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NON-HARMONIC FOURIER SERIES AND APPLICATIONS

by

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ABSTRACT

The main purpose of this thesis is to study the solution systems of delay-differential equations in L^2. An interesting observation is that the systems are closely related to some extreme cases in the theory of non-harmonic Fourier series. We will exploit some fundamental properties of these solutions by developing some theorems on completeness, series expansion, frames and bases.

Specifically, by extending a stability theorem of Sedletskii, we can show that the solution systems are complete in L^2 with excess $E_2 = 1$, but that they can not be Schauder bases. Furthermore we generalize a result of Fujii to the case of two complex sequences located in a curvilinear strip. Also the properties of a *Lambert* W function are employed to discuss the series expansion which is related to $\frac{1}{4}$-theorems. Finally some numerical methods are provided to compute the finite optimal solution of such equations.

The results estabilished in this thesis may have applications to signal and image processing as well as general delay-differential equations.

Key words: completeness, series expansion, frame, basis, approximate solution, Paley-Wiener space, delay-differential equations

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Chapter 1

Non-harmonic Fourier series and the Lambert W functions

1.1 Introduction

It is well known (see, for example, Bellman and Cooke) that the solutions of differential-difference equations can be expressed as a sum of the exponentials $\sum c_n e^{i\lambda_n t}$ where each $i\lambda_n$ is a characteristic root with $\lambda_n \in C$. Here for simplicity, we assume that all the roots are simple. (i.e. that the λ_n are all distinct). The coefficients of this series could not be found systematically until the residue method was developed by Bellman and Cooke [1963]. Verblunsky [1961] introduced the technique of Cauchy exponential series to address a similar problem. With these methods, a solution can be approximated pointwise for a nice initial condition $g(t)$ (say, belonging to C^0, or C^1). This thesis was motivated by the problem of finding an optimal finite numerical solution to a delay-differential equation with a given initial condition in L^2. A reasonable beginning to this study is to look at the structure of the exponential systems $\{e^{i\lambda_n t}\}$ where each $i\lambda_n$ is a characteristic root.

It is worth mentioning that in general, the characteristic function of a delay-differential equation is not of sine-type, but that the imaginary parts of its zeros differ from integers by about $\frac{1}{4}$, and the real parts increase only at a logarithmic rate. We will see that this situation is analogous to several extreme cases in the theory of non-harmonic Fourier series.

Apparently, series of the type $\sum c_n e^{i\lambda_n t}$ were first studied by Paley and Wiener who called them non-harmonic Fourier series to emphasize that they are not trigonometric series. The theory of non-harmonic Fourier series thus contains the study of the completeness and series expansion properties of sets of complex exponentials $\{e^{i\lambda_n t}\}$. Because the study of frames and bases has flourished in recent years, non-harmonic Fourier series have received more attention than ever before. One of the famous early results in this theory (Paley-Wiener [1934] and Levinson [1940]) is that the trigonometric system $\{e^{int}\}_{-\infty}^{\infty}$ is stable in $L^2(-\pi,\pi)$ in the sense that the system $\{e^{i\lambda_n t}\}_{-\infty}^{\infty}$ will form a Riesz basis for $L^2(-\pi,\pi)$ only if $|\lambda_n - n| \leq L$ is small. Furthermore, Kadec [1964], and Redheffer and Young [1983] showed that the optimal perturbation is $L < \frac{1}{4}$. Such $\frac{1}{4}$ theorems have many applications in the theory of completeness, frames, bases and interpolation. On the other hand, many systems in the studies require that the λ_n's be located in a strip parallel to the real axis [Young, 1980] or near by the zeros of a function of sine-type [Adovion, 1988].

This thesis will develop some theorems on completeness, series expansion and stability of frames by exploiting various fundamental properties of these solution systems. Also a new method for the finite optimal solution will be given. Finally, we show that the solution system does not constitute a basis in L^2.

1.2 Definitions and notations

In this paper, we denote by C (respectively by R) the set of all complex (respectively real) numbers. Z denotes the set of all integers, and $\sum'$ and $\prod'$ mean summation and multiplication, respectively, through all the integers except 0.

