij}(k) \ = \ <u_i(k,t)u_j(-k,t)>$$

and that $<u_i(k,t)u_j(j,t)> \ = \ \delta(k+j)Q_{ij}(k)$, the following second moment properties are compared:

$$< \overline{u_i(k,t)} \ \overline{u_j(j,t)} > \ = \ \delta(k+j) \ \overline{Q}_{ij}(k,t) \qquad 8.2\text{-}9$$

$$< \overline{u_i(k,t)} \ \overline{u_j(j,t)} >> \ = \ \overline{u_i(k,t)} \ \overline{u_j(j,t)} + O(\lambda^2) \qquad 8.2\text{-}10$$

$$< u'_i(k,t) \ u'_j(j,t) > \ = \ \delta(k+j) \ Q'_{ij}(k,t) \qquad 8.2\text{-}11$$

For $< u'_i(k,t) \ u'_j(j,t) >>$, the following definition is proposed:

$$< u'_i(k,t) \ u'_j(j,t) >> \ =$$

$$\underbrace{\delta(k+j) \ Q'_{ij}(k,t)}_{8.2\text{-}12a} + \underbrace{\lambda 4 \ M(j) \int \overline{u_i(k-p,t)} \ \delta(j+p) \ Q'_{ij}(j,t) + O(\lambda^2)}_{8.2\text{-}12b} \qquad 8.2\text{-}12$$

$$< \overline{u}_i(k,t) \; \acute{u}_j(j,t) \; >> \quad = \quad \overline{u}_i(k,t) < \acute{u}_j(j,t) \; >> + \; O(\lambda^2) \qquad\qquad 8.2\text{-}13$$

where it is emphasized that the $O(\lambda^2)$ term in equation 8.2-13 has a non-zero average.

$$< \overline{u}_i(k,t) \; \acute{u}_j(j,t) \; > \; = \; \delta(k+j) \; \bar{\acute{Q}}_{ij}(k,t) \qquad\qquad 8.2\text{-}14$$

Consistency with 8.2-8 requires that $\bar{\acute{Q}}_{ij}(k,t)$ is $O(\lambda^2)$.

Based on the above results, the Fourier velocity correlation $Q_{ij}(k,t)$ may be decomposed as:

$$Q_{ij}(k,t) \; = \; \overline{Q}_{ij}(k,t) \; + \; 2 \; \bar{\acute{Q}}_{ij}(k,t) \; + \; \acute{Q}_{ij}(k,t) \qquad\qquad 8.2\text{-}15$$

For stationary flows, the time arguments may be dropped from equation 8.2-15. Imposing the Kolmogorov constraint, each of the three quantities on the left hand side of 8.2-15 must have a component proportional to the Kolmogorov spectrum and may have other components which must cancel in equation 8.2-15, so that this equation may be expressed as:

$$C_k \varepsilon^{2/3} k^{-5/3} \; = \; (\; \overline{C}_k \; + \; 2\bar{\acute{C}}_k \; + \; \acute{C}_k \;) \varepsilon^{2/3} k^{-5/3} \qquad\qquad 8.2\text{-}16$$

It is noted that $u_i u_j$ is a random variable with an average Q_{ij} and a zero mean fluctuating component; $(u_i u_j)'$:

$$u_i(k,t)u_j(k',t) \; = \; \delta(k+k') \; Q_{ij}(k) \; + \; (u_i u_j)'(t) \qquad\qquad 8.2\text{-}17$$

and similarly:

$$\acute{u}_i(k,t)\acute{u}_j(k',t) \; = \; \delta(k+k')\acute{Q}_{ij}(k,t) \; + \; (\acute{u}_i(k,t)\acute{u}_j(k',t))' \qquad\qquad 8.2\text{-}18$$

The significance of equation 8.2-18 is that the RNG method retains $\acute{Q}_{ij}(k)$ but discards $(\acute{u}_i(k,t)\acute{u}_j(k',t))'$ at each step of the iteration. For the RNG method to be valid, it is not required that the rms value of the latter term is smaller than the

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former. What is required is that the cumulative value of the fluctuations decreases while the cumulative value of the $Q'_{ij}(k,t)$ increases. Physically, this may be interpreted in terms of time scales. The characteristic time of $Q_{ij}(k,t)$ is of the same order as the characteristic time of $u(k,t)$, however, the time constant of $\left(u'_i(k,t)u'_j(k',t)\right)'$ is much shorter.

The decomposition 8.2-4 may also be applied to the stirring force:

$$f_i(k,t) = \bar{f}_i(k,t) + f'_i(k,t) \qquad\qquad 8.2\text{-}19$$

However, the stirring force is not expandable in powers of λ. Therefore. if f_i is substituted for u_i in equations 8.2-5 through 8.2-8, all the remainder terms must be zero.

The decomposition 8.2-4, may now be substituted into the Navier-Stokes equations. yielding two coupled momentum equations for each component of 8.2-4:

$$\left(\frac{\partial}{\partial t} + k^2 v_0\right) \bar{u}(k,t) =$$

$$\lambda M(k)\int d^3j \; \bar{u}(k\text{-}j,t)\bar{u}(j,t) + \lambda M(k)\int d^3j \; <u'(k\text{-}j,t)\; u'(j,t)>> + \bar{f}(k,t) + O(\lambda^3) \qquad 8.2\text{-}20$$

$$\left(\frac{\partial}{\partial t} + k^2 v_0\right) u'(k,t) = \lambda 2 M(k)\int \bar{u}(k\text{-}j,t)u'(j,t)$$

$$+ \lambda\, M(k)\int \left\{\; u'(k\text{-}j,t)\; u'(j,t) - <u'(k\text{-}j,t)\; u'(j,t)>> \;\right\} + f'(k,t) + O(\lambda^3) \qquad 8.2\text{-}21$$

Next, the $<u'(k\text{-}j,t)\; u'(j,t)>>$ term in equation 8.2-20 is replaced by the result of equation 8.2-12. Term 8.2-12a vanishes while term 8.2-12b contributes a remainder of $O(\lambda)$ inside the integral or $O(\lambda^2)$ overall. The λ expansion can therefore be justified if the successive remainders generated by iterated partial averaging became smaller in magnitude. This point is elaborated upon in the next section.

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8.3 Partial averaging in terms of the stirring forces

By definition, the stirring forces used in the RNG method are generally considered not to be expandable in powers of λ, (or equivalently, the stirring forces are not expressed as convolution integrals of other variables). Since $f_i(k,t)$ is not expandable in powers of λ, the partial averaging cannot yield a remainder and becomes equivalent to ensemble averaging. The terms of order λ in equation 8.2-14 must be set to zero so that $\langle f_i(k,t) f_j(-k,t) \rangle\rangle = \langle f_i(k,t) f_j(-k,t) \rangle$.

For the RNG procedure, the partial averaging is performed on the high wavenumber stirring forces, $f^>(k,t)$. In this work, the corresponding variable is $f(k,t)$. First, the variable $u(k,t)$ is expanded in powers of λ.

$$u(k,t) = u^0(k,t) + \lambda u^1(k,t) + \lambda^2 u^2(k,t) \, ... \qquad 8.3\text{-}1$$

The above expansion is now extended to the correlation $Q(k,t)$:

$$Q(k,t) = Q^0(k,t) + \lambda Q^1(k,t) + \lambda^2 Q^2(k,t) \, ... \qquad 8.3\text{-}2$$

Again, in the context of Kolmogorov's turbulence, each term of the above series 8.2-20 must have a component proportional to the Kolmogorov spectrum. Also, cross correlations of the various terms must have the Kolmogorov spectrum or must equal zero. To obtain expressions for each term in the series 8.3-1, substitute the above series into both sides of equation 8.2-6 and integrate in time:

$$u^0(k,t) = \int_{-\infty}^{t} e^{-k^2 v(k,\Lambda)(t-\tau)} f(k,\tau) \, d\tau \qquad 8.3\text{-}3$$

$$u^1(k,t) = \int_{-\infty}^{t} e^{-k^2 v(k,\Lambda)(t-\tau)} \, d\tau \times$$

$$M(k) \int d^3 j \left\{ 2 \, \overline{u}(k-j,\tau) u^0(j,\tau) + u^0(k-j,\tau) \, u^0(j,\tau) - \langle u^0(k-j,\tau) \, u^0(j,\tau) \rangle\rangle \right\} \qquad 8.3\text{-}4$$

For a Fourier space correlation of the stirring forces given by equations 4.1-3 and 4.1-4, $\acute{u}^0(k,t)$ as defined by equation 8.3-3 has the spectral correlation given by:

$$< \acute{u}_l^0(k,t) \ \acute{u}_l^0(-k,t) > \ = \ \acute{Q}_{ij}^0(k) \qquad\qquad 8.3\text{-}5$$

with

$$\acute{Q}_{ij}^0(k) \ = \ (2\pi)^4 \ \frac{1}{v(k,\Lambda)} \ D_{ij}(k) \ W \ k^{-3} \ =$$

$$(2\pi)^4 \ \frac{1}{C_v \ \varepsilon^{1/3}\Lambda^{-4/3} \ \tilde{v}(k/\Lambda)} \ D_{ij}(k) \ C_{mg} \ \varepsilon \ k^{-3} \qquad\qquad 8.3\text{-}6$$

The above is not the Kolmogorov spectrum. There are several potential remedies for this problem:

i) Specify the spectral correlation for the stirring forces as $Wk^{-3}(\frac{k}{\Lambda})^{4/3}$.

This option is unsatisfactory, since the stirring forces are intended to represent the effects of the energy cascade, they should depend only on the wavenumber k, not on the cutoff wavenumber, Λ, at least for $\varepsilon = 4$.

ii) Replace $v(k,\Lambda)$ in equation 8.3-3 with a function proportional to $k^{-4/3}$.

The function chosen to replace $v(k,\Lambda)$ may be $v(k,k)$ or its average in wavenumber space (as Smith and Reynolds [5] and Lesieur & Metais [19] interpret the YO eddy viscosity):

$$\frac{1}{(\Lambda - k)} \ \int\limits_k^\Lambda \ f(k)v(k,\Lambda) \ d\Lambda \ = \ v_{YO}(k) \ = \ C_{avg}v(k,k) \qquad\qquad 8.3\text{-}7$$

where f(k) is an unknown weight function and C_{avg} is a constant less than 1 in value. Note that the Yakhot-Orszag eddy viscosity, v_{YO} is a function of Λ_c as $k \sim 0$. $v_{YO}(\Lambda_c)$ but it is a function of k in two instances, as $k \longrightarrow \Lambda_c$ and for the purpose of energy balance as $k \longrightarrow 0$.

For $\acute{u}^0(k,t)$ as an estimate of $\acute{u}(k,t)$, the ad hoc replacement of $v(k,\Lambda)$ with $v(k,k)$ is consistent with the overall procedure when $\acute{u}^0(k,t) = \acute{u}^>(k,t)$ in the wavenumber shell. It will yield the correct k dependance for a Kolmogorov spectrum but not the correct amplitude. However, $\acute{u}^0(k,t) \approx G(v_{YO}(k))f(k)$ recovers the Kolmogorov spectrum based on the work of YO. Hence, both options will be explored in the present study. The variable $\acute{u}^0(k,t)$ may be viewed as a statistical estimate of $u(k,t)$, with the correct representation of the first and second moments of the latter variable. Further discussion of this problem is presented in section 9.5, Compatibility with the YO theory.

It is not clear how to deal with $\acute{u}^1(k,t)$, in terms of the eddy viscosity form for the vertex Green's function. However, for the current study, both $v(k,k)$ and $v_{YO}(k)$ will be tried in place of $v(k,\Lambda)$ in equation 8.3-3. The chief motivation for this step is to avoid non-linearity in the resulting equation for eddy viscosity.

8.4 Partial averaging and λ series expansion of the Navier-Stokes equations

The aim of this section is to discuss various aspects of the series expansion in powers of λ, its relation to partial averaging and the problem of convergence. First, the question is considered whether a series in λ about the stirring force is necessary.

Parts of the RNG treatment of turbulence originated from Renormalized Perturbation Theory developed by Kraichnan, Edwards, Wyld, and others (McComb [13]). These theories utilized a series expansion in λ about a stirring force but no spectral splitting. The terms of this series are functions of stirring forces only, and not of the velocity, which is the unknown variable.

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However, the RNG version of the λ series expands only $u^>$, and the terms of this series include both $f^>$ and $u^<$ (see for example equation 4.4-3 for $u^{>1}$). When this series is substituted into the equation for $u^<$, the effect is to substitute $u^<$ into its own equation, thus increasing the order of the non-linearity of this equation (as in the triple products). The higher order of non-linearity implies that the new equation has additional solutions, spurious from the standpoint of solving the original Navier-Stokes equation. It is hoped that the partial averaging process is able to eliminate the higher order non-linearities, this means that the spurious solutions partial average is zero. Therefore, one would like to select the properties of the partial averaging so as to eliminate the higher order non-linearities at each step but still be consistent with the physics of the problem.

Given this undesirable situation, one may conclude that the formal expansion of $u^>$ as a power series in λ is not necessary. Instead, $u^>$ may be used in its exact form as given by its momentum equation. Consider again the spectrally split, coupled Navier-Stokes equations written below in an abbreviated form and without the stirring forces.

$$u^<(k) \;=\; \lambda G(k) M^<(k) \int u^< u^< \;+\; \lambda 2 G(k) M^<(k) \int u^< u^> \;+\; \lambda G(k) M^<(k) \int u^> u^> \qquad 8.4\text{-}1$$

$$\underbrace{\hspace{3cm}}_{8.4\text{-}1a} \qquad \underbrace{\hspace{3cm}}_{8.4\text{-}1b} \qquad \underbrace{\hspace{3cm}}_{8.4\text{-}1c}$$

$$u^>(k) \;=\; \lambda G(k) M^>(k) \int u^< u^< \;+\; \lambda 2 G(k) M^>(k) \int u^< u^> \;+\; \lambda G(k) M^>(k) \int u^> u^> \qquad 8.4\text{-}2$$

For a high wavenumber shell of thickness $\Delta\Lambda$, if $\frac{k}{\Delta\Lambda} \ll 1$, the integration domain of 8.4-1b is of order k, while the integration domain of 8.4-1c is of order $\Delta\Lambda$ (the YO problem). For this case, equation 8.4-1 may be approximated as:

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$$u^<(k) \;=\; \lambda G(k)M^<(k)\!\int u^<u^< \;+\; \lambda G(k)M^<(k)\!\int u^>u^> \qquad\qquad 8.4\text{-}3$$

<div align="center">

$\underbrace{\qquad\qquad\qquad}_{8.4\text{-}3a}$ $\underbrace{\qquad\qquad\qquad}_{8.4\text{-}3c}$

</div>

A viscous-like term may be obtained by substituting equation 8.4-2 in place of one $u^>$ term but not both in 8.4-3c:

$$8.4\text{-}3c \;=\; \lambda G(k)M^<(k)\!\int u^>u^> \;=\;$$

$$\lambda^2 G(k)M^<(k)\!\int u^>(k\text{-}j)G(j)M^>(j)\!\int\!\big(u^<u^< \;+\; 2\!\int u^<u^> \;+\; \int u^>u^> \big) \qquad 8.4\text{-}4a$$

or

$$\lambda^2 G(k)M^<(k)\!\int u^>(j)G(\,|\,k\text{-}j)M^>(k\text{-}j)\!\int\!\big(u^<u^< \;+\; 2\!\int u^<u^> \;+\; \int u^>u^> \big) \qquad 8.4\text{-}4b$$

Note that the convolution integral cannot be doubled up in this case or double counting of the same variables would result. Therefore, the eddy viscosity may only be obtained from one combination of the expanded and unexpanded $u^>$. However. the series expansion about the stirring force demonstrated in chapter 4 utilized both combinations, and the partial averaging definition 8.2-14 has been derived based on that procedure. Thus, it is concluded that the series expansion about the stirring force is needed if the steps of substitution and partial averaging (8.2-14) are to be interchangeable.

Now the partially averaged spectrum of $u^>u^>$ is taken to be the Kolmogorov spectrum and a remainder of order λ:

$$<\; u^>_i(k)u^>_j(\text{-}k)\; >> \;=\; <\; u^>_i(k)u^>_j(\text{-}k)\; > \;+\; O(\lambda)$$

$$=\; C_k\,\varepsilon^{2/3}\,D_{ij}(k)\;k^{\text{-}11/3} \;+\; O(\lambda) \qquad\qquad 8.4\text{-}5$$

Now applying the partial averaging to 8.4-4a yields:

$$\langle 8.4\text{-}1c \rangle\rangle = \lambda G(k) M^<(k) \int \langle u^> u^> \rangle\rangle =$$

$$\lambda^2 G(k) M^<(k) \int \bigg\{ \quad \langle u^> GM^> \int u^< u^< \rangle\rangle + \langle 2u^> GM^> \int u^< u^> \rangle\rangle + \langle u^> GM^> \int u^> u^> \rangle\rangle \quad \bigg\} \qquad 8.4\text{-}6$$

$$\underbrace{\qquad\qquad}_{8.4\text{-}6a} \qquad \underbrace{\qquad\qquad\qquad}_{8.4\text{-}6b} \qquad \underbrace{\qquad\qquad}_{8.4\text{-}6c}$$

The results of averaging 8.4-6a and 8.4-6c equal 0 plus $O(\lambda)$ remainder, yielding an unknown remainder of $O(\lambda^3)$ in equation 8.4-6. The result of averaging the term 8.4-6b is:

$$\langle \; 8.4\text{-}6b \;\rangle\rangle = \langle 2u^> GM^> \int u^< u^> \rangle\rangle =$$

$$u^<(k) \; C_k \; \varepsilon^{2/3} \; 2\lambda^2 G(k) M^<(k) \int d^3 jG(\,|\,k\text{-}j\,|\,) M^>(k\text{-}j) D(j) j^{-11/3} + O(\lambda^3) =$$

$$u^<(k) \; k^2 \; \Delta v + O(\lambda^3) \qquad\qquad 8.4\text{-}7$$

where the wavenumber labels have been restored.

Clearly, one is able to generate an eddy viscosity correction without resorting to stirring forces, provided that the Kolmogorov constant is given. The method is similar to that used by ZV [10] for the decaying turbulence version of RNG. The Navier-Stokes equation without a source of energy will have a solution with a decaying amplitude.

However, the remainder terms of order $O(\lambda^3)$ generated by all three parts of 8.4-6 are unaccounted for. Consider the term $\langle 8.4\text{-}6a \rangle$ as an example. To generate a term of $O(\lambda^3)$, one is free to substitute for either $u^>$ or one of the $u^<$ velocities with the corresponding momentum equation, 8.4-2 or 8.4-1, respectively. The complete expansion must involve both substitutions giving the relation:

$$\lambda^2 < \text{u}^>\text{GM}^>\int \text{u}^<\text{u}^< \gg\ =\ \lambda^3 < (\text{u}^>=)\ \text{GM}^>\int \text{u}^<\text{u}^< \gg\ +\ \lambda^3 2 < \text{u}^>\ \text{GM}^>\int (\text{u}^<=)\text{u}^< \gg \qquad 8.4\text{-}8$$

where $(\text{u}^>=)$ implies that the momentum equation has been substituted for $\text{u}^>$. It is emphasized that equation 8.4-8 would not be valid without the partial averaging operator due to the double counting of variables. For the partial averaging, the unexpanded variables are equivalent to $\text{u}^0 = \text{Gf}$ for the series expansion about the stirring force. Again, spurious solutions are introduced in the momentum equation by increasing the order of the non-linearity from 3rd order on the left side to 4th order on the right side of 8.4-8.

The substitution for $\text{u}^<$ (or the equivalent series expansion of $\text{u}^<$ in powers of λ as done in the next chapter) appears to be new. To the present author's best knowledge, other forms of RNG do not use it. Here, it is justified by the generalized properties of the partial averaging operator presented in section 8.2. Consider the term $\lambda^3 2 < \text{u}^>\ \text{GM}^>\int (\text{u}^<=)\text{u}^< \gg$:

$$\lambda^3 2 < \text{u}^>\ \text{GM}^>\int (\text{u}^<=)\text{u}^< \gg\ =$$

$$\lambda^3 \text{G(k)M}^<(\text{k})\int\ <\ \text{u}^>\text{GM}^>\int\Big\{ \text{GM}^<\int \text{u}^<\text{u}^<\ +\ 2\text{GM}^>\int \text{u}^<\text{u}^>\ +\ \text{GM}^>\int \text{u}^>\text{u}^> \Big\}\ \text{u}^< \gg\ =$$

$$\lambda^3 \text{G(k)M}^<(\text{k})\int\ \Big\{\ <\ \text{u}^>\text{GM}^>\int \text{u}^<\text{GM}^<\int \text{u}^<\text{u}^< \gg\ +\ <\ \text{u}^>\text{GM}^>\int \text{u}^<\text{GM}^<\int \text{u}^<\text{u}^> \gg\ +$$

$$<\ \text{u}^>\text{GM}^>\int \text{u}^<\text{GM}^<\int \text{u}^>\text{u}^> \gg\ \Big\} \qquad 8.4\text{-}9$$

Again, each of the terms in 8.4-9 is partially averaged, yielding an ensemble average of the $\text{u}^>$ terms plus remainders of $O(\lambda)$ inside the integral or $O(\lambda^4)$ overall. Clearly, the process could be terminated if $\lambda < 1$ or if it could somehow be shown

that the successive remainders were smaller. This however, appears unlikely because λ = 1, and the re-expansion of $u^<$ and $u^>$ to $O(\lambda^m)$ yield terms of the form:

$$\lambda^m GM^< \int u^< GM^< \int u^< ... \int u^< GM^< \int u^< u^< \qquad 8.4\text{-}10$$

$$\underbrace{\hspace{6cm}}_{\text{m convolution integrals}}$$

Apparently, no argument exists to show that terms of the form 8.4-10 reduce in size as m increases. It should be pointed out that it is the modified method of partial averaging which allows $u^<$ to take part in the averaging and causes the apperance of the above terms. For example, the term $\lambda GM^< \int u^< u^<$ may be re-expanded as

$\lambda^3 GM^< \int (u^<=)(u^<=)$ to yield $GM^< \int GM^< \int u^< u^< GM^< \int u^< u^<$. It appears that some constraint must be placed on the re-substitution of the velocity terms. One possibility is that at least one wavenumber in each triad $M \int uu$ is in the $k^>$ domain. This constraint ensures that at least one wavenumber integration will be over a thin shell in wavenumber space. Integrals over two or more thin shell domains will be $O(\Delta\Delta^2)$ or higher and may be discarded as $\Delta\Lambda \longrightarrow 0$, assuming the integrands to be well behaved. This condition on partial averaging is explored in section 9.4. Section 9.1 contains further comments on the λ series convergence and its relation to partial averaging.

8.5 Advective and dynamic temporal frequency ratios

As demonstrated in chapters 4 and 5, the RNG procedure may be carried out in the frequency domain $u(k,\omega)$ or in the temporal domain, $u(k,t)$. Typically. for the frequency method, eliminating the contributions of $u(j,\omega)$ in the shell Λ. YO [3]. Carati [11], and Smith [5] treat the frequency ratio $\frac{\omega}{\Omega}$ as an expansion parameter

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independent from $\frac{k}{\Lambda}$, and carry out series expansions to $\left(\frac{\omega}{\Omega}\right)^0$, $\left(\frac{k}{\Lambda}\right)^n$ (n = 1 for YO. n = 3 for Carati). Here, the assumption of independence between the two expansion parameters will be examined.

The temporal frequency variables, ω and Ω are Eulerian variables. When associated with eddies centered about a certain wavenumber, k, the frequency variable contains two effects: The Galilean shift of the eddies at k occurs due to a background advection velocity plus the combined advective effect of the eddies at smaller wavenumbers. The dynamic effect of the energy flow through the wavenumber k is the Kolmogorov energy cascade and its time constant is proportional to $\varepsilon^{-1/3}k^{-2/3}$. So two time constants are involved, an advective one, $\tau_{ADV}(k)$, and a dynamic one. $\tau_D(k)$. Thus, one may expect two characteristic values of ω; $\omega_{ADV} \propto \tau_{ADV}^{-1}$ and $\omega_D \propto \tau_D^{-1}$. These frequency values may be regarded as an approximate domain of ω for non-zero values of the distribution $u(k,\omega)$.

Let U be a background advection velocity at k = 0, for example the average wind velocity. Let Λ_p represent the wavenumber corresponding to the peak of the turbulent energy spectrum and the start of the inertial range. Consider an idealized eddy (Tennekes and Lumely [21]) centered at the wavenumber k. It is assumed that k is in the inertial range. This eddy may be considered to be advected by the background velocity U plus the root mean square velocity due to the combined turbulent kinetic energy of all the lower wavenumbers:

$$V(k)_{ADV} \sim U + \left\{ \int_0^k E(k)dk \right\}^{1/2}$$

8.5-1

Equation 8.5-1 implies that the integrated energy spectrum will yield a coherent advective velocity. In reality, this velocity would be smaller due to differences in phase of the various eddies in the region 0 to k. Thus, equation 8.5-1 may be considered an 'upper bound' for the advecting velocity. The kinetic energy is composed of the integral of the production spectrum from 0 up to some peak wavenumber Λ_p, plus a portion of the inertial range spectrum from Λ_p to k. Let the energy due to

the production spectrum be E_P. Then, the advective velocity at k may be approximated as:

$$V(k)_{ADV} \sim U + \left\{ E_P + \int_{\Lambda_P}^{k} E(k)dk \right\}^{1/2} \qquad 8.5\text{-}2$$

Substituting the Kolmogorov inertial range relation $E(k) = C_k \varepsilon^{2/3} k^{-5/3}$, and integrating:

$$V(k)_{ADV} \sim U + \left\{ E_P + \varepsilon^{2/3} \frac{2}{3} \left(\Lambda_P^{-2/3} - k^{-2/3} \right) \right\}^{1/2} \qquad 8.5\text{-}3$$

where $C_k \sim 1$.

Taylors 'Frozen Turbulence' approximation [2] yields $\omega_{ADV} = V(k)_{ADV} \bullet k$ may now be used to obtain the advective frequency. Taylor used $V(k)_{ADV} \approx U$, disregarding the turbulent advection.

The dynamic frequency component ω_D is proportional to $\varepsilon^{1/3} k^{2/3}$. To estimate the combined effect of the advection and eddy turnover, consider an approximation where the time variation of the eddy is represented as a product of two sinusoids:

$$\sin(\omega_{ADV}t)\sin(\omega_D t) =$$
$$\frac{1}{2} \left\{ \cos((\omega_{ADV} - \omega_D)t) - \cos((\omega_{ADV} + \omega_D)t) \right\} \qquad 8.5\text{-}4$$

The effective Eulerian frequency of the eddy will range between the sum and difference of the advective and dynamic frequencies:

$$k \bullet \left(U + \left\{ E_P + \varepsilon^{2/3} \frac{2}{3} \left(\Lambda_P^{-2/3} - k^{-2/3} \right) \right\}^{1/2} \pm \varepsilon^{1/3} k^{2/3} \right) \qquad 8.5\text{-}5$$

The next step is to consider another idealized eddy centered about Λ and form a ratio of the two frequencies:

$$\frac{\omega}{\Omega} \propto \frac{k \bullet \left(U + \left\{ E_P + \epsilon^{2/3} \frac{2}{3} \left(\Lambda_P^{-2/3} - k^{-2/3} \right) \right\}^{1/2} \pm \epsilon^{1/3}k^{2/3} \right)}{j \bullet \left(U + \left\{ E_P + \epsilon^{2/3} \frac{2}{3} \left(\Lambda_P^{-2/3} - \Lambda^{-2/3} \right) \right\}^{1/2} \pm \epsilon^{1/3}\Lambda^{2/3} \right)} \qquad 8.5\text{-}6$$

where **j** is a wavenumber such that $j = \Lambda$. Without loss of generality. the angle between **k** and **U** will be taken to equal zero. In the RNG procedure, **j** is integrated about a spherical shell of radius Λ, so that its angle w.r.t. **k** (and **U**) varies from 0 to π. The average value of the dot product on the surface of a sphere is given by:

$$\frac{1}{4\pi} \int_0^\pi \sin(\theta)\cos(\theta)d\theta = \frac{1}{8\pi} \qquad 8.5\text{-}7$$

A number of approximations are now possible. Considering first the case where U and/or $E_P^{1/2}$ is much larger than the inertial range contributions. The equation 8.5-3 simplifies to:

$$\frac{\omega}{\Omega} \propto \frac{k}{\Lambda} \qquad 8.5\text{-}8$$

This is equivalent to Taylors approximation [12].

At the other extreme, assuming that the inertial range energy contribution dominate over background advection and the production energy spectrum. Such a scenario is unlikely [12] but the resulting relation is:

$$\frac{\omega}{\Omega} \propto \frac{1}{8\pi} \frac{k \left\{ \left(\Lambda_P^{-2/3} - k^{-2/3} \right) \right\}^{1/2} \pm k^{2/3}}{\Lambda \left\{ \left(\Lambda_P^{-2/3} - \Lambda^{-2/3} \right) \right\}^{1/2} \pm \Lambda^{2/3}} \qquad 8.5\text{-}9$$

If it is assumed that $\Lambda_P/k \ll 1$ and $\Lambda_p/\Lambda \ll 1$, so that $k^{-2/3}$ and $\Lambda^{-2/3}$ may both be neglected relatively to $\Lambda_P^{-2/3}$, then:

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$$\frac{1}{8\pi} \frac{k^{5/3}}{\Lambda^{5/3}} \leq \frac{\omega}{\Omega} \leq \frac{1}{8\pi} \frac{k^{1/3}}{\Lambda^{1/3}} \qquad\qquad 8.5\text{-}10$$

However, the $\Lambda_p/k \ll 1$ requirement cannot be met for all resolved k which clearly will include Λ_p. A more likely scenario is that only $\Lambda^{-1/3}$ may be discarded as a much smaller quantity than $\Lambda_p^{-1/3}$ while k may range from the same order as Λ_p to several orders of magnitude larger. Relation 8.5-9 then becomes:

$$\frac{\omega}{\Omega} \propto \frac{1}{8\pi} \frac{k}{\Lambda} \frac{\left\{ \left(\Lambda_p^{-2/3} - k^{-2/3} \right) \right\}^{1/2} \pm k^{-1/3}}{\Lambda_p^{-1/3}} \qquad\qquad 8.5\text{-}11$$

According to equation 8.5-11, $\frac{\omega}{\Omega}$ is a function of $\frac{k}{\Lambda}$ and $\frac{k}{\Lambda_p}$ and thus may be treated as independent from $\frac{k}{\Lambda}$ (due to the extra parameter Λ_p) and the expansion due to YO is justified in this case. Equation 8.5-11 is valid only if both the advection velocity U and the production spectrum of the turbulent energy are negligible relative to the inertial range contributions. Also one requires $\frac{\Lambda_p}{\Lambda} \ll 1$, $\frac{k}{\Lambda} \ll 1$, and $\frac{\Lambda_p}{k} < 1$.

If $\frac{k}{\Lambda} \longrightarrow 1$, equation 8.5-10 is again recovered from 8.5-9.

It is concluded that only under very specific conditions may $\frac{\omega}{\Omega}$ be treated as an independent parameter from $\frac{k}{\Lambda}$. Even for flows where the kinetic energy contained in the inertial range is dominant, the independence assumption is only valid for $\frac{k}{\Lambda} \ll 1$. Otherwise, the two parameters are related and are of comparable size.

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8.6 Summary and discussion

General properties of the partial averaging operator have been defined in this chapter. These start with an expansion of the variable a into an invariant component $\bar{a}$ and a fluctuating component a' w.r.t. the partial averaging operator, $<\quad>>$. The conditions for activating the averaging operation must be defined. In general, the partial averaging operation is related to the conventional ensemble averaging by:

$$<a'a'>> \quad = \quad <a'a'> \quad + \quad \lambda(\text{remainder}) \qquad\qquad 8.6\text{-}1$$

Section 8.2 lists the relations for the first and second moments of $\bar{a}$ and a' under partial and ensemble averaging. Relation between partial averaging and λ series expansion or the re-substitution of the momentum variable have been discussed.

The expansion parameter $\frac{\omega}{\Omega}$ has been shown to be related to $\frac{k}{\Lambda}$, except under special circumstances, namely a flow dominated by the inertial range and $\frac{k}{\Lambda} << 1$. As with many other analytical steps of RNG, it is easy to undermine their validity particulary for $\frac{k}{\Lambda} \sim 1$. However, it is difficult if not impossible to substitute a more exact analytical procedure.

ZVH ([8], [10]) presented a formulation of RNG with an exact time dependance (see chapter 5) much too complex to produce a usable turbulence model. These authors subsequently carried out 'Markovian' approximation in the time domain as:

$$u(k,t)\int_{-\infty}^{t} \eta(k,\tau)d\tau \quad \approx \quad \int_{-\infty}^{t} u(k,\tau)\eta(k,\tau)d\tau \qquad\qquad 8.6\text{-}2$$

which amounts to a zero order Taylor series expansion of $u(k,\tau)$ in the τ variable about $\tau = t$ and is equaivalent to the YO calculation with $\left(\frac{\omega}{\Omega}\right)^0$.

As discussed in chapter 3, the RNG method may be interpreted as the gradual elimination of fine scales and the replacement of these scales with an average effect. At each step, the fine scales are decomposed into an average plus a fluctuation. The average is added to the eddy viscosity and the fluctuating remainder is neglected. In order for the RNG method to be valid, it is not required that the rms value of the fluctuating term be smaller than the average. What is required is that the cumulative effect of the unresolved fluctuations on the resolved scales decrease while the cumulative value of the averages increase over a range of eliminated scales. As the smallest resolved scale increases so does the smallest typical time scale. Therefore, one may expect that the effect of fluctuations at much shorter time scale on a much slower variable will approach zero. However, it is possible that the rapid, subgrid fluctuations will alias into 'induced forces' with time scales and magnitudes of the same order as the resolved motions. Indeed, this appears to be the basis of the Iterative Filtering method of Carati [40].

9. Alternatives for partial averaging and eddy viscosity formulations

The RNG procedures (or the iterative filtering of Carati) use an iterative averaging of wavenumber triads involving Fourier coefficients in the narrow waveband about Λ. For the vanishing k case of YO, the averaged triads involve $j = \Lambda$. $|k\text{-}j| = \Lambda$, and $k \sim 0$. For finite k values, the triads involving Λ consist of k, Λ, and $|k\text{-}k_c|$. To date, these triads were averaged whenever both $j \geq \Lambda$ and $|k\text{-}k_c| \geq \Lambda$. If only one of j, or $|k\text{-}j|$ equal to Λ, the triad was either discarded (by a zero result of the partial averaging) or retained until both j and $|k\text{-}k_c|$ were equal to or greater than a reduced value of Λ. The current work explores several different strategies for the removal of triads with only one wavenumber equal to Λ. Since each procedure is expected to yield different results, the question of fundamental importance is: which method more closely represents the physical effects of the small scales on the large scales of Navier-Stokes turbulence? In this chapter, several options for the eddy viscosity will be derived. The numerical results will be evaluated in chapter 10.

9.1 Time scales of wavenumber triads and eddy viscosity

As discussed in chapter 2, an eddy associated with a wavenumber k has a typical frequency represented by $D\varepsilon^{1/3}k^{2/3}$. In addition, each triad of wavenumbers has a typical frequency dominated by the frequency of the highest wavenumber.

The 'eddy damping rate' for the Fourier triad interaction has been proposed by Orszag [24]:

$$\omega_{k,j,k\text{-}j} = \omega_k + \omega_j + \omega_{k\text{-}j} \approx D\varepsilon^{1/3}\left(k^{2/3} + j^{2/3} + |k\text{-}j|^{2/3} \right) \qquad 9.1\text{-}1$$

where D is a coefficient to be determined by a particular turbulence theory (for example $D/C_K^2 = 0.1904$ according to Kraichnan). The $\omega_{k,j,k\text{-}j}$ may be interpreted as an inverse, non-linear time scale for the triple-velocity correlation to evolve toward a quasi-equilibrium state, [14]. This 'quasi-equilibrium' state may be equivalent to a partial average of this particular Fourier triad.

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Figure 9.1 Shell integration domains for finite k in the self-similar region, including the triple products.

Region A

Region C_R

Triple Products Domain CB_R

Region B_R

Interior Region D

Region C_L

Region B_L

Triple Products Domain CB_L
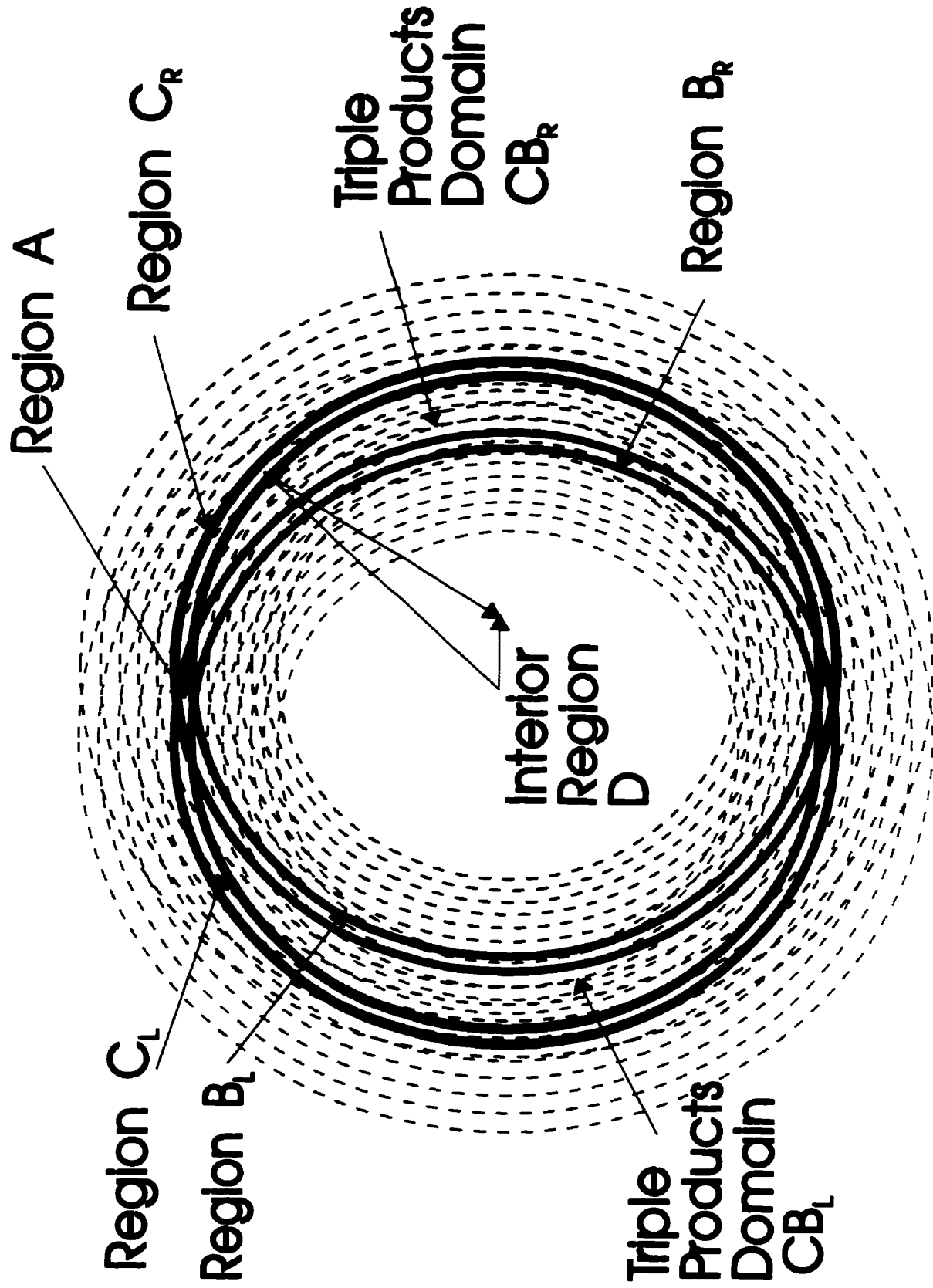
Considering first the case of YO [3], the eddy damping rate of triads (Λ_c, Λ_c, k ~ 0) is approximately proportional to $2\Lambda_c^{2/3}$, where the frequency associated with k has been neglected. This eddy damping rate is also approximately constant around the shell. Since the (j, |k-j|) components of the triangle have much higher frequencies than the 'base' (k), the variables associated with the former wavenumbers may be treated as a statistical average relatively to the latter. However, for finite k, the typical eddy damping rate varies with k and also varies around the shell for a given k. Referring to figure 9.1, region B contains the 'minimum triad' (k,Λ_c,Λ_c-k) while region C contains the 'maximum triad' (k,Λ_c,Λ_c+k). In the B region, the Λ_c-k wavenumber will often be lower than k, while in the C region, both Λ_c and Λ_c+k will always be equal to or higher than k.

The RNG procedure seeks to obtain corrections to momentum equation coefficients and possibly also corrections to source terms like the stirring force. Since the momentum equation is for the **k** Fourier coefficient, eliminating triads over region C is consistent with partial averaging of higher wavenumbers or time filtering of high frequencies. However, the disadvantage of this approach is that the triple velocity products remain at the end of the procedure. These triple products represent interactions where only one wavenumber falls into the subgrid range. As discussed in section 9.3, the triple products are undesirable from the point of view of numerical simulation. In the current study, methods of avoiding the triple products will be investigated by averaging over triads with only one subgrid wavenumber. This process is easier to justify when both j and |k-j| are greater than k. The triads where one wavenumber is subgrid while the other is smaller than k pose special difficulties.

For the λ^2 truncation strategy, one of the wavenumbers (say j) is associated with a random velocity correlation while the other (say |k-j|) is a 'vertex wavenumber', associated with G(|k-j|,Λ_c) and M(k-j). The Green's function is a random variable, with an average and a standard deviation, driven by the random eddy viscosity ν(|k-j|,Λ_c). Hence, the partial averaging process affects both j and |k-j|.

At the start of the RNG procedure, the random variables within the Navier-Stokes equations are the velocity and stirring force Fourier coefficients. However, the

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molecular viscosity, v_0 is statistically sharp. After eliminating a wide range of scales, v_0 is replaced by $v(k,\Lambda_c)$ which is the expected value of a random variable [35]:

$$v(k,\Lambda_c) = \varepsilon^{1/3} \Lambda_c^{-4/3} \tilde{v}(k/\Lambda_c)$$

Examining the eddy viscosity expression for the source of randomness. it is apparent that ε is a random variable which will converge to it's expected value in a wavenumber band about Λ_c if a temporal or spatial filtering is used with sufficiently long averaging interval. The standard deviation may increase with Λ_c leading to considerations of intermittency. The random eddy viscosity will fluctuate about it's mean value with a typical time scale proportional to $\varepsilon^{-1/3}\Lambda_c^{-2/3}$. The dissipation time scale is $k^2 v(k,\Lambda_c)$, and the ratio of the eddy viscosity fluctuation time to the dissipation time is $\dfrac{k^2}{\Lambda_c^2}$. This implies that for $\dfrac{k}{\Lambda_c} \sim 1$, the RNG eddy viscosity may be not well defined in terms of a statistical average.

Returning to the consideration of the averaging of wavenumber triads. for $\dfrac{k}{\Lambda_c} \sim 1$. $|k\text{-}j|$ becomes small when k and j are nearly parallel. $G(|k\text{-}j|,\Lambda_c)$ is well defined with the eddy viscosity approaching $v(0,\Lambda_c)$. However, the equation for the eddy viscosity containing $v(|k\text{-}j|,\Lambda_c)$ is non-linear and difficult to solve. RNG procedure proposed in this study also involve $G(|k\text{-}j|,|k\text{-}j|)$ with a much slower fluctuation frequency than k. Indeed, the vertex Green's function appears to be a problem for the RNG theory presented in this study.

9.2 Alternatives for elimination of wavenumber triads

Three alternatives will be explored in this study for the proposed characteristics of $\acute{u}_i(k,t)$. the ensuing behavior of the partial average, and the RNG iteration procedures.

First, the random variable $\acute{u}_i(k,t)$ is considered to be a response at wavenumber **k** to fluctuations 'originating' from the constant wavenumber band Λ-$\Delta\Lambda$ to Λ. In the context of the RNG methods, the source of the fluctuations is the stirring forces $\hat{f}(j)$ for $|j| = \Lambda$, and equation 8.2-4 is equivalent to the decomposition introduced by Carati for the IF method. The integration contour is region C plus region B. figure 9.1.