We say that an entire function $f(z)$ is of **exponential type** γ if there is a constant $A > 0$ such that

$$|f(z)| \leq Ae^{\gamma|z|},$$

and an entire function f is said to be of **sine type** if it is of exponential type π, its zeros $\{\lambda_n\}$ are separated, i.e. $\inf_{n\neq m} |\lambda_n - \lambda_m| > 0$, and there exist positive constants A, B and H such that

$$Ae^{\pi|y|} \leq |f(x+iy)| \leq Be^{\pi|y|}$$

whenever $|y| \geq H$.

The totality of all entire functions of exponential type at most π that are square integrable on the real axis is known as the **Paley-Wiener space** [Young, 1980] which is a Hilbert space with respect to the inner product $(f,g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$.

In the next chapters, f will always denote an entire function of exponential type $\gamma \leq \pi$. The Phragmén-Lindelöf indicator function $h(\theta)$ of f is defined by

$$h(\theta) = \limsup_{\substack{r\to\infty \\ r\in R}} r^{-1} \log |f(re^{i\theta})|.$$

Definition 1.1 [Duffin]. *Let* $\{\lambda_n\}$, $n \in Z$, *be a sequence of distinct complex numbers. Then the set of functions* $\{\exp(i\lambda_n t)\}$ *is an exponential frame over an interval* $(-\gamma, \gamma)$ *if there exist positive constants* A *and* B, *which depend exclusively on* γ *and the set of functions* $\{\exp(i\lambda_n t)\}$, *such that*

$$A \leq \frac{\frac{1}{2\pi}\sum_n |\int_{-\gamma}^{\gamma} g(t)e^{i\lambda_n t}dt|^2}{\int_{-\gamma}^{\gamma} |g(t)|^2 dt} \leq B$$

for every function $g(t) \in L^2(-\gamma, \gamma)$, *where the index* n *runs through all positive and negative integers and zero. In this case,* $\{\lambda_n\}$ *is called a* **frame sequence**.

For a general frame in a Hilbert space, we have the following definition:

Definition 1.2 [E. Hernandez]. *A collection of distinct elements* $\{g_1, g_2, ...\}$ *in a Hilbert space* H *is said to be a* **frame** *if there exist positive constants* A *and* B *with* $0 < A \leq B < \infty$ *such that*

$$A\|g\|^2 \leq \sum_{n=1}^{\infty} |(g, g_n)|^2 \leq B\|g\|^2$$

for every $g \in H$. *The numbers* A *and* B *are called the* **bounds** *for the frame. When* $A = B$, *we say that the frame is* **tight**.

Definition 1.3. *Two sequences* $\{x_n\}$ *and* $\{y_n\}$ *in a Hilbert space* H *are said to be biorthogonal if*

$$(x_m, x_n) = \delta_{mn}$$

for every m *and* n.

Definition 1.4. *A system* $\{e^{i\lambda_n t}\}$ *of complex exponentials is* **complete** *in* $L^p(-\gamma, \gamma)$,

$1 \leq p < \infty$, if the relations

$$\int_{-\gamma}^{\gamma} f(t)e^{i\lambda_n t}dt = 0$$

for all n with $f \in L^p$, imply that $f = 0$ a.e. In this case $\{\lambda_n\}$ is called a **complete sequence.**

Definition 1.5. *A frame (or complete system) $\{e^{i\lambda_n t}\}$ in L^p that ceases to be a frame (or complete) when any one of its elements is removed is said to be an* **exact frame** *(or complete system). When exactly m elements have to be removed (or added) in order that the new system be exact, then the* **excess** *$E_p(\lambda)$ of the system is m (or $-m$).*

Remark 1.1. As we shall see (Theorem 1.4), the particular functions $e^{i\lambda_n t}$ added or removed are arbitrary; only their number is important.

Definition 1.6. *A sequence $\{\lambda_n\}$ of real or complex numbers has a* **uniform density** *d, $d > 0$, if there are constants L and δ such that*

$$|\lambda_n - \frac{n}{d}| \leq L$$

for any integer n, and

$$|\lambda_n - \lambda_m| \geq \delta > 0$$

if $n \neq m$.