Second, the random variable $\acute{u}_i(k,t)$ is considered to be a response at wavenumber **k** to fluctuations 'originating' from the variable wavenumber band $|k\text{-}j|<$ for $|j| = \Lambda$. For the RNG method, the source of the fluctuations is the stirring forces $f(k\text{-}j,t)$. The contour of integration are the regions B, figure 9.1

Third, the random variable $\acute{u}_i(k,t)$ is considered to be a response at wavenumber **k** to fluctuations 'originating' from the variable wavenumber band $|k\text{-}j|^{>+}$ for $|j| = \Lambda$. where $|k\text{-}j|^{>+}$ exceeds Λ_c. For the RNG method, the source of the fluctuations is the stirring forces $f(k\text{-}j,t)$, the contour of integration are the regions C. figure 9.1

9.2.1 Partial averaging over a constant wavenumber shell (region B plus C)

For the random fluctuations 'originating' from the thin shell, the high wavenumber variable $u^>(k,t)$ may be decomposed into a random part $\acute{u}^>(k,t)$ and a non-random component $\lambda \overline{u}^>(k,t)$; where λ is a marker for the presence of at least one convolution integral. Both $\overline{u}_i^>(k,t)$ and $\acute{u}_i^>(k,t)$ contain convolution integrals but $\acute{u}_i^>(k,t)$ may also contain a source term such as a stirring force. This division is equivalent to the decomposition introduced by Carati [40] for the IF, where $\overline{u}_i^>(k,t)$ contains all the averaged moments of the shell stirring forces while $\acute{u}_i^>(k,t)$ contains the stirring force fluctuations.

$$u_i^>(k,t) = \lambda \overline{u}_i^>(k,t) + \acute{u}_i^>(k,t) \qquad\qquad 9.2\text{-}1$$

One of the objectives of this work is to find physical analogies for the various mathematical steps of the RNG. For the partial averaging over a constant wavenumber, temporal filtering of the flow variable may be appropriate. Since the variable $u_i^>(k,t)$ contains a distribution of energy as a function of ω given by $u_i^>(k,\omega)$, a physical basis may be provided for the partial averaging by writing:

$$u_i^>(k,\omega) = \lambda \overline{u}_i^>(k,\omega^<) + \acute{u}_i^>(k,\omega^>) \qquad\qquad 9.2\text{-}2$$

The boundary between $\omega^<$ and $\omega^>$ will be denoted by ω_Λ. The underlying concept is that the components of $u_i^>(k,\omega)$ at relatively high ω will evolve faster than those at low ω. The partial averaging may now be equated to a sharp cutoff filter in ω space with ω_Λ as the cutoff.

The corresponding low wavenumber series is:

$$u_i^<(k,t) = \overline{u}_i^<(k,t) + \lambda \acute{u}_i^<(k,t) \qquad\qquad 9.2\text{-}3$$

and for $u_i^<(k,\omega)$:

$$u_i^<(k,\omega) = \overline{u}_i^<(k,\omega^<) + \lambda \acute{u}_i^<(k,\omega^>) \qquad\qquad 9.2\text{-}4$$

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The reversed position of the λ factor indicates that the source of the random variation is the high wavenumber shell. It is expected that as $\frac{k}{\Lambda} \ll 1$, the ω domain of $u_i^<(k,\omega^>)$ will go to zero since it is unlikely to encounter a turbulent eddy which varies slowly in space and rapidly in time (see section 8.5). Now the following hypothesis is made:

$$< \overline{\lambda u_i^>(j,t)}^2 > < < \acute{u_i^>}(j,t)^2 >$$

9.2-5

$$< \overline{u_i^<(k,t)}^2 > > < \lambda \acute{u_i^<}(k,t)^2 >$$

9.2-6

$$< \acute{u_i^>}(j,t)^2 > > < \lambda \acute{u_i^<}(k,t)^2 >$$

9.2-7

A physical argument to support the above assertion is that rapid fluctuations in time tend to be associated with rapid fluctuations in space. Let ω_j represent the frequency region $C_k^{1/2}\varepsilon^{1/3}j^{2/3} < \omega < U\bullet j$, typical for eddies with a spatial fluctuation centered about j. If now the spatial wavenumber is reduced from j to k where $\frac{k}{j} \ll 1$ but the frequency is held constant at ω_j, it is reasonable to expect that the rms value of $u(k,\omega_j) < u(j,\omega_j)$. If correct, this hypothesis may provide a physical basis for the convergence of the λ series.

Although the above hypothesis is feasible, it may nonetheless be wrong due to the non-linear nature of the Navier-Stokes equations.

Returning to the time domain and substituting the results of equations 9.1-1 and 9.1-3 into the Navier-Stokes equations for $u^<(k,t)$ and averaging the $\acute{u}$ variables according to the results of section 8.2 gives:

$$\left(\frac{\partial}{\partial t} + k^2 v_0\right)< \bar{u}^<(k,t) + \lambda \acute{u}^<(k,t) >> =$$

$$\lambda M^<(k) \int\limits_{D1} d^3j \; \bar{u}^<(k-j,t)\bar{u}^<(j,t) + 2\lambda^2 M^<(k) \int\limits_{Br1} d^3j \; \bar{u}^<(k-j,t)\bar{u}^>(j,t) +$$

$$2\lambda^2 M^<(k) \int\limits_{A1} d^3j \; \bar{u}^>(k-j,t)\bar{u}^>(j,t) + 2\lambda^2 M^<(k) \int\limits_{Br1} < \acute{u}^<(k-j,t) \; \acute{u}^>(j,t) >> +$$

$$\lambda \; M^<(k) \int\limits_{A1} < \acute{u}^>(k-j,t) \; \acute{u}^>(j,t) >> + f^<(k,t) + O(\lambda^3) \qquad\qquad 9.2\text{-}8$$

The eddy viscosity correction, Δv_{B1}, is obtained from the term

$$\int\limits_{Br1} < \acute{u}^<(k-j,t)\acute{u}^>(j,t)>> \text{ (see figure 9.1):}$$

$$2\lambda^2 M^<(k) \int\limits_{Br1} < \acute{u}^<(k-j,t) \; \acute{u}^>(j,t) >> =$$

$$2\lambda^2 \int\limits_{Br1} d^3j \; M^<(k-j) \int d\tau \; G_0(\,|k-j\,|^<,t-\tau) \; \acute{Q}^>(j,\tau) \; \bar{u}^<(k,\tau) \approx$$

$$\bar{u}^<(k,t) \; 2\lambda^2 \int\limits_{Br1} d^3j \; M^<(k-j) \int d\tau \; G_0(\,|k-j\,|^<,t-\tau) \; \acute{Q}^>(j,\tau) = \qquad\qquad 9.2\text{-}9a$$

$$\bar{u}^<(k,t) \; \Delta v_B \qquad\qquad 9.2\text{-}9b$$

The accuracy of the approximation in equation 9.1-4a is questionable, particularly when $\frac{k}{\Lambda} \longrightarrow 1$. This approximation is sometimes called Markovian (Orszag, [24]) since the result of equation 9.1-4a is assumed to depend only on $\bar{u}^<(k,t)$, rather than on the range of values $\bar{u}^<(k,\tau)$ within the 'memory' integral $\int d\tau$. Its validity will be further discussed later on in this study.

The resulting momentum equation after the elimination of the first shell is:

$$\left(\frac{\partial}{\partial t} + k^2 v_1\right) \bar{u}^<(k,t) =$$

$$\lambda M^<(k) \int_{DI} d^3j \ \bar{u}^<(k\text{-}j,t)\bar{u}^<(j,t) +$$

$$2\lambda^2 M^<(k) \int_{BrI} d^3j \ \bar{u}^<(k\text{-}j) \int G_0(j,t\text{-}\tau)d\tau \ M^>(j) \int d^3p \ \bar{u}^<(j\text{-}p)\bar{u}^<(p) + f^<(k,t) \qquad 9.2\text{-}10$$

where $v_1 = v_0 + \Delta v_{BI}$.

9.2.2 The RNG iteration procedure and differential equation for $v(k,\Lambda_c)$

The iteration commences by setting $\bar{u}^<(k,t) \longrightarrow u(k,t)$ on the reduced wavenumber interval $k^< \longrightarrow k$. The result is:

$$\left(\frac{\partial}{\partial t} + k^2 v_1\right) u(k,t) =$$

$$\lambda \ M(k) \int_{DI} \frac{d^3j}{(2\pi)^3} \ u(k\text{-}j,t)u(j,t) +$$

$$2\lambda^2 \ M(k) \int_{BrI} \frac{d^3j}{(2\pi)^3} \ u(k\text{-}j) \int d\tau \ G_0(j,t\text{-}\tau) \ M^>(j) \int \frac{d^3p}{(2\pi)^3} \ u(j\text{-}p)u(p) + f(k,t) \qquad 9.2\text{-}1$$

The new aspect of equation 9.2-1 is the triple product term residing in integration volume B_{rI}. This term is now re-expanded on the second shell, resulting in the following equation:

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$$\lambda^2 \, M(k) \int_{Brl} \frac{d^3j}{(2\pi)^3} \, u(k\text{-}j) \int G_0(j,t\text{-}\tau)d\tau \, M^>(j) \int \frac{d^3p}{(2\pi)^3} \, u(j\text{-}p)u(p) =$$

$$2\lambda^2 \, M(k) \int_{Crl^2} \frac{d^3j}{(2\pi)^3} \, u^>(k\text{-}j) \int G_0(j,t\text{-}\tau)d\tau \, M^{>l}(j) \int \frac{d^3p}{(2\pi)^3} \, u^<(j\text{-}p)u^>(p) \; +$$

$$2\lambda^2 \sum_{i=3}^{3+m} M(k) \int_{Brl^i} \frac{d^3j}{(2\pi)^3} \, u^<(k\text{-}j) \int G_0(j,t\text{-}\tau)d\tau \, M^{>l}(j) \int \frac{d^3p}{(2\pi)^3} \, u^<(j\text{-}p)u^<(p) \qquad \text{9.2-2}$$

where $m = m(n) = \frac{1}{\Delta l} \ln\left(1 + \frac{k}{\Lambda_n} \right)$.

The triple product expansion in equation 9.2-2 yields an additional contribution to the eddy viscosity from the term in the domain crl^2, for $n \geq 2$. $v_2 = v_1 + \Delta v_{B2} + \Delta v_{C2}$. The eddy viscosity due to re-expanding the triple products has been initially proposed by ZVH for finite shell thickness and extended to the differential limit by Carati [11] as discussed in chapters 5 and 6. After eliminating a range of wavenumbers from Λo to Λo - k (which may be neglected due to the low energies associated with high wavenumbers) the region C becomes a full arc with an angular range from $\arccos(\frac{k}{2\Lambda})$ to 0 and its contributions to eddy viscosity become self-similar $(n \geq m+1)$. Consider the self-similar portion of region C, after n discrete shells have been removed. At the n+1 step of the recursion, expanding 9.2-1, substituting relation 9.2-2, and retaining terms up to order λ^2 gives:

$$\left(\frac{\partial}{\partial t} + k^2 v_n\right) u(k,t) = \lambda \, M(k) \int_{Dn} \frac{d^3j}{(2\pi)^3} \, u(k\text{-}j,t)u(j,t) + f(k,t) +$$

$$\sum_{j=n+1\text{-}m}^{n} \sum_{i=n+1}^{j+m} 2\lambda^2 \, M(k) \int_{Brj^i} \frac{d^3j}{(2\pi)^3} \, u(k\text{-}j) \; \times$$

$$\int G_j(j,t\text{-}\tau)d\tau \, M^{>j}(j) \int \frac{d^3p}{(2\pi)^3} \, u(j\text{-}p)u(p) \qquad \text{9.2-3}$$

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The following notation is used for the Green's function:

$$G_n(k,t,\tau) = e^{-V_n[k]k^2(t-\tau)}$$

9.2-4

The eddy viscosity, $v_n[k]$ is expected to be of the form:

$$v_n[k] = v_0 + \sum_{i=1}^{n} \Delta v_i[k] \approx \epsilon^{1/3} \Lambda_n^{-4/3} \tilde{v}[\frac{k}{\Lambda_n}]$$

9.2-5

where the contribution of v_0 is considered negligible for sufficiently large n and the function $v_n[k]$ is the average, viscous-like effect on $u(k,t)$ due to motions at wavenumbers $\geq \Lambda_n$.

Thus, the eddy viscosity contribution Δv_{n+1}, for finite k values originates from the equation:

$$k^2 \Delta v_{n+1} u^<(k,t) =$$

$$2\lambda M(k) \int_{B_{m+1}} \frac{d^3j}{(2\pi)^3} <u^<(k-j,t)u^{>0}(j,t)>> +$$

$$2\lambda^2 M(k) \int_{A_{n+1}} \frac{d^3j}{(2\pi)^3} <u^{>1}(k-j,t)u^{>0}(j,t)>> + 4\lambda^2 M(k) \times$$

$$\sum_{j=n+1-m}^{n} \int_{C_{r_{n+1}^j}} \frac{d^3j}{(2\pi)^3} <u^{>0}(k-j,t) \; G_j(j,t-\tau)M^{>j}(j) \int \frac{d^3p}{(2\pi)^3} u^<(j-p,\tau)u^{>0}(p,\tau)>>$$

9.2-6

It is emphasized that the eddy viscosity decreases along the arc Cr of figure 9.1 as j increases along this contour.

The contribution of the first term on the right hand side of 9.2-6 appears to be new while the eddy viscosity composed of the last two terms have been examined by ZVH

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[8] and Carati [11]. The various integration regions are indicated in figure 9.1. For finite k values, the magnitude of region A_n becomes negligible in the differential limit of the shell thickness and may be dropped. Expanding the first and third terms to $O(\lambda^2)$ and anticipating the results of the averaging based on the results of section 8.3:

$$k^2 \Delta v_{n+1} u^<(k,t) = k^2 u^<(k,t)\left(\Delta v^{Br}_{n+1} + \Delta v^{Cr}_{n+1}\right) =$$

$$4\lambda^2 M(k) \sum_{\substack{j=n+2 \\ Br^j_{n+1}}}^{n+1+m} \int \frac{d^3j}{(2\pi)^3} M^<(k-j) \int_{-\infty}^{t} d\tau\, G_j(|k-j|,t-\tau) \times$$

$$\int \frac{d^3p}{(2\pi)^3} u^<(k-j-p,\tau) \int_{-\infty}^{\tau} d\tau' G_n(p,\tau-\tau') < f^>(p,\tau') \int_{-\infty}^{t} d\tau'' G_n(j,t-\tau'') f^>(j,\tau'')>> + \qquad 9.2\text{-}7a$$

$$4\lambda^2 M(k) \sum_{\substack{j=n+1-m \\ Cr^j_{n+1}}}^{n} \int \frac{d^3j}{(2\pi)^3} \int_{-\infty}^{t} d\tau'' \, G_n(|k-j|,t-\tau'') < f^>(k-j,\tau'') \times$$

$$M^{>j}(j) \int_{-\infty}^{t} d\tau\, G_j(j,t-\tau) \int \frac{d^3p}{(2\pi)^3} u^<(j-p,\tau) \int_{-\infty}^{\tau} d\tau' G_n(p,\tau-\tau') f^>(p,\tau')>> \qquad 9.2\text{-}7b$$

where Δv^{Br}_{n+1} and Δv^{Cr}_{n+1} are the eddy viscosity contributions of the two integration regions. Replacing $G_n(|k-j|)$ with $G_j(|k-j|)$ in 9.2-8a is consistent with replacing $v(k,\Lambda)$ with $v(k,k)$ in the $\acute{u}^l(k,t)$ term as discussed in section 8.2.

Carrying out the partial averaging with the force correlation tensor:

$$< f^>(k,t)f^>(k',t') >> = 2\, W\, k^{-y}\, D_{\alpha\beta}(k)\, \delta(k+k')\delta(t-t')\, (2\pi)^3$$

and evaluating the wavenumber and temporal integrals containing Dirac delta functions gives:

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$$k^2 \Delta v_{n+1} u^<(k,t) =$$

$$8\lambda^2 W \; \frac{1}{2} \; M(k) \; D(k) \int_{Bm+1} \frac{d^3 j}{(2\pi)^3} \; j^{-3} \; D(j) \; M^<(k-j) \; \times$$

$$\int_{-\infty}^{t} d\tau \; G_j(|k-j|,t-\tau) \; u^<(k,\tau) \int_{-\infty}^{\tau} d\tau' G_n(j,\tau-\tau') \; G_n(j,t-\tau') \; +$$

$$8\lambda^2 W \; \frac{1}{2} \; M(k) \; D(k) \sum_{j=n+1-m}^{n} \int_{Cr_{n+1}^j} \frac{d^3 j}{(2\pi)^3} \; M^{>j}(j) \; |k-j|^{-3} \; D(k-j) \; \times$$

$$\int_{-\infty}^{t} d\tau \; G_j(j,t-\tau) \; u^<(k,\tau) \int_{-\infty}^{\tau} d\tau' G_n(|k-j|,\tau-\tau')G_n(|k-j|,t-\tau') \qquad \text{9.2-8}$$

where the term $D(k)$ has been inserted to facilitate contraction of tensor indices prior to the spherical integration in wavenumber space and $\frac{1}{2}$ is inserted since the trace of $D(k)$ is $d-1 = 2$.

Using the relations introduced by Carati [11]:

$$A(k,j) = P_{imn}(k)P_{mrs}(j)D_{is}(k)D_{nr}(k-j) \qquad \text{9.2-9}$$

$$A(k,k-j) = P_{imn}(k)P_{mrs}(k-j)D_{is}(k)D_{nr}(j) \qquad \text{9.2-10}$$

where $M_{imn}(k) = -\frac{i}{2} P_{imn}(k)$. Substituting 9.2-9 and 9.2-10 into 9.2-8 one has the result:

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$$k^2 \Delta v_{n+1} u^<(k,t) =$$

$$8\lambda^2 W \frac{1}{2} \int_{Bm+1} \frac{d^3j}{(2\pi)^3} j^{-3} A(k,k-j) \times$$

$$\int_{-\infty}^{t} d\tau\, G_j(|k-j|,t-\tau)\, u^<(k,\tau) \int_{-\infty}^{\tau} d\tau' G_n(j,\tau-\tau')\, G_n(j,t-\tau') +$$

$$8\lambda^2 W \frac{1}{2} \sum_{j=n+1-m}^{n} \int_{Cr_{n+1}^j} \frac{d^3j}{(2\pi)^3} |k-j|^{-3} A(k,j) \times$$

$$\int_{-\infty}^{t} d\tau\, G_j(j,t-\tau)\, u^<(k,\tau) \int_{-\infty}^{\tau} d\tau' G_n(|k-j|,\tau-\tau') G_n(|k-j|,t-\tau') \qquad 9.2\text{-}8$$

The following identities may be recognized in regions Brn and Crn:

a) Region Brn

$$A(k,j) = j_\perp^2 \left\{ \frac{(nd-1)k^2}{|k-j|^2} - \frac{2kj_\parallel}{j^2} \right\} =$$

$$\left\{ \Lambda^2 - \left\{ \frac{\Lambda^2 + k^2 - \Lambda^{<2}}{2k} \right\}^2 \right\} \left\{ \frac{(nd-1)k^2}{\Lambda^{<2}} - \frac{\Lambda^2 + k^2 - \Lambda^{<2}}{\Lambda^2} \right\}$$

$$= \Lambda^2\, S_{Br}(l,\zeta^<)\, B(1,\zeta^<,l) \qquad 9.2\text{-}11$$

where $l = \frac{k}{\Lambda}$, $\zeta^< = \frac{\Lambda^<}{\Lambda}$ are dimensionless variables. The symbols $j_\perp$ and $j_\parallel$ represent the components of the vector j perpendicular to k, and parallel to k, respectively.

$$A(k,k-j) = |k-j|_\perp^2 \left\{ \frac{(nd-1)k^2}{j^2} - \frac{2k|k-j|_\parallel}{|k-j|^2} \right\} =$$

$$\left\{ \ \Lambda^2 - \left\{ \ \frac{\Lambda^2 + k^2 - \Lambda^{<2}}{2k} \ \right\}^2 \ \right\} \left\{ \ \frac{(nd-1)k^2}{\Lambda^2} - \frac{\Lambda^{<2} + k^2 - \Lambda^2}{\Lambda^{<2}} \ \right\}$$

$$= \Lambda^2 \ S_{Br}(l,\zeta^<) \ B(\zeta^<,1,l) \qquad\qquad 9.2\text{-}12$$

b) Region Crn

$$A(k,j) = j\bot^2 \left\{ \ \frac{(nd-1)k^2}{|k\text{-}j|^2} - \frac{2kj}{j^2}\bigg\|{}_{_{_{\parallel}}} \ \right\} =$$

$$\left\{ \ \Lambda^{>2} - \left\{ \ \frac{\Lambda^{>2} - \Lambda^2 - k^2}{2k} \ \right\}^2 \ \right\} \left\{ \ \frac{(nd-1)k^2}{\Lambda^2} - \frac{\Lambda^{>2} + k^2 - \Lambda^2}{\Lambda^{>2}} \ \right\}$$

$$= \Lambda^2 \ S_{Cr}(\zeta^>,l) \ B(\zeta^>,1,l) \qquad\qquad 9.2\text{-}13$$

$$A(k,k\text{-}j) = |k\text{-}j|\bot^2 \left\{ \ \frac{(nd-1)k^2}{j^2} - \frac{2k|k\text{-}j|}{|k\text{-}j|^2}\bigg\|{}_{_{_{\parallel}}} \ \right\} =$$

$$\left\{ \ \Lambda^{>2} - \left\{ \ \frac{\Lambda^{>2} - \Lambda^2 - k^2}{2k} \ \right\}^2 \ \right\} \left\{ \ \frac{(nd-1)k^2}{\Lambda^{>2}} - \frac{\Lambda^2 + k^2 - \Lambda^{>2}}{\Lambda^2} \ \right\}$$

$$= \Lambda^2 \ S_{Cr}(\zeta^>,l) \ B(1,\zeta^>,l) \qquad\qquad 9.2\text{-}14$$

where $\zeta^> = \dfrac{\Lambda^>}{\Lambda}$.

The factors S_{Br} and S_{Cr} represent $\sin^2\theta$, where θ is the angle between vectors j and k. These factors must always be positive or zero. However, depending on the combination of ζ and l, the factor B may be negative or positive. This is significant because some 'eddy viscosity' contributions from regions B_r and C_r become negative for certain range of l due to the changing sign of the factor B. Thus, the factor B is responsible for the 'backscatter' that is energy flow from the small scales to the large scales, reverse form the average flow of the energy cascade.

The above relations may be summarized in the following table:

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	Bm	Cm
A(k,j)	$\Lambda^2\ S_{Br}(l,\zeta^<)\ B(1,\zeta^<,l)$	$\Lambda^2\ S_{Cr}(\zeta^>,l)\ B(\zeta^>,1,l)$
A(k,k-j)	$\Lambda^2\ S_{Br}(l,\zeta^<)\ B(\zeta^<,1,l)$	$\Lambda^2\ S_{Cr}(\zeta^>,l)\ B(1,\zeta^>,l)$

The volume integrals are given by:

$$\int_{B_{m+1}} d^3j \; = \; 2\pi\Delta\Lambda \; \frac{\Lambda}{k} \int_{\Lambda-k}^{\Lambda} \Lambda^< d\Lambda^< \; + \; O(\Delta\Lambda) \qquad\qquad 9.2\text{-}15$$

and

$$\sum_{j=n+1-m}^{n} \int_{C^j_{r_{n+1}}} d^3j \; = \; \int_{C_{r_{n+1}}} d^3j \; = \; 2\pi\ \Delta\Lambda \; \frac{\Lambda}{k} \int_{\Lambda}^{\Lambda+k} \Lambda^> d\Lambda^> \; + \; O(\Delta\Lambda) \qquad\qquad 9.2\text{-}16$$

where $\Delta\Lambda$ is the incremental thickness of the shell radius, and $\Lambda = \Lambda_{n+1}$. It is noted that as $k \longrightarrow 0$, formulas 9.2-15 and 9.2-16 reduce to the volumes of half spherical shells of thickness $\Delta\Lambda$ to order $\Delta\Lambda$, $2\pi\Lambda^2\Delta\Lambda$. However, as indicated in figure 9.1, the regions B and C exist due to the offset of size k between the shells $j = \Lambda$ and $|k\text{-}j| = \Lambda$ and must vanish with k. This incorrect limiting behavior of 9.2-15 and 9.2-16 suggest that the above formulation is inapplicable in the low k limit. As discussed in chapter 6, ZV [9] explained that the limits $\Delta\Lambda \longrightarrow 0$ and $k \longrightarrow 0$ are not interchangeable. The latter limit must be performed first to obtain the correct $v(0,\Lambda)$ eddy viscosity behavior. This assertion means that it is impossible to obtain a transition between the YO formulation ($k \longrightarrow 0$ limit first) and the Carati formulation ($\Delta\Lambda \longrightarrow 0$ limit first) and that a formula valid in both regions does not exist. For $\frac{k}{\Delta\Lambda} \leq 1$, the YO formulation applies with a correction $O(k)$, for $\frac{k}{\Delta\Lambda} \geq 1$, the Carati formulation is valid. In the present work, the observation is made that for $\frac{k}{\Delta\Lambda} \leq 1$, the shell thickness factor $\Delta\Lambda$ must be replaced by $0.5k$ in the formulas 9.2-15 and 9.2-16 to obtain volumes of regions B and C valid to $O(k)$ and $O(\Delta\Lambda)$ and equal to $\pi\Lambda^2 k$. Clearly, $\frac{k}{\Delta\Lambda} = 1$ is the switchover point which is identifiable if $\Delta\Lambda$ is finite.

Since the present study is based on the Carati formulation, the evaluation of $v(0.\Lambda)$ presents a problem which is further discussed in the next chapter.

Changing the summation of discrete volume increments into a continuous integral. necessitates the transition from a discrete to a continuous eddy viscosity function. $v_n(k) \longrightarrow v(k,\Lambda)$, so that the Green's function becomes:

$$G_n(k,t,\tau) \longrightarrow G(k,\Lambda,t,\tau) = e^{-v(k,\Lambda)k^2(t-\tau)}$$

9.2-17

The change of notation from a discrete to a continuous function may be generalized as follows:

$$f_n(k) \longrightarrow f(k,\Lambda)$$

Substituting appropriate variables in each region gives the result:

$$k^2\Delta v_{n+1}u^<(k,t) = \frac{8\lambda^2 W}{(2\pi)^3} \frac{1}{2} 2\pi \ \Delta\Lambda \ \frac{\Lambda}{k} \int_{\Lambda-k}^{\Lambda} \Lambda^< d\Lambda^< \ \Lambda^{-3} \left(\frac{-i}{2}\right)^2 \Lambda^2 \ S_{Br}(l,\zeta^<) \ B(\zeta^<.1.l) \times$$

$$\int_{-\infty}^{t} d\tau \ e^{-v(\Lambda^<,\Lambda^<)\Lambda^{<2}(t-\tau)} \ u^<(k,\tau) \int_{-\infty}^{\tau} d\tau' \ e^{-v(\Lambda,\Lambda)\Lambda^2(t+\tau-2\tau')} \ +$$

$$\frac{8\lambda^2 W}{(2\pi)^3} \frac{1}{2} 2\pi \ \Delta\Lambda \ \frac{\Lambda}{k} \int_{\Lambda}^{\Lambda+k} \Lambda^> d\Lambda^> \ \Lambda^{-3} \left(\frac{-i}{2}\right)^2 \Lambda^2 \ S_{Cr}(\zeta^>,l) \ B(\zeta^>,1,l) \times$$

$$\int_{-\infty}^{t} d\tau \ e^{-v(\Lambda^>,\Lambda^>)\Lambda^{>2}(t-\tau)} \ u^<(k,\tau) \int_{-\infty}^{\tau} d\tau' \ e^{-v(\Lambda,\Lambda)\Lambda^2(t+\tau-2\tau')}$$

9.2-18

Integrating first with respect to τ':

$$k^2 \Delta v_{n+1} u^<(k,t) = \frac{-\lambda^2 W}{(2\pi)^3} \, 2\pi \, \Delta\Lambda \, \frac{\Lambda}{k} \int_{\Lambda-k}^{\Lambda} \Lambda^< d\Lambda^< \, \Lambda^{-3} \, \Lambda^2 \, S_{Br}(l,\zeta^<) \, B(\zeta^<,1,l) \times$$

$$\int_{-\infty}^{t} d\tau \, e^{-v(\Lambda^<,\Lambda^<)\Lambda^{<2}(t-\tau)} \, u^<(k,\tau) \, e^{-v(\Lambda,\Lambda)\Lambda^2(t-\tau)} \, \frac{-1}{(\,2v(\Lambda,\Lambda)\Lambda^2\,)} \, +$$

$$\frac{-\lambda^2 W}{(2\pi)^3} \, 2\pi \, \Delta\Lambda \, \frac{\Lambda}{k} \int_{\Lambda}^{\Lambda+k} \Lambda^> d\Lambda^> \, \Lambda^{-3} \, \Lambda^2 \, S_{Cr}(\zeta^>,l) \, B(\zeta^>,1,l) \times$$

$$\int_{-\infty}^{t} d\tau \, e^{-v(\Lambda^>,\Lambda^>)\Lambda^{>2}(t-\tau)} \, u^<(k,t) \, e^{-v(\Lambda,\Lambda)\Lambda^2(t-\tau)} \, \frac{-1}{(\,2v(\Lambda,\Lambda)\Lambda^2\,)}$$

Now integrating with respect to τ and simplifying:

$$k^2 \Delta v_{n+1} = - \frac{\lambda^2 W}{(2\pi)^2} \, \Delta\Lambda \, \frac{\Lambda}{k} \, \frac{1}{2v(\Lambda,\Lambda)\Lambda^2} \times \Big\{$$

$$\int_{\Lambda-k}^{\Lambda} \Lambda^< d\Lambda^< \, \Lambda^{-3} \, \Lambda^2 \, S_{Br}(l,\zeta^<) \, B(\zeta^<,1,l) \, \frac{1}{(\,v(\Lambda^<,\Lambda^<)\Lambda^{<2} + v(\Lambda,\Lambda)\Lambda^2\,)} \, +$$

$$\int_{\Lambda}^{\Lambda+k} \Lambda^> d\Lambda^> \, \Lambda^{-3} \, \Lambda^2 \, S_{Cr}(\zeta^>,l) \, B(\zeta^>,1,l) \, \frac{1}{(\,v(\Lambda^>,\Lambda^>)\Lambda^{>2} + v(\Lambda,\Lambda)\Lambda^2\,)} \, \Big\} \qquad 9.2\text{-}19$$

The recursion relation in steps of $\Delta\Lambda$ is converted into a differential equation according to:

$$\frac{\partial v(k,\Lambda)}{\partial \Lambda} = \lim_{\Delta\Lambda \to 0} \frac{\Delta v(k,\Lambda)}{\Delta\Lambda} \qquad 9.2\text{-}20$$

Thus, the differential equation is:

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$$\frac{\partial v(k,\Lambda)}{\partial \Lambda} = -\frac{\lambda^2 W}{(2\pi)^2} \frac{1}{2v(\Lambda,\Lambda)\Lambda^2} \frac{\Lambda}{k^3} \Bigg\{$$

$$\int_{\Lambda-k}^{\Lambda} \Lambda^< d\Lambda^< \; \Lambda^{-3} \; \Lambda^2 \; S_{Br}(l,\zeta^<) \; B(\zeta^<,1,l) \; \frac{1}{(\; v(\Lambda^<,\Lambda^<)\Lambda^{<2} + v(\Lambda,\Lambda)\Lambda^2 \;)} \; +$$

$$\int_{\Lambda}^{\Lambda+k} \Lambda^> d\Lambda^> \; \Lambda^{-3} \; \Lambda^2 \; S_{Cr}(\zeta^>,l) \; B(\zeta^>,1,l) \; \frac{1}{(\; v(\Lambda^>,\Lambda^>)\Lambda^{>2} + v(\Lambda,\Lambda)\Lambda^2 \;)} \; \Bigg\} \qquad 9.2\text{-}21$$

Substituting $W = \pi^2 C_{mg}\varepsilon$, $\lambda = 1$, and also using the relation $v(k,\Lambda)\Lambda^2 = \varepsilon^{1/3} \; \tilde{v}(l)\Lambda^{2/3}$ and differentiating with respect to Λ:

$$\Lambda^{-7/3} \; \varepsilon^{1/3} \left(-\frac{4}{3} \tilde{v}[l] - l\frac{d}{dl} \tilde{v}[l] \right) = -\frac{1}{8} \frac{C_{mg}\; \varepsilon}{v(\Lambda,\Lambda)\Lambda^2} \frac{\Lambda}{k^3} \Bigg\{$$

$$\int_{\Lambda-k}^{\Lambda} \Lambda^< d\Lambda^< \; \Lambda^{-3} \; \Lambda^2 \; S_{Br}(l,\zeta^<) \; B(\zeta^<,1,l) \; \frac{1}{(\; v(\Lambda^<,\Lambda^<)\Lambda^{<2} + v(\Lambda,\Lambda)\Lambda^2 \;)} \; +$$

$$\int_{\Lambda}^{\Lambda+k} \Lambda^> d\Lambda^> \; \Lambda^{-3} \; \Lambda^2 \; S_{Cr}(\zeta^>,l) \; B(\zeta^>,1,l) \; \frac{1}{(\; v(\Lambda^>,\Lambda^>)\Lambda^{>2} + v(\Lambda,\Lambda)\Lambda^2 \;)} \; \Bigg\} \qquad 9.2\text{-}22$$

Converting to non-dimensional variables:

$$\Lambda^{-7/3} \left(-\frac{4}{3} \tilde{v}[l] - l\frac{d}{dl} \tilde{v}[l] \right) = -\frac{1}{8} \frac{C_{mg}}{\tilde{v}_1(1)^2} \frac{\Lambda}{k^3} \Lambda^{-4/3} \Bigg\{$$

$$\Lambda^2 \int_{1-l}^{1} \zeta d\zeta \; \Lambda^{-3} \; \Lambda^2 \; S_{Br}(l,\zeta) \; B(\zeta,1,l) \; \frac{1}{(\; \zeta^{2/3} + 1 \;)} \; +$$

$$\Lambda^2 \int_{1}^{1+l} \zeta d\zeta \; \Lambda^{-3} \; \Lambda^2 \; S_{Cr}(\zeta,l) \; B(\zeta,1,l) \; \frac{1}{(\; \zeta^{2/3} + 1 \;)} \; \Bigg\} \qquad 9.2\text{-}23$$

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$$\frac{4}{3}\,\tilde{v}[l] + l\frac{d}{dl}\,\tilde{v}[l] = \frac{1}{8}\,\frac{C_{mg}}{\bar{v}(1)^2}\,\frac{1}{l^3}\,\bigg\{$$

$$\int_{1-l}^{1} \zeta d\zeta\; S_{Br}(l,\zeta)\; B(\zeta,1,l)\;\frac{1}{(\,\zeta^{2/3} + 1\,)} +$$

$$\int_{1}^{1+l} \zeta d\zeta\; S_{Cr}(\zeta,l)\; B(\zeta,1,l)\;\frac{1}{(\,\zeta^{2/3} + 1\,)}\;\bigg\}$$

<div align="right">9.2-24</div>

$$\frac{4}{3}\,\tilde{v}_F(l) + l\frac{d}{dl}\,\tilde{v}_F(l) = \frac{1}{8}\,\frac{C_{mg}}{\bar{v}_S(1)^2}\,\frac{1}{l^3}\,I_F(l)$$

<div align="right">9.2-25</div>

where $\tilde{v}(l)$ has now been labelled $\tilde{v}_f(l)$ to indicate that it had been derived using iterative averaging over wavenumber magnitude fixed at Λ. The corresponding right hand side forcing term $I_f(l)$ is given below. The relations for $S_{Cr}(\zeta,l)$ and $B(l,\zeta)$ have been substituted:

$$I_f(l) = \int_{1-l}^{1} \zeta d\zeta\;\frac{1}{(\,\zeta^{2/3} + 1\,)}\;\times$$

$$\bigg\{\,1 - \Big\{\frac{1 + l^2 - \zeta^2}{2l}\Big\}^2\,\bigg\}\,\bigg\{\,2l^2 - \frac{\zeta^2 + l^2 - 1}{\zeta^2}\,\bigg\} +$$

$$\int_{1}^{1+l} \zeta\,d\zeta\;\frac{1}{(\,\zeta^{2/3} + 1\,)}\;\times$$

$$\bigg\{\,\zeta^2 - \Big\{\frac{\zeta^2 - 1 - l^2}{2l}\Big\}^2\,\bigg\}\,\bigg\{\,2l^2 - \frac{\zeta^2 + l^2 - 1}{\zeta^2}\,\bigg\}$$

<div align="right">9.2-26</div>

It is noted that $I_f(l) = I_{fB}(l) + I_{fC}(l)$, contributions from regions B and C respectively to the fixed wavenumber stirring force. The integral $I_f(l)$ may be evaluated analytically. *Mathematica* gives a rather unwieldy expression which will not be reproduced here. The explicit solution for $\tilde{v}_f(l)$ is:

<div align="center">153</div>

$$\tilde{v}_f(l) = \frac{1}{8} \frac{C_{mg}}{\tilde{v}_f(1)^2} l^{-4/3} \int_0^l d\xi \; \xi^{-8/3} \; I_f(\xi) + l^{-4/3}C_f \qquad 9.2\text{-}27$$

To the knowledge of the present author, the integration in equation 9.2-27 must be performed numerically. Numeric evaluation of 9.2-26 shall be deferred to the next chapter.

It should be emphasized that after eliminating all the supergrid wavenumbers, the final form of the momentum equation will still contain the triple products. This is undesirable for the purpose of simulation. This momentum equation is presented below:

$$\left(\frac{\partial}{\partial t} + k^2 v_1(k,\Lambda)\right) u(k,t) =$$

$$\lambda \; M(k) \int_{D(k,\Lambda)} d^3j \; u(k\text{-}j,t)u(j,t) + f(k,t) +$$

$$\lambda^2 M(k) \int_{CB(k,\Lambda)} d^3j \; u(k\text{-}j) \int_{-\infty}^{t} G_j(j,t\text{-}\tau)d\tau \; M^{>j}(j) \int_{D(j,\Lambda)} d^3p \; u(j\text{-}p)u(p) \qquad 9.2\text{-}28$$

where the symbol $M^{>j}(j)$ indicates that $j > \Lambda$. Here, the integration domain of the vector j for the triple products is the region between contours C and B, denoted by $CB(k,\Lambda)$:

$$\text{volume} \left(CB(k,\Lambda) \right) = 2\pi k(\Lambda^2 - \frac{k^2}{12}) \qquad 9.2\text{-}29$$

while the volume of region $D(k,\Lambda)$ is given by:

$$\text{volume} \left(D(k,\Lambda) \right) = \pi(\frac{4\Lambda^3}{3} + \frac{k^3}{12} - k\Lambda^2) \qquad 9.2\text{-}30$$

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as indicated in figure 9.1. In the limit as $k \longrightarrow \Lambda$, volume of $CB(\Lambda,\Lambda) = \frac{11}{6}\pi\Lambda^3$ while the volume of region $D(\Lambda,\Lambda)$ is $\frac{5}{12}$ $\pi\Lambda^3$. Hence, the integration volume of the triple product region is over 4 times that of the standard Navier-Stokes convolution integral on the resolved wavenumber range. This implies a large computational burden.

According to Zhou and Vahala [10], a simulation of equation 9.2-28 has not been carried out as of 1993 and to the knowledge of this author, this situation has not changed. From the point of view of the numerical simulation, the problem resides with the high wavenumber $j^>$ which ranges in magnitude from the final cutoff, Λ to Λ + k. Also, as pointed out earlier in connection with the λ series substitution, the higher order of non-linearity now present implies an additional, spurious solution to the momentum equation.

The triple velocity product term of equation 9.2-24 merits further discussion. This term may be written as:

$$\lambda^2 M(k) \int_{CB(k,\Lambda)} d^3j \; u(k\text{-}j) \int_{-\infty}^{t} G_j(j,t\text{-}\tau)d\tau \; M^{>j}(j) \int_{D(j,\Lambda)} d^3p \; u(j\text{-}p)u(p)$$

$$= \lambda^2 M(k) \int_{CB(k,\Lambda)} d^3j \; u(k\text{-}j,t) \; u_{tp}^{>j}(j,t) \qquad\qquad 9.2\text{-}31$$

where $u_{tp}^{>j}(j,t)$ is given by:

$$u_{tp}^{>j}(j,t) = \int_{-\infty}^{t} G_j(j,t\text{-}\tau)d\tau \; M^{>j}(j) \int_{D(j,\Lambda)} d^3p \; u(j\text{-}p)u(p) \qquad\qquad 9.2\text{-}32$$

It is noted that $u^{>j}(j,t)$ is a subgrid variable, $u_{tp}^{>j}(j,t)$ is that component of $u^{>j}(j,t)$ which can be expressed as a function of the resolved variables. This variable suggests the need to decouple the partial averaging from the spectral splitting.

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The integration volume of $u_{tp}^{>j}(j,t)$ is the intersection of two spheres of radius Λ separated by $j > \Lambda_n$. This volume equals $D(j,\Lambda)$ where $D(2\Lambda,\Lambda) = 0$. Thus, the variable $u_{tp}^{>j}(j,t)$ is a component of $u^{>j}(j,t)$ in the region $\Lambda < j < 2\Lambda$. Since $u_{tp}^{>j}(j,t)$ is not affected by the partial averaging, a connection may be made with equation 9.1-2 and it may be speculated that:

$$u_{tp}^{>j}(k,t) = u^{>}(k,\omega^{>})$$

9.2-33

A second problem with retaining the triple products in the momentum equation is their lack of Galilean invariance as mentioned in section 6.2.

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9.3 Partial averaging over the B regions

In this section, the objective is to average the wavenumber triads where one side length equals Λ and falls into the shell to be eliminated, while the other 'leg' of the triangle is less than Λ. The Navier-Stokes equations in Fourier space are reiterated below:

$$\left(\frac{\partial}{\partial t} + k^2 v\right) u(k,t) = M(k)\int u(k-j,t)u(j,t) + f(k,t) \qquad 9.3\text{-}1$$

The condition for partial averaging is that at least one wavenumber in the triad k, j, $k-j$ is in the shell Λ to $\Lambda-\Delta\Lambda$. The spectrally split variables are given below:

$$u^>(k,t) = \acute{u}^>(k,t) \qquad 9.3\text{-}2$$

$$u^<(j,t) = \begin{cases} \overline{u}^<(j,t) \ \text{if} \ k \ \text{and} \ |k\text{-}j| \ < \ \Lambda \\[2mm] \acute{u}^<(j,t) \ \text{if} \ k \ \text{and/or} \ |k\text{-}j| \ \geq \ \Lambda \end{cases} \qquad 9.3\text{-}3$$

where it is noted that $\overline{u}^>(j,t) = 0$. For the stirring forces, since these are not expressed as non-linear convolutions of other wavenumbers, one has that $\overline{f}_i(k,t) = f_i^<(k,t)$ and $f_i(k,t) = f_i^>(k,t)$ as in 8.2. Expanding the Navier-Stokes equation 9.3-1 based on 9.3-2 and 9.3-3 and inserting the partial averaging operator wherever $\acute{u}$ appears:

$$\left(\frac{\partial}{\partial t} + k^2 v_0\right)\left(\overline{u}^<(k,t) + <\acute{u}^<(k,t)> \right) = \lambda M(k)\int_{DI} d^3j \ \overline{u}^<(k\text{-}j,t) \ \overline{u}^<(j,t) \ +$$

$$\lambda 2 M(k)\int_{BI} d^3j \ <\acute{u}^<(k\text{-}j,t) \ \acute{u}^>(j,t)> \ + \ \lambda 2 M(k)\int_{AI} d^3j \ < \ \acute{u}^>(k\text{-}j,t) \ \acute{u}^>(j,t) \ > \ + \ \overline{f}(k,t) \qquad 9.3\text{-}4$$

The region A will become a second order differential upon taking the differential limit and thus is neglected. Taking the partial average of equation 9.3-4 according to equations 8.2-5 to 8.2-12 yields:

$$\left(\frac{\partial}{\partial t} + k^2 v_0\right) \overline{u^<}(k,t) = \lambda M(k) \int_{Dl} d^3j\ \overline{u^<}(k\text{-}j,t)\ \overline{u^<}(j,t) + \overline{f}(k,t) - k^2 \Delta v_1 \overline{u^<}(k,t) \qquad 9.3\text{-}5$$

where

$$- k^2 \Delta v_1 \overline{u^<}(k,t) \approx \lambda 2 M(k) \int_{Brl} d^3j\ \langle \acute{u}^<(k\text{-}j,t)\ \acute{u}^>(j,t) \rangle\rangle \qquad 9.3\text{-}6$$

Now, the partial averaging principle in terms of the stirring forces yields:

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$$\langle \acute{u}^{<}(k\text{-}j,t)\ \acute{u}^{>}(j,t)\rangle\rangle\ =\ \langle \acute{u}^{<}(k\text{-}j,t)\ \acute{u}^{>}(j,t)\rangle\ +$$

$$\langle\ (\ \acute{u}^{<0}(k\text{-}j,t)\ +\ \lambda\acute{u}^{<1}(k\text{-}j,t)\ )(\ \acute{u}^{>0}(j,t)\ +\ \acute{u}^{>1}(j,t)\ )\ \rangle\ +\ O(\lambda^3)\ =$$

$$\lambda^2 8\ M(k)\int_{Brl} d^3j\ \int d\tau\ G(j,t\text{-}\tau)\ M^{>}(j)\ \bar{u}(k,\tau)\ \acute{Q}^{<0}(|k\text{-}j|,t)\ +$$

$$\lambda^2 8\ M(k)\int_{Brl} d^3j\ \int d\tau\ G(|k\text{-}j|,t\text{-}\tau)\ M^{<}(k\text{-}j)\ \bar{u}(k,\tau)\ \acute{Q}^{>0}(j,t)\ +\ O(\lambda^3) \qquad 9.3\text{-}7$$

and $\acute{Q}^0(j,t)$ is given by equation 8.2-20. Carrying out a series of steps analogous to section 9.2, the resulting solution for the eddy viscosity is:

$$\frac{4}{3}\ \tilde{v}_n(l)\ +\ l\frac{d}{dl}\ \tilde{v}_n(l)\ =\ \frac{1}{8}\ \frac{C_{mg}}{\tilde{v}_n(1)^2}\ \frac{1}{\beta}\ I_n(l) \qquad 9.3\text{-}8$$

where

$$I_n(l) = \int_{1-l}^{1} \zeta d\zeta \; \frac{1}{(\zeta^{2/3} + 1)} \; \frac{1}{\zeta^{11/3}} \times$$

$$\left\{ 1 - \left\{ \frac{1 + l^2 - \zeta^2}{2l} \right\}^2 \right\} \left\{ 2\frac{l^2}{\zeta^2} - (1 + l^2 - \zeta^2) \right\} +$$

$$\left\{ \int_{1-l}^{1} \zeta d\zeta \; \frac{1}{(\zeta^{2/3} + 1)} \times \right.$$

$$\left. \left\{ 1 - \left\{ \frac{1 + l^2 - \zeta^2}{2l} \right\}^2 \right\} \left\{ 2l^2 - \frac{\zeta^2 + l^2 - 1}{\zeta^2} \right\} \right.$$ 9.3-9

As in the case of $I_f(l)$, $I_n(l)$ may be evaluated analytically but this expression is omitted here to save space. It is noted that $0 < l < 1$ and so the above equation contains a singularity as $l \longrightarrow 1$ since $\zeta \longrightarrow 0$ and the terms $\frac{1}{\zeta^{11/3}}$ and $\frac{1}{\zeta^{11/3}} \frac{l^2}{\zeta^2}$ become large. Physically, the term $\zeta^{-11/3}$ corresponds to the Kolmogorov energy density on the wavenumber sphere of radius ζ, so as the wavenumber tends to zero, the kinetic energy becomes large.