Definition 1.7. *A complex sequence $\{\lambda_n\}$ is of density d (> 0) with* **deviation** *$\phi(n)$ if*

$$|\lambda_n - \frac{n}{d}| \leq \phi(n)$$

and

$$|\lambda_n - \lambda_m| \geq \delta > 0 \quad (n \neq m)$$

for any integer n, *where* $\phi(n) = o(n) \to \infty$, *and* δ *is a constant.*

Definition 1.8. *A sequence* $\{f_1, f_2, ...\}$ *in an infinite-dimensional Banach space* X *is said to be a* **Schauder basis** *for* X *if for each* $f \in X$, *there is a unique sequence of scalars* $\{c_1, c_2, ...\}$ *such that*

$$f = \sum_{n=1}^{\infty} c_n f_n,$$

i.e

$$\| f - \sum_{i=1}^{n} c_i f_i \| \to 0 \quad as \quad n \to \infty.$$

Henceforth, the term basis will always mean a Schauder basis.

Definition 1.9. *A basis* $\{e_k\}$ *in Hilbert space is called a* **Riesz basis** *if there are constants* A *and* B *where* $0 < A \leq B < \infty$ *such that for each* $x = \sum_{k=1}^{\infty} a_k e_k$

$$A(\sum_{k=1}^{\infty} |a_k|^2)^{\frac{1}{2}} \leq \|x\| \leq B(\sum_{k=1}^{\infty} |a_k|^2)^{\frac{1}{2}}.$$

Note: a Riesz basis is a basis which is also a frame [see Young, 1980].

1.3 Some important known theorems

The following result of Paley and Wiener plays a key role to connect sampling theory with frames:

Theorem 1.1. *If $f(z)$ is an entire function of exponential type γ and f, considered as a function on the real line, belongs to $L^2(-\infty,\infty)$, then there is a function $g(t) \in L^2(-\gamma,\gamma)$ such that*

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\gamma}^{\gamma} g(t) e^{izt} dt. \tag{1.1}$$

Furthermore,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\gamma}^{\gamma} |g(t)|^2 dt.$$

Conversely, a function f in the form (1.1) is an entire function of exponential type γ satisfying $f \in L^2(-\infty,\infty)$.

This theorem shows that any function in the Paley-Wiener space can be expressed in the form (1.1). In the next theorem, due to Levinson, the form (1.1) is associated to an incomplete system:

Theorem 1.2[Levinson, 1940]. *For the system $\{e^{i\lambda_n t}\}$ to be incomplete in $C(-\gamma,\gamma)$ (or in $L^p(-\gamma,\gamma)$, $1 < p < \infty$), it is necessary and sufficient that there exists a non-trivial entire function $f(z)$, which vanishes at every λ_n and is expressible in the form:*

$$f(z) = \int_{-\gamma}^{\gamma} e^{izt} d\omega(t),$$

where $\omega(t)$ is of bounded variation on $(-\gamma,\gamma)$ (or $\omega(t) \in L^q(-\gamma,\gamma)$, $1/p + 1/q = 1$).

To prove the incompleteness of the system $\{e^{i\lambda_n t}\}$, we often construct a function as an infinite product involving the $\{\lambda_n\}$ and then show that it is in the Paley-Wiener space. To reach this goal, we will employ the following lemma due to Sedletskii [1985].

Lemma 1.1. *Suppose $G(z)$ is in the Paley-Wiener space and its zero set is $\{z_n\}_{n=0}^{\infty}$. Then $\lim_{r\to\infty} \sum_{0<|z_n|<r} \frac{1}{z_n}$ exists.*

Furthermore, suppose the Hadamard's factorization of $G(z)$ is

$$G(z) = Be^{az}(z - z_0) \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}).$$

Assume the complex sequence $\{w_n\}$ satisfies $w_0 = z_0$, $\lim_{n\to\infty} \sum_{|w_n|<n} \frac{1}{w_n}$ exists and $\lim_{n\to\infty} \frac{z_n}{w_n} = 1$. Set

$$P(z) = Be^{az}(z - w_0) \prod_{n=1}^{\infty} (1 - \frac{z}{w_n}).$$

If $|P(z)| \leq k|G(z)|$ on some horizontal line $\Im z = h$ with constants k and h, then there is a real constant α such that the function $e^{i\alpha z}P(z)$ is in the Paley-Wiener space. Consequently, $\{e^{iw_nt}\}_{n=0}^{\infty}$ is incomplete in $L^2(-\pi, \pi)$.