The formal solution is:

$$\tilde{v}_n(l) = l^{-4/3} \int_{0}^{l} d\xi \; \xi^{-8/3} \; I_n(\xi) + l^{-4/3}C_n$$ 9.3-10

A plot of $\tilde{v}_n(l)$ as computed numerically will be shown in the next chapter.

9.4 Partial averaging over the C regions

In this section, partial averaging will be carried out over the C regions only. This procedure is similar to the iterative filtering procedure of Carati in that the stirring forces acting in region B are retained until these can be incorporated into successive C regions. The detailed derivation is analogous to that of section 9.2 and will not be presented here to save space. Instead, only the final results are presented:

$$\frac{4}{3}\, \tilde{v}_{fh}(l) + l\frac{d}{dl}\, \tilde{v}_{fh}(l) \;=\; \frac{1}{8}\, \frac{C_{mg}}{\tilde{v}_{fh}(1)^2}\, \frac{1}{l^3}\, I_{fh}(l) \qquad\qquad 9.4\text{-}1$$

where $\tilde{v}(l)$ has now been labelled $\tilde{v}_{fh}(l)$ to correspond to $I_{fh}(l)$. Substituting the relations for $Sc_f(\zeta,l)$ and $B(l,\zeta)$:

where I_{fh} is given as:

$$I_{fh}(l) \;=\; \int_{1}^{1+l} \zeta d\zeta\, \frac{1}{(\zeta^{2/3}+1)}\left\{\; \zeta^2 - \left\{\frac{\zeta^2 - 1 - l^2}{2l}\right\}^2\; \right\} \times$$

$$\left\{\; \frac{1}{\zeta^{2/3}}\left\{\, 2\frac{l^2}{\zeta^2} - 1 - l^2 + \zeta^2 \,\right\} + \left\{\, 2l^2 - \frac{\zeta^2 + l^2 - 1}{\zeta^2} \,\right\}\; \right\} \qquad 9.4\text{-}2$$

The results will be presented in chapter 10. This formulation of the eddy viscosity will also leave triple products of the resolved velocities in the momentum equation. See section 9.2 for details.

9.5 Compatibility with the YO theory

The YO version of the RNG procedure for 'vanishing k' yields several self-consistent relations between turbulence quantities. These relations, reiterated below for convenience are mutually dependent:

$$C_{mg} \; \varepsilon \; = \; \frac{2WS_d}{(2\pi)^d}$$

4.9-4

with $C_{mg} \approx 1.59$. From the above relation YO obtained

$$E(k) \; = \; 1.61\varepsilon^{2/3}k^{-5/3}$$

4.9-5

and

$$v_{YO}(k) \; = \; 0.49\varepsilon^{1/3}k^{-4/3}$$

4.9-6

The resulting relation for $W = 1.59\pi^2\varepsilon$ and also $v(k)$ has been used in the derivation of the YO result for C_k. The latter result has been obtained from the approximation:

$$E(k) \; = \; \frac{1}{2}S_dk^2 \; \frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} d\omega \; \text{Trace} \; \langle u^0{}_\alpha(k,\omega)u^0{}_\beta(-k,-\omega)\rangle$$

9.5-1

with

$$u^0{}_\alpha(k,\omega) \; = \; G(k,\omega)f_\alpha(k,\omega)$$

and

$$G(k,\omega) \; = \; \frac{1}{\left(-i\omega \; + \; v_{YO}(k)k^2\right)}$$

It is noted that $G(k,\omega)$ contains $v_{YO}(k)$ from 4.9-6. The derivations in sections 9.2, 9.3, and 9.4 of this study use the relation $W = 1.59\pi^2\varepsilon$, but $G(k,\omega)$ contains $v(k,k)$ instead of $v_{YO}(k)$. Since it is expected that $v(k,k) > v_{YO}(k)$, $u^0{}_\alpha(k,\omega)$ is expected to be overdamped and inconsistent with the YO theory. The energy spectrum evaluated according to 9.5-1 will have the correct k dependence but too low amplitude.

9.6 Summary

Several RNG eddy viscosity formulations have been presented in this chapter. First, the general properties of partial averaging were specified to act only on the stirring forces at Λ for each stage of the calculation. The resulting calculation included the fixed stirring force contribution from both the B and C regions of diagram 9.1. Also, the derivation of eddy viscosity from regions C for both fixed and high components of the stirring force has been presented. The region C triads have both averaged wavenumbers in the subgrid region. In both cases, the final version of the momentum equation includes the remaining triple products of the low wavenumber velocities. This term is undesirable for numerical simulations. As a remedy, a different averaging strategy was proposed, including the fixed and low wavenumber stirring forces from regions B of diagram 9.1. In this case, one is averaging over triads with only one subgrid wavenumber. It is noted that one cannot formulate an eddy viscosity v_{LH} as this would involve double counting of the same physical interactions. The three versions of the eddy viscosity each model the subgrid interactions in a different way, attempting to reproduce the physical effects of the small scales on the large scales. The question which is correct may be easier to answer once the explicit forms of these functions are obtained in the next chapter.

10. Numerical Results and validation

The objective of this chapter is to obtain numerical solutions for the energy drain function and the eddy viscosity as governed by differential equations derived in chapter 9. Then, it is required to verify if any of the solutions are suitable for use as an eddy viscosity function in a large eddy simulation. The criteria here is that for a cutoff wavenumber in the inertial range of the original flow energy spectrum, the simulated energy spectrum will also have the same amplitude right up to the cutoff. In order to check the suitability of the various proposed solutions. the validation criteria will be discussed first.

10.1 Criteria for validation of potential eddy viscosity functions

It is required to validate the various candidates for the universal eddy viscosity function for isotropic, high Reynolds number turbulence. Since the present knowledge of turbulent energy transfer is incomplete, sufficient validation may only be achieved with extensive large eddy simulation tests which are beyond the scope of the current work. Instead, this study examines theoretical considerations which provide necessary but not sufficient conditions that must be met by the eddy viscosity. Also. a limited comparison to published literature is carried out since there is a consensus about the qualitative shape of the eddy viscosity plot.

10.1.1 Theoretical constraints on the eddy viscosity function

The eddy viscosity function is expected to meet the following criteria: achieve the correct energy drain rate, ε while maintaining the correct energy spectrum function $E(k)$ (proportional to $k^{-5/3}$ in the inertial range. $\Lambda_p < k < \Lambda_c$). The former of these requirements may be characterized as global over the resolved wavenumber range, the latter is local for each wavenumber mode. The global requirement is expressed by the equation:

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$$\Lambda_c$$

$$\int_0^{\Lambda_c} dk \ 2v(k,\Lambda_c)k^2 \ E(k) \ = \ \varepsilon \qquad\qquad 10.1\text{-}1$$

Substituting $E(k) = C_k \varepsilon^{2/3} k^{-5/3}$ into 10.1-1:

$$\int_0^{\Lambda_c} dk \ 2v(k,\Lambda_c) \ C_k \ \varepsilon^{2/3} k^{1/3} \ = \ \varepsilon \qquad\qquad 10.1\text{-}2$$

Equation 10.1-2 is necessary but not sufficient to uniquely determine $v(k,\Lambda_c)$. An infinite number of functions $v(k,\Lambda_c)$ will satisfy 10.1-1 subject to the constraint that $E(k) = C_k \varepsilon^{2/3} k^{-5/3}$. Conversely, an infinite number of possible spectra $E(k)$ will satisfy 10.1-1 for a given $v(k,\Lambda_c)$. Therefore, if a large eddy simulation is performed using an eddy viscosity which satisfies 10.1-2, the resulting $E(k)$ may deviate from the inertial range form. It will satisfy 10.1-1 but not necessarily 10.1-2.

The above discussion is illustrated with an example. Substituting $v(k,\Lambda_c) = \varepsilon^{1/3} \Lambda_c^{-4/3} \ \tilde{v}(l)$ into 10.1-2, one obtains:

$$\int_0^1 dl \ \tilde{v}(l) \ l^{1/3} \ = \ \frac{1}{2C_k} \qquad\qquad 10.1\text{-}3$$

For $C_k = 1.61$ as obtained by the YO application of RNG, the expected value for 10.1-3 is 0.31. The constant $\tilde{v}(l) = 0.414$ will satisfy 10.1-3. However, constant values of eddy viscosity yield $E(k)$ spectra which decay closer to k^{-2} rather than $k^{-5/3}$ [14]. This implies overprediction of low wavenumber Fourier velocity components and underprediction of high wavenumber components.

If the eddy viscosity is too low, particularly at wavenumbers near the cutoff Λ_c, $E(k)$ will form a cusp near the cutoff, [14], rising above the required values. This will result in overprediction of the Fourier velocity components near the cutoff. The analytical form of such a cusp is unknown. It may be that the eddy viscosity cusp is

required to prevent the cusp of the energy spectrum. If the eddy viscosity is too high, $E(k)$ will form a dissipation region where the spectrum decay is faster than $k^{-5/3}$. The complete inertial-dissipation spectrum may be approximated by the relation due to Pao [13]:

$$E(k) = C_k \varepsilon^{2/3} k^{-5/3} \exp\left(- \frac{3C_k}{2} \left(\frac{k}{\Lambda_0}\right)^{4/3} \right)$$ 10.1-4

The generalized Kolmogorov wavenumber Λ_0 maybe defined as:

$$\Lambda_0(k, \Lambda_c) = \left(\frac{\varepsilon}{v(k, \Lambda_c)^3}\right)^{1/4}$$ 10.1-5

The relations 10.1-4 and 10.1-5 are normally applicable to a fully developed spectrum with a constant viscosity as opposed to a LES truncated one. However, they may indicate cases where the inertial range spectrum cannot be maintained due to overdamping. Substituting $v(k, \Lambda_c) = \varepsilon^{1/3} \Lambda_c^{-4/3} \tilde{v}(l)$, one has the result:

$$\Lambda_0(k, \Lambda_c) = \Lambda_c \tilde{v}(l)^{-3/4}$$ 10.1-6

Previous studies indicate that the eddy viscosity which follows a plateau-cusp behavior as a function of l may be expected to faithfully reproduce the intertial energy spectrum. Accordingly, some criteria for a suitable plateau and cusp characteristics are discussed below:

10.1.2 The plateau region

The plateau region is the constant portion of the eddy viscosity curve at low wavenumbers ($l < 0.3$ according to literature). Thus, for low wavenumbers, the effect is the same as increasing the molecular viscosity of the fluid while maintaining the power input per unit mass. The Reynolds number is reduced and the new Kolmogorov wavenumber is given by:

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$$\Lambda_0(0,\Lambda_c) \quad = \quad \Lambda_c \; \tilde{v}(0)^{-3/4} \qquad\qquad\qquad 10.1\text{-}7$$

As may be seen from figure 2.1, (see also reference [13] for a compilation of experimental data), noticeable deviation from the $k^{-5/3}$ slope occurs for wavenumbers greater than $\gamma\Lambda_0$ where $\gamma \approx 0.1$. Hence, in order to maintain the $k^{-5/3}$ slope. k should be less than $0.1\Lambda_0$, leading to the result:

$$\gamma^{4/3} \; l^{-4/3} \; \geq \; \tilde{v}(0) \qquad\qquad\qquad 10.1\text{-}8$$

The published range of l for the plateau region is generally $l < 0.3$. Substituting $l = 0.3$ and $\gamma = 0.1$ into 10.1-8 yields:

$$0.231 \; \geq \; \tilde{v}(0) \qquad\qquad\qquad 10.1\text{-}9$$

This upper bound limit on $\tilde{v}(0)$ is somewhat smaller than the typical literature value of 0.29 [14] and clearly depends on where one wants the plateau to end.

A similar requirement for the eddy viscosity in the plateau region may be arrived at by comparing the eddy turnover time versus the viscous diffusion time. The chief difficulty with time arguments in turbulence is that while proportional relationships are easily established, the constants of proportionality are somewhat arbitrary. Prior to any renormalization, at a given wavenumber k in the inertial range of turbulence. the eddy turnover time scale is much smaller than the viscous time scale. so that:

$$\frac{D \; \varepsilon^{1/3} \; k^{2/3}}{v_0 k^2} \; \gg \; 1 \quad (=) \quad D \; \varepsilon^{1/3} \; k^{-4/3} \; \gg \; v_0 \qquad 10.1\text{-}10$$

where $\tau_k \approx D^{-1}\varepsilon^{-1/3}k^{-2/3}$ is a typical eddy turnover time and $D \approx 0.1904 \; C_k^2$ according to an analysis due to Kraichnan [25]. Using the YO value of $C_k = 1.61$. $D \approx 0.494$.

After the elimination of a wide range of scales, the v_0 may be neglected and replaced by $v(k,\Lambda_c)$. If the resulting momentum equation is still to display inertial range turbulence, one now has for $k/\Lambda_c \ll 1$:

$$\frac{D\ \varepsilon^{1/3}\ k^{2/3}}{v(k,\Lambda_c)k^2} \gg 1 \quad (=) \quad D\ \varepsilon^{1/3}\ k^{-4/3} \gg v(k,\Lambda_c)$$

$$D\ \varepsilon^{1/3}\ k^{-4/3} \gg \varepsilon^{1/3}\ \Lambda_c^{-4/3}\ \tilde{v}(0)$$

$$D\ l^{-4/3} \gg \tilde{v}(0) \qquad\qquad\qquad\qquad\qquad 10.1\text{-}11$$

At the end of the plateau region, published studies indicate that $l \approx 0.3$, so that it is required that $\tilde{v}(0) \ll 2.46$.

10.1.3 The cusp

In comparison to the plateau region, the cusp portion of the eddy viscosity requires much more information to describe. There is the onset l value, the rate of increase, and the peak cusp value. Unfortunately, theoretical constraints cannot be used without further approximations. At the cutoff wavenumber, Λ_c, an energy conservation equation may be written:

$$(\ 2v(\Lambda_c,\Lambda_c)\Lambda_c^2\ )E(\Lambda_c) = \lambda \int_0^{\Lambda_c} \int_{\Delta k} dp\ dq\ S_E(k_c,p,q) + P(\Lambda_c) \qquad 10.1\text{-}12$$

where $S_E(k,p,q)$ contains triple correlations of $u(k)$, $u(p)$, and $u(q)$ subject to the constraint that k_c, p, and q form a triangle. Also, $|k_c| = \Lambda_c$ and $P(\Lambda_c)$ is the external power input into mode Λ_c.

Inserting the inertial range forms for the energy spectrum and the eddy viscosity, and setting $P(\Lambda_c) = 0$ in the inertial range, the following relation results:

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$$2 \ C_k \ \tilde{v}(1) \ \frac{\varepsilon}{\Lambda_c} \ = \ \lambda \int_0^{\Lambda_c} \int_{\Delta k} dp \ dq \ S_E(k,p,q) \qquad\qquad 10.1\text{-}13$$

However, the right hand side of equation 10.1-13 is unknown and must be approximated in terms of the λ expansion and stirring forces. Such approximations compromise the potential for validation of the eddy viscosity result.

At the cutoff, (l = 1), the eddy turnover time should equal the 'eddy-viscous' time. Equation 10.1-11 leads to:

$$D \approx \tilde{v}(1) \approx 0.494 \qquad\qquad 10.1\text{-}14$$

It is noted that the requirement of equation 10.1-14 does not confirm the Kraichnan cusp value of about 1.52, but instead it agrees with the result of YO. whose constant eddy viscosity equals $0.49 \ \varepsilon^{1/3} \ \Lambda_c^{-4/3}$.

10.1.4 The limit $\tilde{v}(l\rightarrow 0)$

It is the present author's opinion that the derivations presented in chapter 9 are inapplicable in the limit of $l \longrightarrow 0$. As already discussed in section 9.2. the volumes of regions B and C do not go to zero with k. Instead, the above derivation is limited to a range $\Delta l < l < 1$. For the range $0 < l \leq \Delta l$, the dominant contribution will be from region A instead of B and C and a modified version of the YO calculation will be applicable. It is not clear what value should be assigned to Δl. Unfortunately, it appears to be impossible to obtain a transition formula from the 'vanishing k' to the 'finite k' derivations since the sequence of the limits (k $\longrightarrow$ 0 limit first for the former and $\Delta\Lambda \longrightarrow$ 0 limit first for the latter) is not interchangeable [9].

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Smith and Reynolds [5] have pointed out that YO use their value of $v(0,\Lambda)$ in place of $v(\Lambda,\Lambda)$ at each stage of the elimination process, so that the differential equation introduced in chapter 4 as 4.8-5:

$$\frac{dv(0,\Lambda)}{dl} = A_d \frac{\lambda^2 W}{v(0,\Lambda)^2 \Lambda(l)^\in}$$

4.8-5

should instead be written as:

$$\frac{dv(0,\Lambda)}{dl} = A_d \frac{\lambda^2 W}{v(\Lambda,\Lambda)^2 \Lambda(l)^\in}$$

10.1-15

Since $\Lambda = \Lambda_0 e^{-l}$,

$$\frac{dv(0,\Lambda)}{dl} = \frac{\partial v(0,\Lambda)}{\partial \Lambda} \frac{d\Lambda}{dl} = - \frac{\partial v(0,\Lambda)}{\partial \Lambda} \Lambda$$

and using:

$$\frac{\partial v(k,\Lambda)}{\partial \Lambda} = \Lambda^{-7/3} \varepsilon^{1/3} \left(- \frac{4}{3} \tilde{v}(l) - l\frac{d}{dl} \tilde{v}(l) \right)$$

One has to first order in $\in$ expansion:

$$\frac{lim}{l \longrightarrow 0} \varepsilon^{1/3} \Lambda^{-4/3} \left(\frac{4}{3} \tilde{v}(l) + l\frac{d}{dl} \tilde{v}(l) \right) =$$

$$\frac{d^2 - d}{2d(d + 2)} \frac{C_{mg} \varepsilon}{2 \varepsilon^{2/3} \Lambda^{-8/3} \tilde{v}(l)^2 \Lambda^4}$$

which simplifies to:

$$\frac{lim}{l \rightarrow 0} \left(\frac{4}{3} \tilde{v}(l) + l\frac{d}{dl} \tilde{v}(l) \right) = \frac{1}{10} \frac{C_{mg}}{\tilde{v}(1)^2}$$

10.1-16

The solution is:

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$$\tilde{v}(0) \;=\; \frac{3}{4}\,\frac{1}{10}\,\frac{C_{mg}}{\tilde{v}(1)^2} \quad (=) \quad \tilde{v}(0) \;=\; \frac{0.119}{\tilde{v}(1)^2}$$

10.1-17

The above equation provides a relation between $\tilde{v}(0)$ and $\tilde{v}(1)$ based on the YO derivation, and thus the spherical iteration shell used therein. From the three derivations in chapter 9, the B+C region also has a spherical shell. However, relation 10.1-17 will be tested on all three values of $\tilde{v}(1)$. It is noted that the YO value (0.49) is a 'break-even' point where $\tilde{v}(0) = \tilde{v}(1)$. For $\tilde{v}(1) > 0.49$, $\tilde{v}(0) < 0.49$ and vice-versa.

In comparison, Kraichnan obtained $\tilde{v}(0) = 0.29$, and $\tilde{v}(1) = 1.52$, [13], leading to $\tilde{v}(0) = 0.67/\tilde{v}(1)^2$, far off from 10.1-17, suggesting a fundamentaly different theory.

10.1.5 Comparison with literature

The spectral eddy viscosity results obtained in this study will be compared to two published forms, that due to Kraichnan [23] (the first one to propose the cusp - plateau behaviour) and that due to the cumulative work of Chollet, Métais, and Lesieur (referred herein as CML) who proposed the following analytical form for the eddy viscosity:

$$v(k,\Lambda_c) \;=\; 0.441 C_k^{-3/2} \left[\frac{E(\Lambda_c)}{\Lambda_c}\right]^{1/2} \left(1 + v_n^* \left[\frac{k}{\Lambda_c}\right]^{3.7}\right)$$

10.1-18

Subject to the constraint 10.1-2, and using the YO value of $C_k = 1.61$, equation 10.1-13 may be solved to yield $v_n^* \approx 1.93$. The resulting equation for $\tilde{v}(l)$ is:

$$\tilde{v}(l) \;=\; \frac{0.441}{1.61}\left(1 + 1.93\; l^{3.7}\right)$$

10.1-19

Many other spectral eddy viscosity results exist [13] but with a wide discrepancy among each other.

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10.2 Eddy viscosity solution from averaging B-C regions with fixed wavenumber averaging

The equation for $\tilde{v}_f(l)$ is repeated here for convenience:

$$\tilde{v}_f(l) = \frac{1}{8} \frac{C_{mg}}{\tilde{v}_f(1)^2} \, l^{-4/3} \int_0^l d\xi \, \xi^{-8/3} \, I_f(\xi) + l^{-4/3}C_f \qquad 9.2\text{-}27$$

As a first step in the examination of the characteristics of 9.2-27, the plots of I_{fB} and I_{fC}, and their sum, I_f are indicated in figure 10.1. An equation equivalent to 9.2-27 but containing only I_f has been presented by Carati [11].

Referring to figure 10.1, the term I_{fC} is negative and relatively small in magnitude over the range $0 < l < 0.519$. It is this negative range of I_{fC} which causes the negative regions in the numeric plots of the eddy viscosity as generated by Carati [11]. The negative region implies a backscatter of energy, that is energy flow from small to large scales for motions contained in region C and with l in the above range. $I_{fC}(l=1)$ is well-defined and equals 1.42.

I_{fB} is positive for $0 < l < 1$, reaching a peak value $I_{fB}(l\approx0.854) \approx 0.1255$, then decreasing to approximately 0.1133 for $l \longrightarrow 1$. It is noted that $I_{fB}(l=1)$ is indeterminate but the limit is well defined.

Including the region B forces through I_{fB} almost compensated the negative region of I_f. The total, $I_f(l)$ is negative for $0 < l < 0.062$ but very near zero, with a minimum value of the order of 1×10^{-6} at $l \approx 0.048$. For $l \geq 0.062$, $I_f(l)$ is positive and rises to a cusp, commencing at approximately $l \approx 0.3$. The peak value is 1.5332.

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Next, in anticipation of possible singularities, the behavior of I_f and its components as $l \longrightarrow 0$ is investigated. A series expansions of I_{fB}, I_{fC}, and I_f about the point $l = 0$ have been evaluated to $O(l^8)$ with the aid of *Mathematica* as follows:

$$I_{fB}(l) = 0.25 \ l^2 + 0.311111 \ l^3 - 0.5 \ l^4$$
$$- 0.0261023 \ l^5 + 0.0516332 \ l^6 + 0.0125787 \ l^7 + 0.00537194 \ l^8 + O(l^9) \qquad 10.2\text{-}1$$

$$I_{fC}(l) = - 0.25 \ l^2 - 0.355556 \ l^3 + 1.16667 \ l^4$$
$$+ 0.862787 \ l^5 - 0.0326003 \ l^6 + 0.0504975 \ l^7 - 0.0380794 \ l^8 + O(l^9) \qquad 10.2\text{-}2$$

and the sum:

$$I_f(l) =$$
$$- 0.0444444 \ l^3 + 0.666667 \ l^4 + 0.836684 \ l^5 +$$
$$+ 0.0190329 \ l^6 + 0.0630762 \ l^7 - 0.0327075 \ l^8 + O(l^9) \qquad 10.2\text{-}3$$

Clearly, $I_f(l=0) = 0$. It is noted that the coefficient of l^2 in the expansion for $I_f(l)$ is zero, thus eliminating a possible singularity in the solution for $v_f(l)$ Inspection of equation 9.2-27 indicates that $I_f(\zeta)$ is multiplied by $\zeta^{-8/3}$, integrated with respect to l and multiplied by $l^{-4/3}$. For the net effect near $l = 0$, this is equivalent to a multiplication of $I_f(l)$ by l^{-3} yielding -0.0444444 as the lowest order term. Thus, the singularity identified by Carati [11] has been eliminated by introducing region B forces.

The numerical evaluation of the integrand $\xi^{-8/3} I_f(\xi)$ in equation 9.2-27 yields:

$$\int_0^1 d\xi \ \xi^{-8/3} \ I_f(\xi) = 0.516 \qquad 10.2\text{-}4$$

Substituting this result into 9.2-27 and setting $l = 1$, one obtains the result:

$$\tilde{v}_f(1) = \left(0.516 \ \frac{C_{mg}}{8} \right)^{1/3} = 0.468 \qquad 10.2\text{-}5$$

where $C_{mg} = 1.59$ has been used and the integration constant C_f has been set to zero in order to avoid singularity as $l \longrightarrow 0$. The result is close to the YO value of 0.49. The equation for $\tilde{v}_f(l)$ may now be expressed as:

$$\tilde{v}_f(l) = l^{-4/3} \frac{1.59}{8(0.468^2)} \int_0^l d\xi \; \xi^{-8/3} \; I_f(\xi)$$

10.2-6

The numerically generated solution is plotted in figure 10.2. Note that the plot scales are linear. The $\tilde{v}_f(l)$ increases monotonically with l at a slightly higher than linear rate. There is no noticeable plateau-cusp behavior. The $\tilde{v}_f(l)$ becomes negative at $l = 0.1$. Extrapolating the plot, the y-intercept should be approximately -0.04. This value does not conform to 10.1-17. For very small values of l, the plot of $\tilde{v}_f(l)$ appears to have a negative singularity. This anomaly is due to the fact that the derivation of $\tilde{v}_f(l)$ is based on the geometry of two thin, spherical shells with well defined intersection points. As l becomes small, the intersection points are no-longer well defined. Therefore, in the series expansion of I_f as generated by *Mathematica*, in addition to the terms listed in equation 10.2-3, there are spurious terms such as $O(10^{-16}) l^{-2}$ etc. These terms affect the solution for small values of l. This problem affects all of the eddy viscosity models derived in this section.

Figure 10.1 Plots of integrals with fixed stirring forces in B and C regions as a function of l.
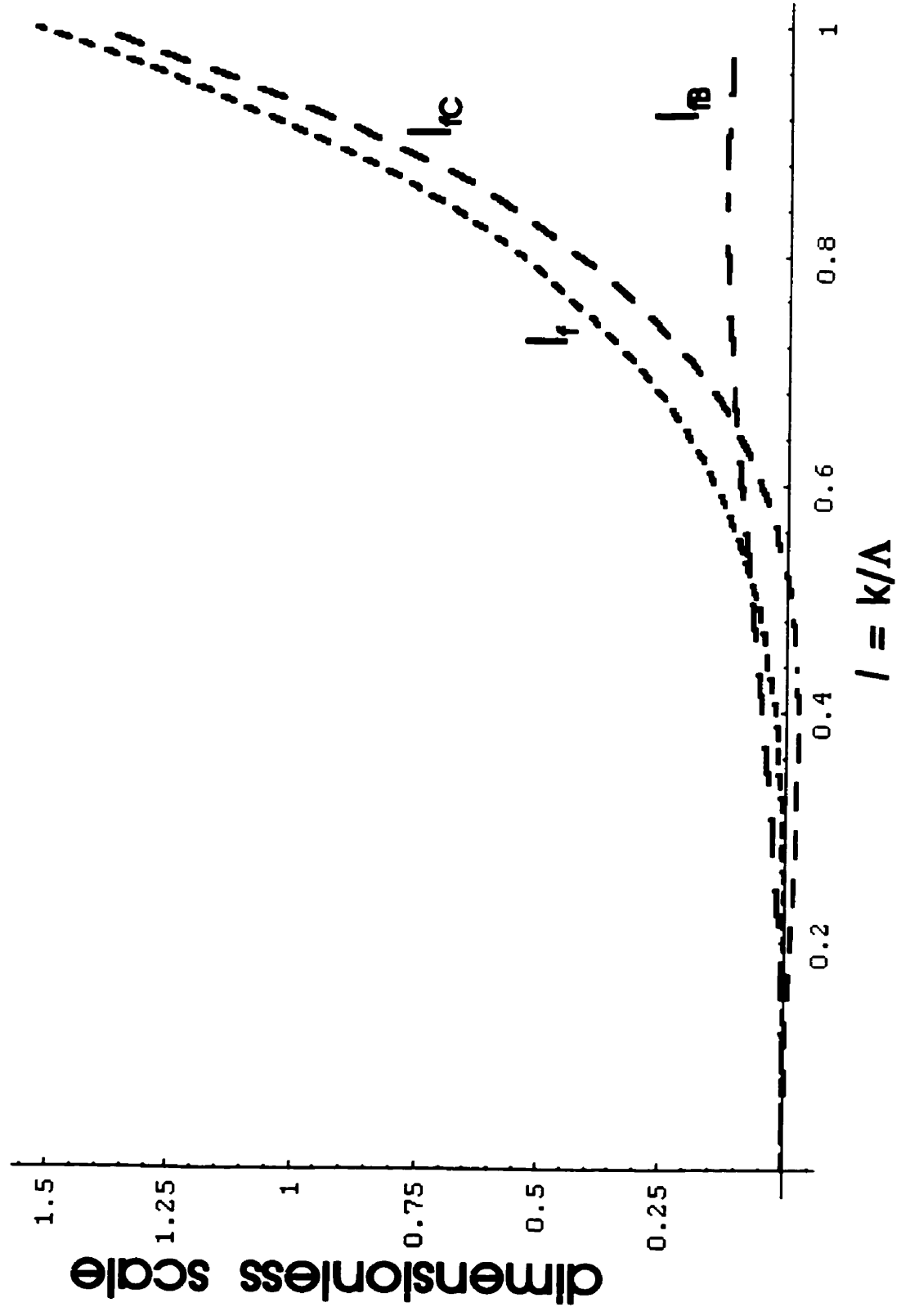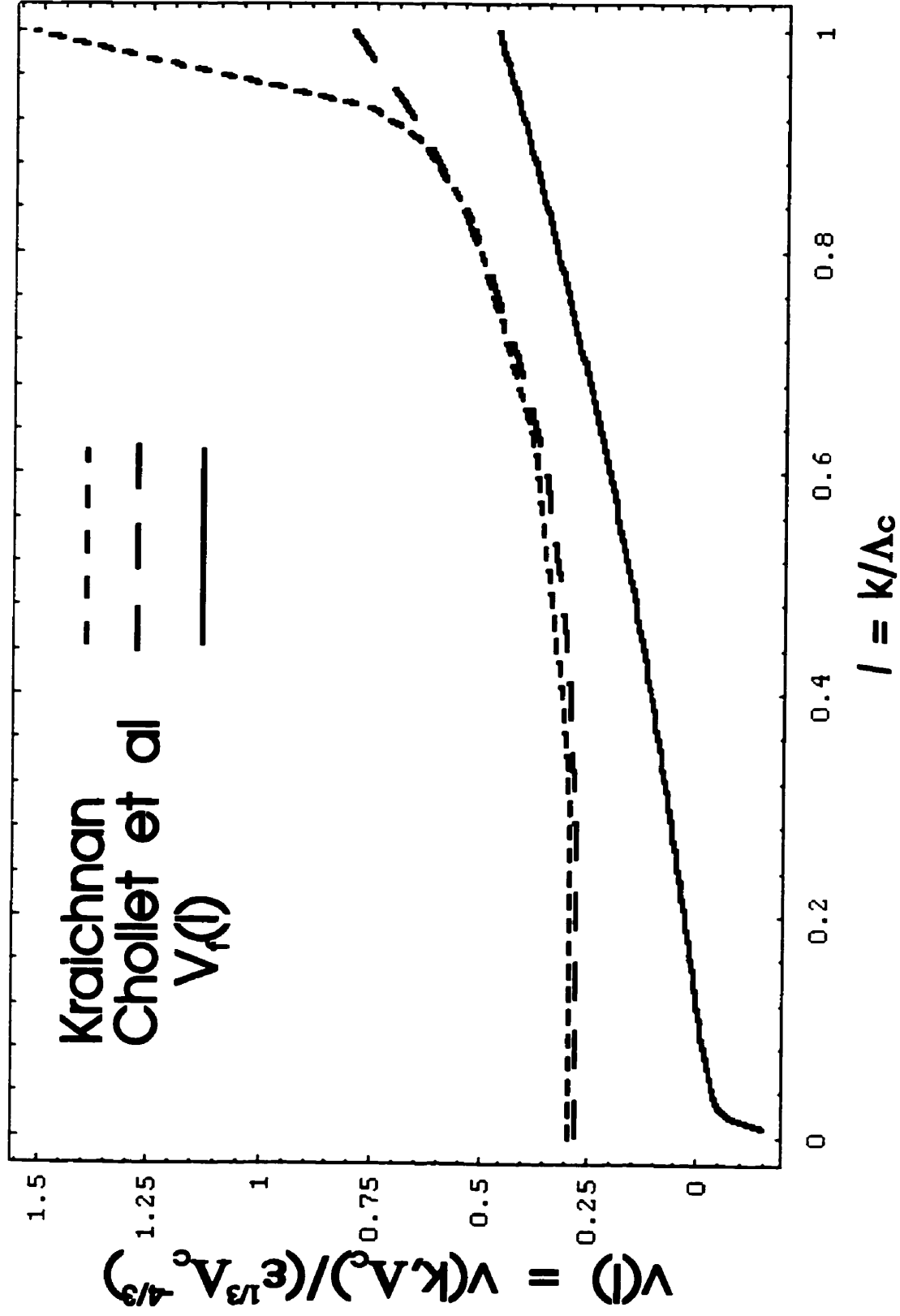
Figure 10.2 Plot of v(l) vs. wavenumber ratio. Averaged wavenumber fixed at Λ_c.

$$v(l) = v(k\Lambda_c)/(\varepsilon^{1/3}\Lambda_c^{-4/3})$$

$l = k/\Lambda_c$

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10.2.1 Validation tests of $\tilde{v}_f(l)$

Numerical integration of the dissipation integral gives:

$$\int_0^1 dl \ \tilde{v}_f(l) \ l^{1/3} = 0.158 \qquad\qquad 10.2\text{-}7$$

which is approximately half of the expected value of 0.31. Therefore, not enough damping is provided by $\tilde{v}_f(l)$. As this test is a necessary condition for a realistic eddy viscosity, it is concluded that $\tilde{v}_f(l)$ is unsuitable as a subgrid model for turbulence.

10.3 Eddy viscosity solution from averaging fixed plus low wavenumbers

Now one considers the equation for $\tilde{v}_{fl}(l)$ as derived in section 9.3, repeated here for convenience:

$$\tilde{v}_{fl}(l) = \frac{1}{8} \frac{C_{mg}}{\tilde{v}_{fl}(1)^2} \ l^{-4/3} \int_0^l d\xi \ \xi^{-8/3} \ I_{fl}(\xi) + l^{-4/3}C_{fl} \qquad 9.3\text{-}8$$

where

$$I_{fl}(l) = \frac{1}{l^3} \int_{1-l}^{1} \zeta d\zeta \ \frac{1}{(\zeta^{2/3} + 1)} \frac{1}{\zeta^{11/3}} \times$$

$$\left\{ 1 - \left\{ \frac{1 + l^2 - \zeta^2}{2l} \right\}^2 \right\} \left\{ 2 \frac{l^2}{\zeta^2} - (1 + l^2 - \zeta^2) \right\} +$$

$$\frac{1}{l^3} \left\{ \int_{1-l}^{1} \zeta d\zeta \ \frac{1}{(\zeta^{2/3} + 1)} \times \right.$$

$$\left\{ 1 - \left\{ \frac{1 + l^2 - \zeta^2}{2l} \right\}^2 \right\} \left\{ 2l^2 - \frac{\zeta^2 + l^2 - 1}{\zeta^2} \right\} \qquad 9.3\text{-}9$$

The above relation may be written as $I_n(l) = I_{fB}(l) + I_{lB}(l)$, where $I_{fB}(l)$ is the same as in section 10.3. It is noted that the integrand of $I_{lB}(l)$ is singular at $\zeta = 0$, (corresponding to $l = 1$, the lower integration limit).

A series expansion of $I_{lB}(l)$ and the total, $I_n(l)$, about $l = 0$ gives:

$$I_{lB}(l) = -0.25l^2 + 0.133333l^3 + 0.62963l^4 + 0.975661l^5 + 1.26938l^6 + 1.53486l^7 + 1.78223l^8 + O(l^9)$$

10.3-1

and

$$I_n(l) = 0.44444l^3 + 0.12963l^4 + 0.949559l^5 + 1.32101l^6 + 1.54743l^7 + 1.7876l^8 + O(l^9)$$

10.3-2

Inspection of 10.3-1 indicates that there will not be a singularity at $l \longrightarrow 0$ and also that $\tilde{v}_n(0)$ is expected to be positive since:

$$\tilde{v}_n(0) \sim 0.444444 \frac{C_{mg}}{8 \, \tilde{v}_n(1)^2} > 0$$

10.3-3

The significance of this result is the indication that eddy viscosity models averaging over triads with only one subgrid wavenumber are not expected to exhibit backscatter at low wavenumbers.

However, it is expected that a singularity appears at $l \longrightarrow 1$. A series expansion of $\tilde{v}_n(l)$ about $l = 1$ gives:

$$\tilde{v}_n(l) = \frac{-0.654545}{(l-1)^{5/3}} + \frac{1.33333}{(l-1)} + \frac{-0.654545}{(l-1)^{2/3}} - \frac{5.14286}{(l-1)^{1/3}} - 14.0216 + O(l-1)$$

10.3-4

The $\tilde{v}_n(l)$ function diverges as $\frac{-0.654545}{(l-1)^{5/3}}$ as $l \longrightarrow 1$. Therefore it is expected that:

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$$\int_0^1 d\xi \; \xi^{-8/3} \; I_{fl}(\xi) \; = \; \infty \qquad\qquad 10.3\text{-}5$$

Numerical calculations confirm that the above integral diverges. The general remedy for this situation is to provide a low integral limit cutoff as a function of l to prevent ζ from reaching 0. The selection and justification of such a cutoff limit is crucial to the resulting form of $\tilde{v}_{fl}(l)$.

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10.3.1 Eddy viscosity solution from averaging fixed plus low wavenumber limited to k.

The cutoff may be justified by restricting the variable averaging of $f(|k-j|.t)$ to $|k-j| > k$. Allowing the averaging to extend to $|k-j| \le k$ will require that $u(k.t)$ now participate in the averaging and yield a zero contribution. Therefore, the definition of $I_{fl}(l)$ is now modified as follows:

$$
I'_{fl}(l) = \begin{cases} \displaystyle\int_{1-l}^{1} i_{fl}(l,\zeta)\, d\zeta & \text{for } 0 < l < 0.5 \\[2em] \displaystyle\int_{l}^{1} i_{fl}(l,\zeta)\, d\zeta & \text{for } 0.5 \le l \le 1 \end{cases} \qquad\qquad 10.3\text{-}6
$$

where i_{fl} is the combined integrand from equation 9.3-9.

Equation 10.3-1 indicates that ζ should not be smaller than l for partial averaging. For $\zeta \ge l$, both l and ζ participate in the averaging, yielding a zero result for the remaining portion of the integration contour (between l and $1-l$).

Note that both $I_{fB}(l)$ and $I_{lB}(l)$ are modified due to the above definition, even though there is no potential singularity in $I_{fB}(l)$. However, the form of this integral is $\int u^{>0} G(|k-j|)M^<(k-j)\int u^<u^{>0}$ and the term $G(|k-j|)$ is a random variable due to $v(|k-j|,|k-j|)$ with a time scale proportional to $|k-j|^{-2/3}$ and implying an average over wavenumbers less than k.

$I_{fB}(l)$, $I_{lB}(l)$, and the sum $I'_{fl}(l)$ with the limits given by 10.3-6. were evaluated analytically using *Mathematica* and plotted in figure 10.3. It is noted that $I_{lB}(l)$ is negative over $0 < l < 0.38$ approximately. The plot of $I_{fl}(l)$ increases to a peak at about $l = 0.62$, then decreases again.

The numerical integration of $I_{fB}(l)$ as given by equation 10.3-6. may be used to evaluate the peak

$$\int_0^1 d\xi \; \xi^{-8/3} \; I_n'(\xi) \; = \; 0.405 \qquad\qquad\qquad 10.3\text{-}7$$

As before, one can evaluate $\tilde{V}_n(1)$:

$$\tilde{V}_n(1) \; = \; \left(\; 0.405 \; \frac{C_{mg}}{8} \; \right)^{1/3} \; = \; 0.432 \qquad\qquad 10.3\text{-}8$$

where the integration constant has been set to zero.

The eddy viscosity $\tilde{V}_n(l)$ as given by equation 9.3-8 was evaluated numerically and plotted in figure 10.4. Due to the peak of $I_n(l)$, the plot of $\tilde{V}_n(l)$ also has a peak value of approximately 0.62 at $l = 0.63$ and then decreases to a value of about 0.44. Ignoring the supurious downturn, $\tilde{V}_n(l \sim 0) \approx 0.35$. This value does not satisfy equation 10.1-17.