The following result presents some intrinsic properties of the functions in the Paley-Wiener space:

Theorem 1.3 [Levinson, 1935]. *Suppose $f(z)$ is an entire function of exponential type a and $\{\lambda_n\}$ is its zero set. Let $n^+(r)$ be the counting function for its zeros in the right half disc $|z| < r$, and $n^-(r)$ for the left half one. If*

$$\lim_{\substack{|x|\to\infty \\ x\in R}} \frac{\log|f(x)|}{|x|} \leq 0$$

and

$$\int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1+x^2} dx < \infty,$$

then the following hold:

1) $\sum |\Im \frac{1}{\lambda_n}| < \infty$;

2) $\lim_{r\to\infty} \frac{n^+(r)}{r} = \lim_{r\to\infty} \frac{n^-(r)}{r} = A$ *for some constant A.*

Finally, we introduce two theorems on exponential sequences.

Theorem 1.4 [Levinson, 1940]. *The completeness of the system* $\{e^{i\lambda_n t}\}$ *in* $L^p(-\gamma,\gamma)$, $1 \leq p < \infty$ *or in* $C(-\gamma,\gamma)$ *is unaffected if a single* λ_n *is replaced by another number which is not in the set* $\{\lambda_n\}$.

Theorem 1.5 [Duffin, 1952]. *The removal of a vector from a frame leaves either a frame or an incomplete set.*

In the next section, a special function will be introduced, which applies to the extreme cases of various theorems.

1.4 A modification of the Lambert W functions

The entire function $g(z) = z - ae^{-z}$ is the characteristic function of the delay-differential equation

$$y'(t) = ay(t-1).$$

and the distribution of its zeros is typical to those of general differential-difference equations. So a thorough understanding of the locations of its zeros is not only necessary for the study of this kind of equation, but also useful for further generalization.

Note that when $a \neq -\frac{1}{e}$, all the zeros of $g(z)$ are simple; when $a = -\frac{1}{e}$, there is a double zero at $z = -1$. We first assume $a \neq -\frac{1}{e}$ and write the zero z of $g(z)$ as

$$z = \eta_1 + i\eta_2 = re^{i\phi} \qquad (-\pi < \phi \leq \pi).$$

Then from $z + \log z = \log a$, we get that

$$\log r + i\phi + \eta_1 + i\eta_2 \equiv \log a \qquad (\text{mod } 2\pi i).$$

This yields that for $-\pi < \arg a \leq \pi$

$$\eta_1 = \log |a| - \log r \qquad \eta_2 = \arg a + 2\pi k - \phi. \tag{1.2}$$

Now suppose a is real. It is known (see [Wright, 1959] or [Verblunsky, 1961]) that

1) If a does not satisfy $-\frac{1}{e} \leq a \leq 0$, then there is a zero corresponding to each integer k for all $k \in Z$.

2) If a satisfies $-\frac{1}{e} < a < 0$, then there is a zero corresponding to each integer k for all k in $Z \setminus \{-1, 0\}$, and to $k = 0$ there correspond two zeros z_0 and z_{-1} satisfying $-1 < z_0 < 0$ and $z_{-1} < -1$. Note that from (1.2) the case $k = -1$ cannot happen when $a < 0$ and $z < 0$. So we still have the one-one correspondence between the zeros and all the integers.

3) if $a = -\frac{1}{e}$, there is a zero of order two at -1, and thus after naming the double zeros $z_0 = z_{-1} = -1$, the zeros are again in exact correspondence with all integers.

From the above classification, we have that when $a > 0$ or $a < -\frac{1}{e}$

$$\eta_1^{(k)} = \log |a| - \log r_k \qquad \eta_2^{(k)} = \arg a + 2\pi k - \phi_k. \tag{1.3}$$

for all $k \in Z$; when $-\frac{1}{e} \leq a < 0$, we have that z_0 and z_{-1} are real, and (1.3) holds for all $k \in Z \setminus \{-1, 0\}$. In summary, to each integer k, there corresponds a unique zero of g which depends on the variable a. This defines a function $W_k(a)$ called the

Lambert W function (see [Corless, 1995]). The following asymptotic property of these functions was proved by Verblunsky [1961].