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Figure 10.3 Plots of integrals with fixed and low stirring forces in regions B as a function of l.
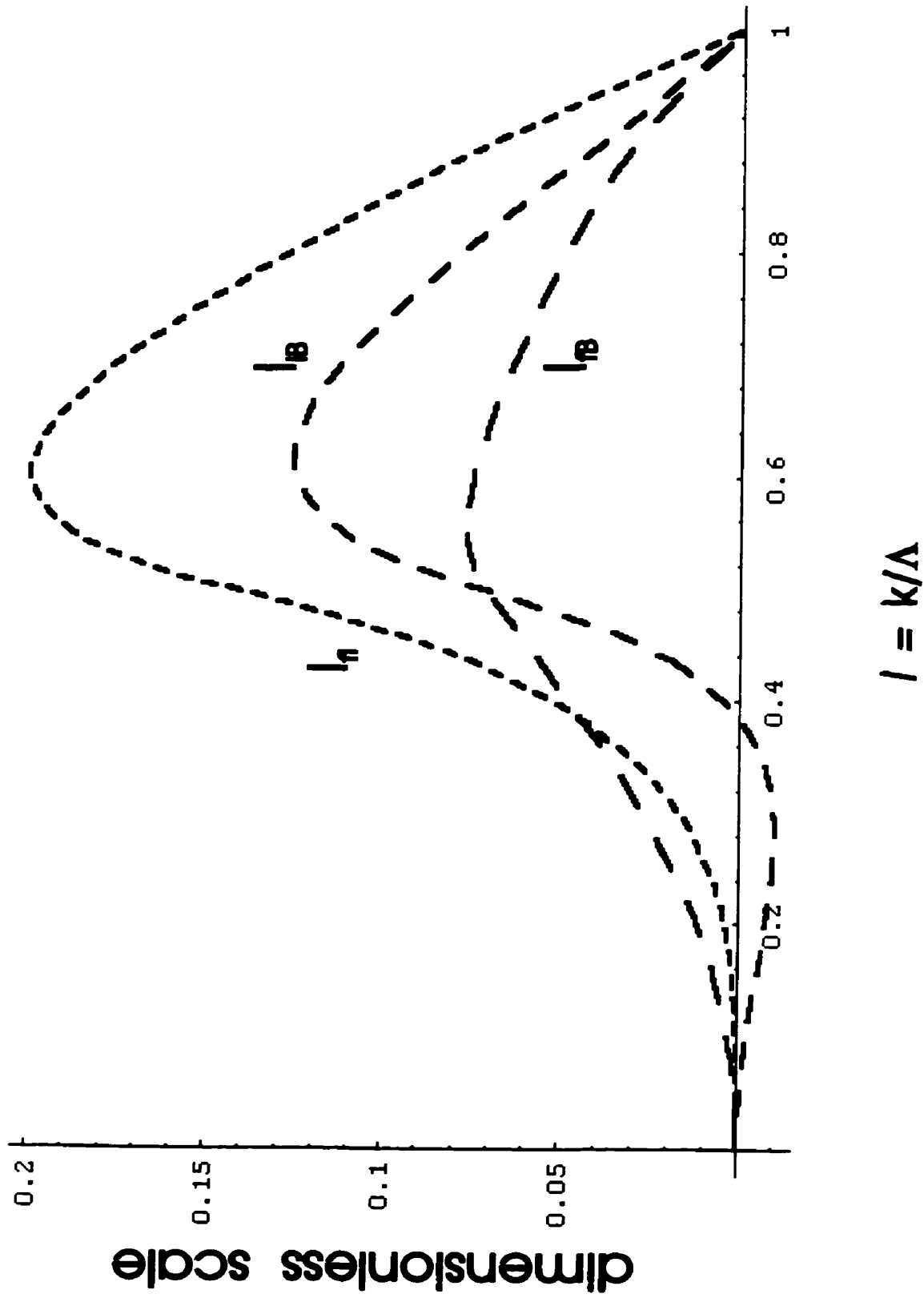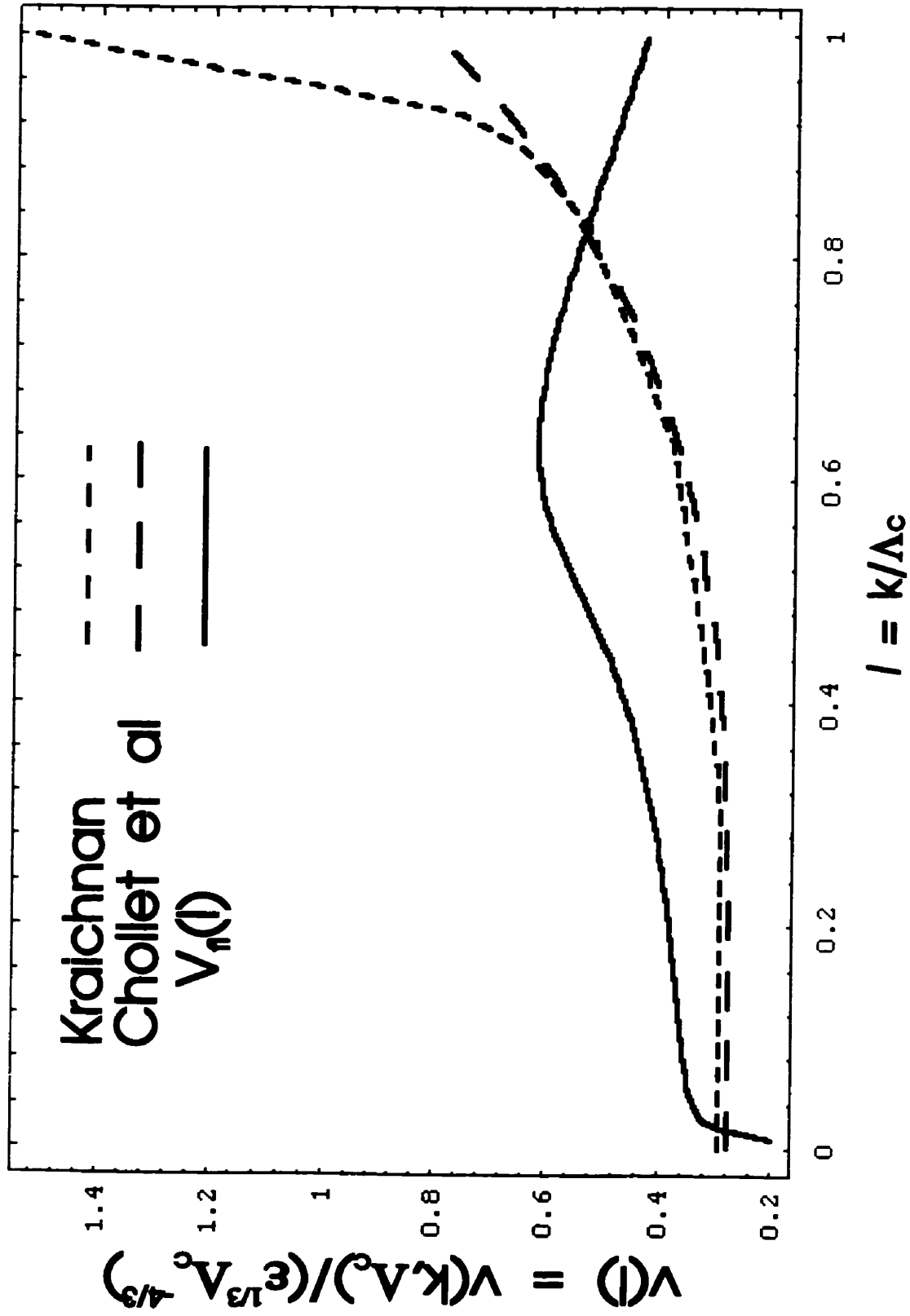
$l = k/\Lambda$

I_{fB}

I_{lB}

I_{fl}

dimensionless scale

Figure 10.4 Plot of v(l) vs. wavenumber ratio.
Averaged wavenumbers limited to k.

10.3.1.1 Validation tests of $\tilde{v}_{fl}(l)$

Numerical integration of the dissipation integral gives:

$$\int_0^1 dl \; \tilde{v}_{fl}(l) \; l^{1/3} = 0.373 \qquad\qquad 10.3\text{-}9$$

which is somewhat larger than the expected value of 0.31. The result of 10.2-8 corresponds to a Kolmogorov constant of 1.34 which is outside the accepted range. Therefore, too much damping is provided by $\tilde{v}_{fl}(l)$.

Comparison of the plots for $\tilde{v}_{fl}(l)$ to these of Kraichnan and CML published literature is clearly unsatisfactory for the expected shape of the plot, since a plateau followed by a cusp is expected. It is concluded that the variable averaging with a cutoff at $|\mathbf{k\text{-}j}| = k$ does not yield the correct eddy viscosity behavior.

10.3.2 Eddy viscosity solution from averaging regions B with fixed plus low wavenumber averaging adjusted for dissipation

An alternative for specifying the cutoff on the minimum value of the averaged wavenumber is a limit Λ_L so that the partial averaging is allowed to extend to $\Lambda_L <$ $|\mathbf{k\text{-}j}| < \Lambda$, even if $|\mathbf{k\text{-}j}| < k$. In that case, the RNG theory needs to be extended to self-consistently provide the Λ_L value.

The limit Λ_L could equal Λ_P, the peak of the energy spectrum. Theories that relate Λ_L and $\tilde{v}(l)$ have appeared before, [22]. As pointed out by Leslie. $\tilde{v}(l)$ should be determined locally, not globally in wavenumber space. To be self-similar. $\tilde{v}(l)$ must depend only on k and Λ. Therefore Λ_L must be a function of k and/or Λ. Unfortunately. the current formulation of the RNG does not provide an explicit relation for Λ_L. However, any proposed form of the lower limit should yield a dissipation integral that complies with 10.1-3 and it will be selected on that basis in this work.

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Two simple and arbitrary forms of the low limit of the partial average are γl (proportional to k) or γ (proportional to Λ_c) where γ maybe selected from $0 < \gamma \leq 1$.

a. Low limit proportional to k.

To satisfy the dissipation integral (10.1-3), the low limit proportional to k is γl. Equation 10.3-6 is thus modified:

$$
I'_n(l) = \begin{cases} \displaystyle\int_{l-l}^{1} i_n(l,\zeta)\ d\zeta & \text{for } 0 < l < \dfrac{1}{1+\gamma} \\[20pt] \displaystyle\int_{\gamma l}^{1} i_n(l,\zeta)\ d\zeta & \text{for } \dfrac{1}{1+\gamma} \leq l \leq 1 \end{cases}
\qquad 10.3\text{-}10
$$

For γ equal to approximately 0.6, the dissipation integral 10.1-2 yields 0.310 and the $C_k \approx 1.61$. The plot of $\tilde{v}_n(l)$ is presented in figure 10.5. The function rises to a peak value of $\tilde{v}_n(l=0.85) = 0.59$, then decreases in value to $\tilde{v}_n(1) = 0.55$. It is noted that the peak moves closer to $l = 1$, relatively to figure 10.4, and the plot becomes closer to that of CML also presented in figure 10.5. It is noted that since the CML plot satisfies 10.2, the higer cusp of the function due to Kraichnan appears to be too dissipative for a $C_k \approx 1.61$.

Figure 10.5 Plot of v(l) vs. wavenumber ratio.
Averaged wavenumbers limited to 0.6k.
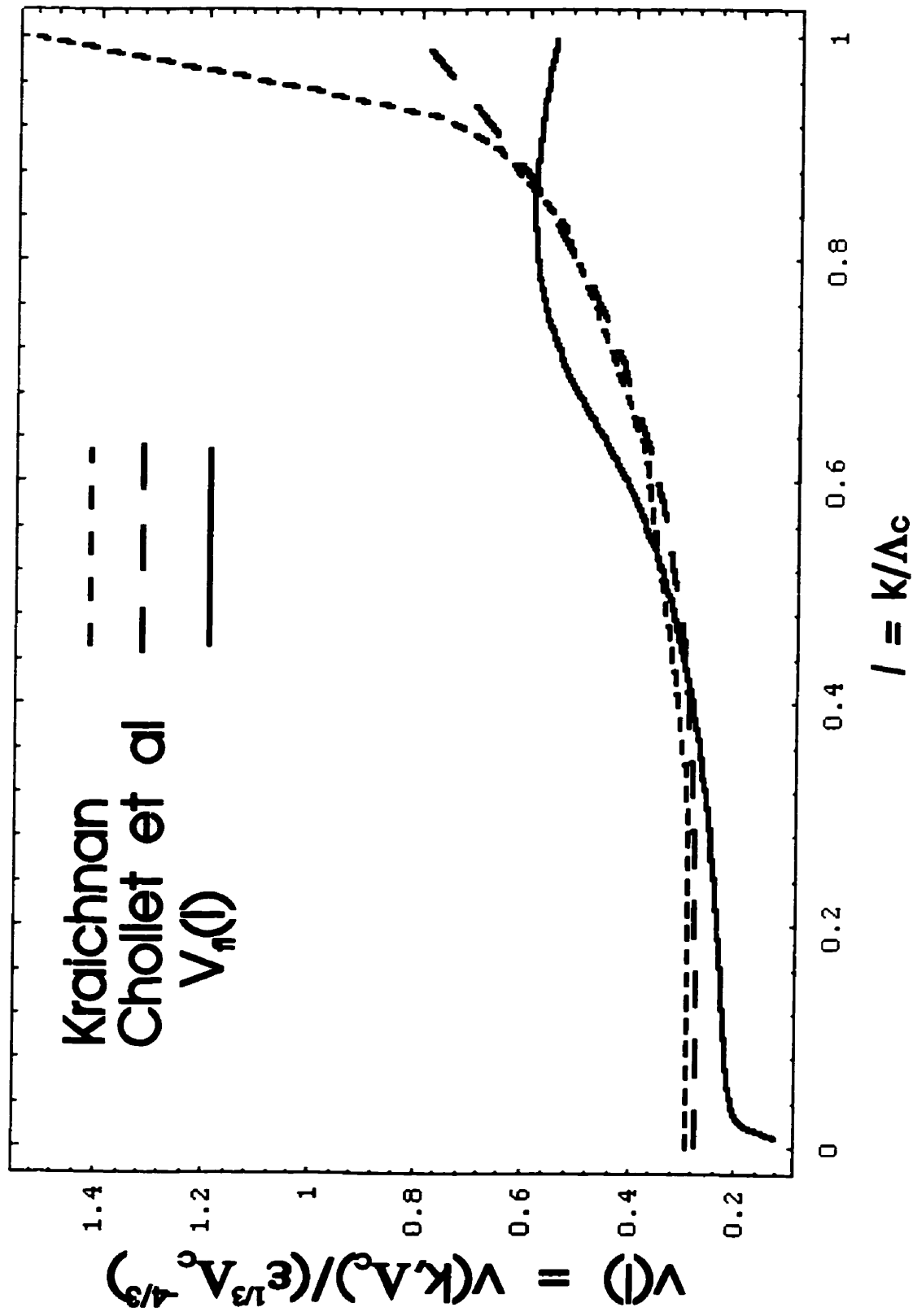
b. Low limit proportional to Λ_c.

To satisfy the dissipation integral (10.1-3), the low limit proportional to Λ_c is γ. Equation 10.3-6 is thus modified:

$$I'_n(l) = \begin{cases} \displaystyle\int_{1-l}^{1} i_n(l,\zeta) \, d\zeta & \text{for } 0 < l < 1 - \gamma \\[20pt] \displaystyle\int_{\gamma}^{1} i_n(l,\zeta) \, d\zeta & \text{for } 1 - \gamma \leq l \leq 1 \end{cases} \qquad 10.3\text{-}10$$

For the constant limit γ equal to approximately 0.55, the dissipation integral 10.1-2 yields 0.310 and the $C_k \approx 1.61$. The plot of $\bar{v}_n(l)$ is presented in figure 10.6. The function rises to a peak value of $\tilde{v}(l{=}0.89) = 0.51$, then decreases in value to $\tilde{v}_n(1) = 0.50$.

Figure 10.6 Plot of $v(l)$ vs. wavenumber ratio. Averaged wavenumbers limited to $0.55\Lambda_c$.

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$v(l) = v(k,\Lambda_c)/(\varepsilon^{1/3}\Lambda_c^{-4/3})$

$l = k/\Lambda_c$
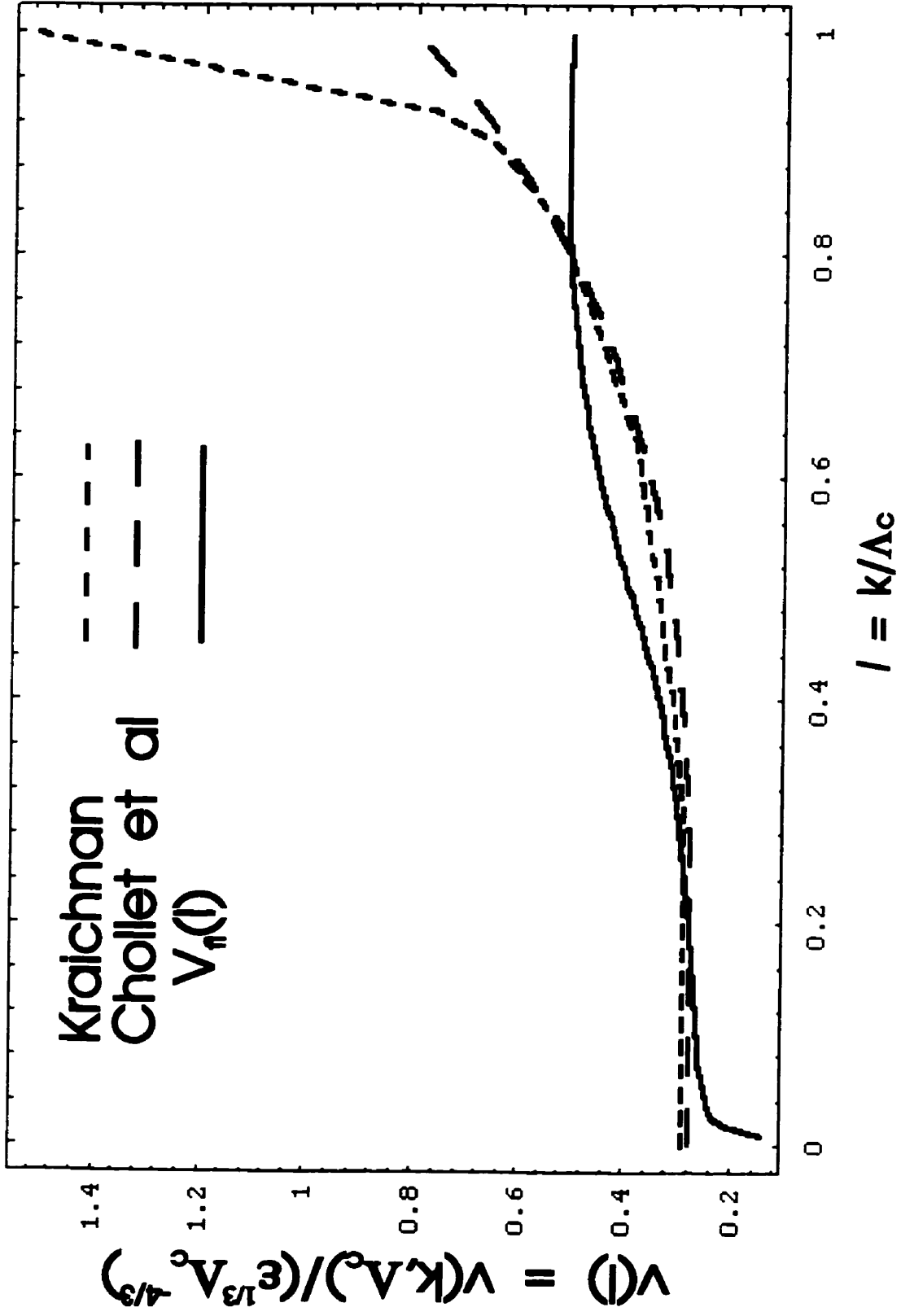
10.4 Eddy viscosity solution from averaging over regions C

Again, the expression for $I_{fh}(l)$ is reiterated below for convenience.

$$I_{fh}(l) = \int_{1}^{1+l} \zeta d\zeta \frac{1}{(\zeta^{2/3} + 1)} \left\{ \zeta^2 - \left\{ \frac{\zeta^2 - 1 - l^2}{2l} \right\}^2 \right\} \times$$

$$\left\{ \frac{1}{\zeta^{2/3}} \left\{ 2\frac{l^2}{\zeta^2} - 1 - l^2 + \zeta^2 \right\} + \left\{ 2l^2 - \frac{\zeta^2 + l^2 - 1}{\zeta^2} \right\} \right\} \qquad 9.4\text{-}2$$

Expanding $I_{fh}(l)$ in powers of l about $l = 0$, gives:

$$\begin{aligned}
I_{fh}(l) = \ & 0.844444l^3 + 1.78704l^4 - 0.17425l^5 + 0.963649l^6 \\
& - 0.910493l^7 + 0.893298l^8 + O(l^9)
\end{aligned} \qquad 10.4\text{-}1$$

The above relation may be written as $I_{fh}(l) = I_{fC}(l) + I_{hC}(l)$, where $I_{fC}(l)$ is the same as in section 10.3. The three components are plotted in figure 10.7. Integrating the eddy viscosity integral:

$$\int_{0}^{1} d\xi \ \xi^{-8/3} \ I_{fh}(\xi) = 1.472 \qquad 10.4\text{-}2$$

This yields a value of $\tilde{v}_{fh}(1) = 0.664$. The plot of $\tilde{v}_{fh}(l)$ (figure 10.8) appears nearly a straight line. If one disregards the apparently spurious downturn near the y axis, the asymptotic y intercept would occur near 0.28. This value is close to the value of 0.27, predicted by equation 10.1-17.

10.4.1 Validation tests of $\tilde{v}_{fl}(l)$

Numerical integration of the dissipation integral gives:

$$\int_{0}^{1} dl \ \tilde{v}_{fl}(l) \ l^{1/3} = 0.370 \qquad 10.4\text{-}3$$

which is somewhat larger than the expected value of 0.31. The result of 10.3-9 corresponds to a Kolmogorov constant of 1.35 which is outside the accepted range. Therefore, too much damping is provided by $\tilde{v}_{fl}(l)$. The shape of the plot does not exhibit the plateau-cusp behavior.

Figure 10.7 Plots of integrals with fixed and high wavenumber stirring forces in the C regions.
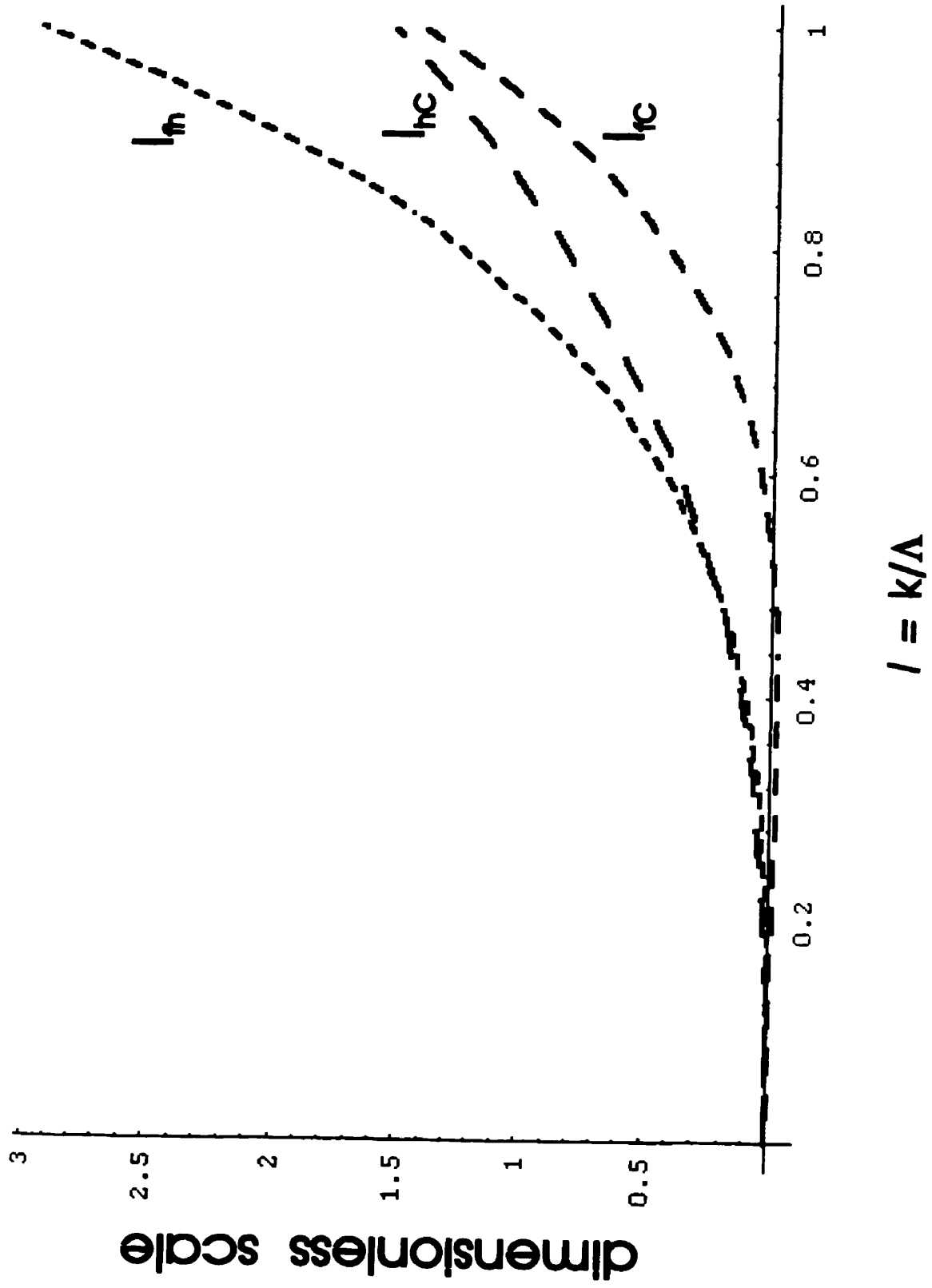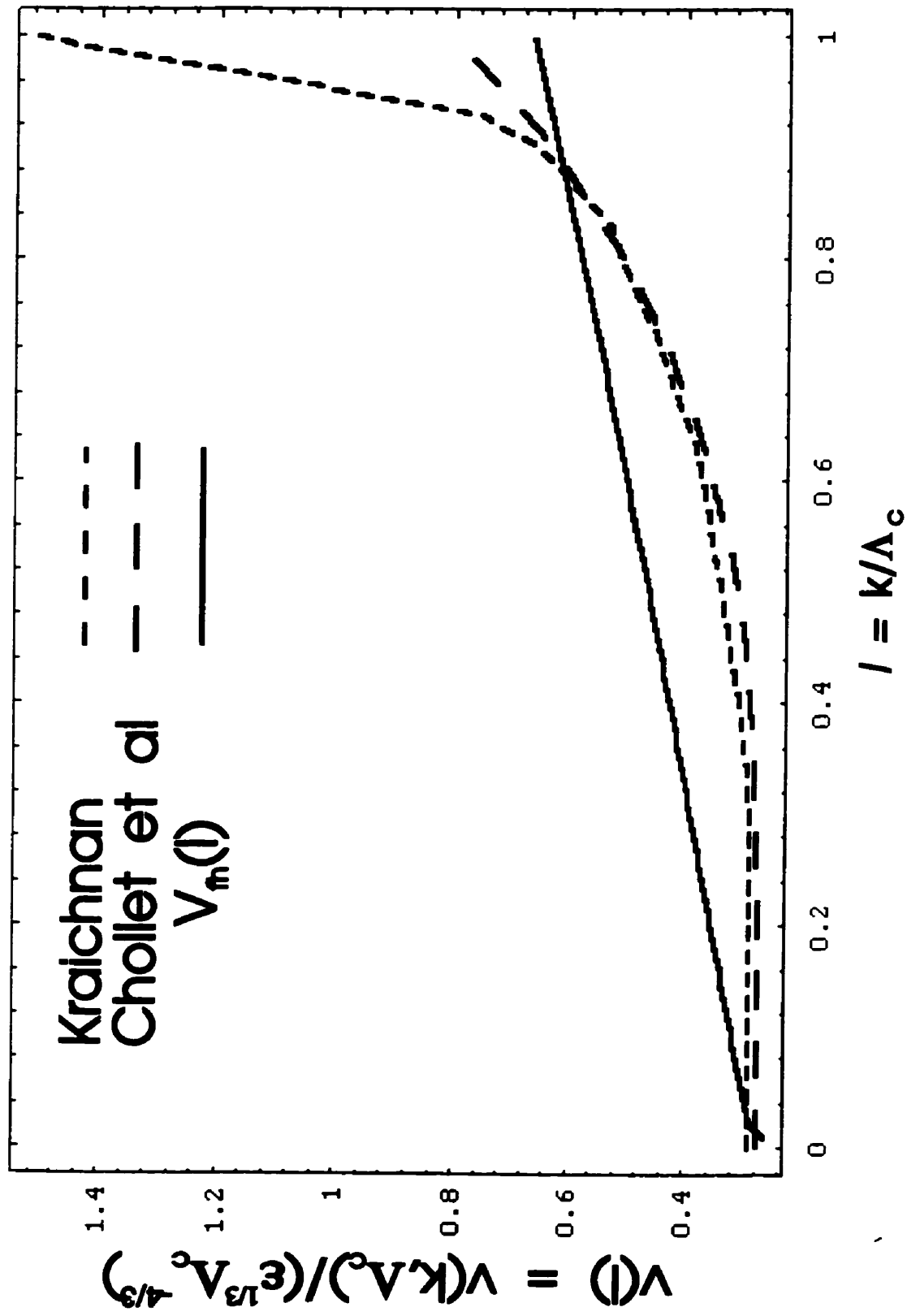
Figure 10.8 Plot of v(l) vs. wavenumber ratio for constant and high averaged wavenumbers.

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$V_m(l)$

$l = k/\Lambda_c$

$$V(l) = v(k,\Lambda_c)/(\varepsilon^{1/3}\Lambda_c^{-4/3})$$

10.5 Summary and discussion

Plots of the eddy viscosity integrands I_{fC}, I_{hC}, I_{fB}, and I_{lB} were presented in this chapter (figures 10.1, 10.3, and 10.8). The integrands I_{fC}, (figures 10.1. 10.7) and I_{lB} (figure 10.3) are negative over a range of l values, $0 < l < 0.52$ for I_{fC} and $0 < l < 0.38$ for I_{lB}. The negative values are significant because an eddy viscosity formed from these terms alone (as is the case for ZVH and others [19]) will also be negative over the low range of l. This results in the phenomenon of backscatter reported by some researchers in this field ,[19], that is a net energy flux from small to large scales across the cutoff Λ_c. Such backscatter had been reported at low l values for the non-local interactions. These findings contradict the theory of YO although the latter may be viewed as a type of average eddy viscosity [7].

None of the eddy viscosity models analyzed in this chapter have been in qualitative agreement with the results due to Kraichnan [23] or CML [14]. The eddy viscosity derived by partial averaging of low wavenumbers limited to 0.6k appears the closest to the model of CML. This model has been adjusted to satisfy the dissipation integral.

The eddy viscosity results due to Kraichnan are treated as a kind of a benchmark in the literature quoted herein. Detailed examination of Kraichnans work is beyond the scope of this study. however several observations are germane to the current effort. The eddy viscosity relation due to Kraichnan using the Renormalized Perturbation Theory (RPT) (McComb [13]) is given below:

$$v(k/\Lambda_c) = \frac{Ck\varepsilon^{1/3}}{4\pi\beta} k^{-2} \int_{j \geq \Lambda_c} d^3j \ L(k,j) \ |k-j|^{-11/3} \frac{k^{-11/3} - j^{-11/3}}{k^{2/3} + j^{2/3} + |k-j|^{2/3}} \qquad 10.5\text{-}1$$

where β is a constant related to C_{mg}, and $L(k,j)$ is a function of k. j. and the cosine of the angle between k and j, similar to $A(k,j)$. It is noted that the term $|k-j|^{-11/3}$ will be less than Λ for values of k near Λ. Since $|k-j|^{-11/3}$ is the statistical average energy density, RPT utilizes averaging of wavenumbers less than Λ like the variable averaging version of RNG. Also $|k-j| = 0$ when $k = j = \Lambda_c$. causing a

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possible singularity in 10.3-2, similarly to the variable averaging RNG. However, it may be shown ([13]) that $k^{-11/3}$ - $j^{-11/3}$ goes to zero fast enough for the integral to converge as $|k-j|$ —→ 0. Unfortunately, the same is not true for the present RNG formulation. Indeed, Kraichnan identified $k^{-11/3}$ as an output term and $j^{-11/3}$ as an input term in his theory [23]. His physical explanation for the occurence of the cusp was that the high wavenumbers k and j are subject to coherent straining excitation by the random shear associated with $|k-j|$. Kraichnan labelled this a diffusion process in wavenumber. The following is a quotation from [23]. Note that the notation in the below quotation has been altered to match that used herein.

'...in which there is a two-way exchange, by stretching and unstretching, across the
boundary Λ_c. The input term ... then describes the "unstretching" whereby excitation
at wavenumbers slightly greater than Λ_c is transformed to excitation at k, and the
output term describes the opposite process. The two terms nearly cancel for low $|k-j|$
triads and the slight excess of output over input gives the net contribution ...
which is responsible for the rise of $v(k,\Lambda_c)$ to a finite cusp at $k = \Lambda_c$.'

Thus, the eddy viscosity cusp appears to be caused by correlations of small wavenumbers. This is also the case for the RNG models derived in this study. However, it is difficult to reconcile the above theory with the theory of vortex stretching by similar size eddies. A key component of the latter is that that the most effective stretching of a smaller vorter (higher wavenumber) by a larger vortex (lower wavenumber) occurs when the strain rate ratio of the larger to the smaller is about one half, [21]. According to Kraichnan, this ratio approaches zero.

The spatial behavior of motions associated with a very low wavenumber (as in $|k-j|$ —→ 0) features a very large kinetic energy and a negligible spatial variation. Negligible spatial variation should mean negligible stretching. The effect is similar to the background advecting velocity discussed in section 8.5. The present author finds it difficult to understand why the local transfer of energy is apparently driven by very non-local wavenumbers. Further study is required to resolve this issue.

The DNS data available so far concerns turbulent flows at relatively low Reynolds numbers and so cannot be used to provide a definite verification of the required eddy

viscosity behavior. The presence of the cusp has been demonstrated by Métais and Lesieur [14] in a spectral LES of decaying turbulent flow. Given the real resolution cutoff wavenumber, Λ_c, these researchers considered an artificial cutoff at $\Lambda_c/2$. It was demonstrated that an eddy viscosity defined with respect to $\Lambda_c/2$ must have a cusp in order to simulate the increased energy transfer from wavenumbers just below $\Lambda_c/2$ to the region between $\Lambda_c/2$ and Λ_c. However, the peak cusp value was lower than predicted by the Eddy Damped Quasi Normal Markovian method (E.D.Q.N.M.) or the model due to Kraichnan. Also, a plateau value was found that was close to the 0.267 theoretical value from (E.D.Q.N.M.).

The above spectral eddy viscosity models are intended for high Reynolds number flows, with an inertial range in the energy spectrum. Examples of the experimental energy spectra of such flows are given by McComb [13]. Further, the LES cutoff wavenumber should be inside the inertial range. The correct shape of the eddy viscosity near the cutoff wavenumber is important to correctly simulate the statistics of the resolved scales near the cutoff. However, an eddy viscosity which satisfies the energy dissipation requirement (equation 10.1-2) will correctly represent the statistics of the large eddies, far from the cutoff. Therefore, the problem is usually not critical from an engineering standpoint.

11. Conclusions and recommendations for future work

This chapter contains a summary and evaluation of the achievement of objectives set out in chapters 1 and 7. Aside from providing a review of the pertinent body of literature, these objectives were (i) to examine if the RNG analytical tools are suitable to represent local interactions, and (ii) to attempt to obtain an eddy viscosity function $v(k,\Lambda)$, hopefully with a cusp behavior as reported in literature dealing with other theories. The results of the current work produced some insights into the first problem but no conclusive answers. The second aim also met with qualified success. Several proposed forms of eddy viscosity have been presented in chapter 10 but no agreement was reached with other published data. Contributions of the current study are summarized in section 11.1 while some unresolved issues are outlined in section 11.2. Plans and recommendations for future work are discussed in section 11.3.

11.1 What has been achieved?

The single most important contribution of this study is the alteration of the partial averaging of the product $u^<u^>{}^0$. To the present author's knowledge, all the published literature uses the property:

$$< u^<(k-j)u^>{}^0(j) >> = 0 \qquad\qquad 11.1-1$$

so that the contributions to the eddy viscosity arise only from the term:

$$< u^>{}^!(k-j)u^>{}^0(j) >> = \lambda k^2 \Delta v u^<(k) \qquad\qquad 11.1-2$$

In place of 11.1-1, the current work proposes the equation:

$$< u^<(k-j)u^>{}^0(j) >> = 0 + O(\lambda) \qquad\qquad 11.1-3$$

where the $O(\lambda)$ term is an additional contribution to Δv, analogous to 11.1-2.

The consequence of equation 11.1-3 is to cancel a negative term proportional to k which occurs in the function derived solely from 11.1-2, as indicated by Carati. This

development indicates that the RNG method may be able to capture local and semi-local interactions in wavenumber space.

For the purpose of partial averaging, the convolution product $u^<(k-j)u^>(j)$. contains a vertex wavenumber $k-j$ and an averaged wavenumber j. The vertex wavenumber is named after the vertex term $M(k-j)$ which appears when the momentum equation is substituted for $u^<(k-j)$. The random variable associated with this vertex is the eddy viscosity $v(|k-j|, \Lambda_c)$. Unfortunately, in this form, the differential-integral equation for eddy viscosity becomes non-linear and difficult to solve. In the current work, this term has been replaced with $v(|k-j|, |k-j|)$. Further study is required as to the justification of this substitution. The effect is to restore linearity and symmetry to the eddy viscosity equation. This symmetry causes the first odd power of k to vanish so that one is able to extract the k^2 diffusion factor.

In the 'fixed wavenumber' averaging method, the stirring forces at Λ are averaged at each iteration. In the B region one allows the vertex Green's function wavenumber $|k-j|$ to be less than Λ while averaging over variables at j. This poses the following dilemma; since Λ is greater than k, the forces at Λ in region B will have an average effect on $u(k)$, however the intervening Green's function (defined as an average) has fluctuations on a time scale smaller than k.

The second important modification of the partial averaging operation is the proposal to average the effects of all interactions where at least one wavenumber is greater than Λc. This implies that for the product $u^<(k-j)u^>(j)$, the vertex wavenumber and the averaged wavenumber are now interchangeable and both combinations contribute to the eddy viscosity. The consequence of this modification is allowing the averaged wavenumber magnitude Λ' (equal to $|k-j|$ or j) to be smaller than Λc. Several options are available to specify the extended range of averaging. If $k < \Lambda' \leq \Lambda c$, the physical justification is that the averaged motions are still smaller (albeit resolved) scales than the wavenumber k under consideration. The resulting eddy viscosity is positive but reaches a maximum for $k < \Lambda c$ and so it is not in qualitative agreement with the expected shape. If $0 \leq \Lambda' \leq \Lambda c$, the eddy viscosity increases without bound as $\Lambda' \longrightarrow 0$. The lower averaging limit as a function of k or

Λc may be specified so the resulting eddy viscosity satisfies an additional relation or a physical argument. The inertial range energy balance has been used for this purpose.

The appeal of the 'low variable averaging' method is that it does not introduce triple products of the resolved velocity into the momentum equation. In that sense it may be considered a more genuine Renormalization Group method, since the form of the Navier-Stokes equations remains unaltered at each step. The difficulties of numerical simulation of an equation containing triple products have been acknowledged in the literature [9] and in the current study (end of section 9.2).

A further contribution of the current work is a generalized formulation of the partial averaging process and its relation to the λ series. The formulation has some elements in common with that due to McComb and Watt [33], [13]. However, these authors restrict their version of partial averaging to the narrow wavenumber bandwidth only. An attempt has been made to relate this operation to the physically realizable procedure of temporal filtering. The consequence of this work is to improve the theoretical framework of partial averaging and RNG for future research.

The factors responsible for negative components of the 'eddy viscosity' have been identified (see figures 10.1 and 10.3) at least for methods that use a λ^2 closure. These negative 'backscatter terms' cause an inverse energy cascade observed by some researchers of LES with eddy viscosity [13]. However, these terms are nearly or completely cancelled by the equal and opposite terms generated by the modified partial averaging procedure introduced in this work. A widely accepted physical interpretation is that energy flows both ways across the subgrid cutoff [19] but the net balance is the energy cascade equal to ε in the inertial range. Indeed. direct numerical simulation of low Reynolds number turbulence, has indicated that the inverse cascade of energy may be comparable to or larger than the net transfer rate [46]. Therefore, it is apparently an open question whether a small energy flux may reverse direction at very non-local (low k/Λ) interactions.

The temporal frequency expansion parameter, $\frac{\omega}{\Omega}$, has been shown to depend on the wavenumber expansion ratio $\frac{k}{\Lambda}$, although these two variables are generally treated independently in the literature. The issue is one of consistent accuracy for the RNG expansion which generally retains $(\frac{\omega}{\Omega})^0$ and $(\frac{k}{\Lambda})^1$, [7]. Taking $\frac{k}{\Lambda} < 1$, the present study has shown that $\frac{\omega}{\Omega}$ will generally vary between $(\frac{k}{\Lambda})^{5/3} < \frac{\omega}{\Omega} < (\frac{k}{\Lambda})^{2/3}$ when both k and Λ are in the inertial range (see section 8.5 for further details). In fact. $\frac{\omega}{\Omega}$ may be considered to approach zero independently of $(\frac{k}{\Lambda})$ only if rather unlikely conditions are met. First, the kinetic energy contained in the inertial range of the flow must be much larger than the energy contained in the production range or the background advecting velocity that usually accompanies real turbulence. Otherwise $\frac{\omega}{\Omega} \propto \frac{k}{\Lambda}$ using Taylor's Frozen Turbulence hypothesis [12]. Second. the values of $\frac{k}{\Lambda}$ considered must be sufficiently low. Only under these circumstances is the RNG perturbation expansion as carried out by YO [3] and Smith and Woodruff [7] consistent. This conclusion suggests that the RNG is suitable only for $\frac{k}{\Lambda} \ll 1$ and only for a special class of turbulent flows without background advection. This point contradicts the qualified success in deriving eddy viscosity for local interactions.

It has been pointed out that the physical explanation of the cusp in the eddy viscosity plot as offered by Kraichnan [23] contradicts the generally accepted mechanism of the local energy transfer in wavenumber space. Kraichnan attributes the cusp to the balance between stretching and unstretching of the high wavenumbers near the cutoff by very low wavenumbers. However, the physical characteristics associated with low wavenumbers are a high kinetic energy and very low spatial gradients. Small vortexes associated with high wavenumbers are likely to be advected, not stretched by the former. Indeed, the usual explanation of the local energy transfer is by a mechanism of stretching the smaller vortex by a slightly larger vortex. The conflict between these two theories has not been resolved in the current study.

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The above is a summary of the contributions of the current work to the RNG treatment of Navier-Stokes equations driven by a stirring force selected to reproduce the statistics of inertial range turbulence. Some important problems in the RNG method which had been attempted without success during this investigation will be discussed next.

11.2 What has not been achieved?

Considerable effort went into investigating the convergence properties of the λ expansion series about a stirring force. Many of the terms of $O(\lambda^n)$ may be eliminated due to repeated convolution integrals over thin shells, but some $O(\lambda^n)$ terms will contain only convolutions over the $k^<$ domain and cannot be assumed to be small. This problem is aggravated for finite wavenumbers by using the modified partial averaging equation 11.1-3 instead of 11.1-1.

As discussed in section 8.4, an alternative to the λ expansion series is to simply substitute the momentum equation for $u^>$ wherever this variable appears in the momentum equation for $u^<$. The problem resulting from this course of action is that the order of the substitution and partial averaging is now not interchangeable. Partial averaging must be done first if the results are to match those obtained from the λ series expansion about $u^{>0}$.

Both the series expansion and the substitution method increase the order of the non-linearity in the momentum equation and so spurious solutions are expected. For the YO case, where the triple velocity products have a negligible domain of integration, the partial averaging eliminates higher order non-linearities at each step. For the finite k values considered in the current study, the same is true for the variable wavenumber averaging which does not generate triple products. For the constant wavenumber averaging, the triple products remain and so potentially does a spurious solution of the flow field. Further analysis of this problem is beyond the scope of this work.

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The subject of background advection velocity raised in section 11.1 brings to mind the lack of Galilean invariance of the triple velocity products as discussed in section 6.2 [30]. The significance of this unphysical behavior has not been explored.

Recovery of the YO limit or the EDQNM plateau value of $\tilde{v}(\ l \to 0)$ from the eddy viscosity formulation has not succeeded. In particular, the YO method attempts to capture a type of turbulent interaction that may be described as very non-local so that $\frac{k}{\Delta\Lambda} \sim 0$. A paradox of this version of RNG is that it successfully models distant interactions on the assumption that the interactions are for the most part local in wavenumber space so that the partial averaging is a good approximation even for nearby scales. Indeed, for $\frac{k}{\Delta\Lambda} \sim 0$, the interactions between **k-j** and **j** are confined to the shell bandwidth $\Delta\Lambda$. In contrast, this study deals with finite $\frac{k}{\Lambda}$ values. where the range of involved interactions is spread out over the range Λ-k to Λ+k. It is now attempted to model local interactions near $\frac{k}{\Lambda} \longrightarrow 1$ by averaging over interaction triads which span from 0 to 2Λ. It is the YO paradox in reverse.

Probably the most important shortcoming of this study is the lack of an answer to the fundamental question: is the Renormalization Group applicable to the Navier-Stokes equations? At each stage, the removed shell is replaced by statistically averaged corrections to the coefficients. If the solution for the remaining range of variables is the same as before the shell removal, then the equation is exactly renormalizable. However, it is certain that the partial averaging produces errors at the interface between the resolved and unresolved wavenumbers at each averaged shell. Do the errors produced at each stage accumulate or cancel out as the procedure progresses? What are the effects of these errors on resolved scales?

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11.3 Recommendations for future work

Further work on the RNG treatment of Navier-Stokes turbulence could consist of both theoretical and computational parts. Following the experiences of the present study, the recommendations for future work in the theoretical branch of this research must be made with caution since they may prove impossible to carry out. Unfortunately, due to several difficult-to-justify approximations which must be made in the initial stages of the analysis, further series of self-consistent. analytical steps may yield meaningless results.

It is the recommendation of the present author that further theoretical investigation of the problems outlined in this study be scaled down to simpler models such as the Burger's equation [42] or the problem of scalar transport in a known velocity field. For example, one simplified version of RNG is discussed by Smith and Woodruff [7]. It is also of interest to investigate the relationship between the RNG and multiple scale analysis as discussed by Frisch [16] and the EDQNM theory [13].

The validity of the partial averaging approximation is the size of the fluctuating residual and, more importantly, the accumulation or cancelation of the residuals from subsequent shells. Such a study, tantamount to an analysis of induced forces and related to the work of Carati [40], would improve our understanding of the limitations of the partial averaging procedure.

A study of the behavior of $u(k,\omega)$ in isotropic, high Reynolds number turbulence should be made, relying on experimental data. In particular, it is important to verify the hypothesis of section 9.1 that $u(j,\omega_j) > u(k,\omega_j)$ as it may justify the λ series expansion.

On the computational side, the LES simulation of flow fields using the eddy viscosity functions developed in this study could be attempted without including the triple velocity products.

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As it stands, the RNG treatment of Navier-Stokes turbulence appears insufficient to make further progress. On the one hand, very limited representation of the turbulence is captured by the second statistical moment of the stirring force. For example, there is no information on intermittency or coherent structures. On the other hand, even for this simple model, the mathematical analysis is quite involved. However, the RNG contains many useful analytical tools such as spectral splitting, partial averaging, the gradual increments of the eddy viscosity, the induced forces and so on. There is potential for parts of the RNG method to be combined with other analytical tools to make further progress in the analysis of Navier-Stokes turbulence.

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