Lemma 1.2. *Suppose $W_k(a)$ is the Lambert W function for $k \in Z$, and $\alpha = \frac{|a|}{2\pi}$, then*

(1) when $a > 0$, for all $k \neq 0$, we have

$$\begin{aligned} W_k(a) &= \{\log\frac{\alpha}{|k|} + \frac{1}{4k}sign(k) + O\big(\frac{\log k}{k}\big)^2\} \\ &+ i\{2\pi[k - \frac{1}{4}sign(k)] + \frac{\log\frac{\alpha}{|k|}}{2\pi k} + O\big(\frac{\log|k|}{k^2}\big)\}, \end{aligned}$$

and $W_0(a) > 0$;

(2) when $-\frac{1}{e} < a < 0$, for all $k \in Z \setminus \{-1, 0\}$, we have

$$W_k(a) = \{\log\frac{\alpha}{|k|} - (\frac{1}{2k} - \frac{1}{4k}sign(k)) + O\big(\frac{\log k}{k}\big)^2\} \tag{1.4}$$

$$+ \; i\{2\pi[k + \frac{1}{2} - \frac{1}{4}sign(k)] + \frac{\log\frac{\alpha}{|k|}}{2\pi k} + O\big(\frac{\log|k|}{k^2}\big)\}, \tag{1.5}$$

and $-1 < W_0(a) < 0$, $W_{-1}(a) < -1$;

(3) when $a = -\frac{1}{e}$, we have (1.4) for all $k \in Z \setminus \{-1, 0\}$, and $W_0(a) = W_{-1}(a) = -1$;

(4) when $a < -\frac{1}{e}$, we have (1.4) for all $k \in Z \setminus \{0\}$, and $W_0(a)$ is not real.

Now if we suppose that a is real and that z_k satisfies $z_k e^{z_k} = a$, then $\bar{z_k}$ also satisfies $\bar{z_k} e^{\bar{z_k}} = a$. In fact we have

1) if $a > 0$, $W_0(a)$ is real, and $\overline{W_k(a)} = W_{-k}(a)$ for all $k \geq 1$;

2) if $-\frac{1}{e} \leq a < 0$, only $W_0(a)$ and $W_{-1}(a)$ are real, and the others satisfy $\overline{W_k(a)} = W_{-k-1}(a)$ for all $k \geq 1$.

3) if $a < -\frac{1}{e}$, none of the $W_k(a)$ are real and we have that $\overline{W_k(a)} = W_{-k-1}(a)$ for all $k \geq 0$.

To obtain more symmetry in the index, we modify the $W_k(a)$ in the following ways:

when $a > 0$, let

$$V_n(a) = -\frac{i}{2\pi} W_n(a) \quad \text{for} \quad n \in Z,$$

when $a < 0$, let

$$V_n(a) = \begin{cases} -\frac{i}{2\pi} W_n(a) & n > 0 \\ 0 & n = 0 \\ -\frac{i}{2\pi} W_{n-1}(a) & n < 0. \end{cases}$$

Furthermore set $V_n(a) = \rho_n(a) + i\sigma_n(a)$.

After the above modification, we find that $\{V_n(a)\}$ has one element less than $\{W_k(a)\}$ for $a < 0$. But it gives us a nice symmetrical index on both of $\{\rho_n(a)\}$ and $\{V_n(a)\}$. From the definition of $V_n(a)$ and Lemma 1.2, we immediately get that (see [Verblunsky, 1961])

Proposition 1.1. *Under the definition above, $V_n(a)$ and ρ_n satisfy that $\overline{V_n(a)} = -V_{-n}(a)$ and $\rho_n(a) = -\rho_{-n}(a)$ for all integers n and real a. Moreover it has the following asymptotic property:*

when $a > 0$,

$$V_n(a) = \begin{cases} \{n - \frac{1}{4} - \epsilon_n^{(1)}(a)\} + i\{\frac{1}{2\pi} \log|n| + O(1)\} & n > 0 \\ 0 & n = 0 \\ \{n + \frac{1}{4} + \epsilon_n^{(1)}(a)\} + i\{\frac{1}{2\pi} \log|n| + O(1)\} & n < 0; \end{cases}$$

when $a < 0$,

$$V_n(a) = \begin{cases} \{n + \frac{1}{4} - \epsilon_n^{(2)}(a)\} + i\{\frac{1}{2\pi}\log|n| + O(1)\} & n > 0 \\ 0 & n = 0 \\ \{n - \frac{1}{4} + \epsilon_n^{(2)}(a)\} + i\{\frac{1}{2\pi}\log|n| + O(1)\} & n < 0; \end{cases}$$

where $\epsilon_n^{(j)}(a) = O(\frac{\log|n|}{|n|})$ *and* $\epsilon_n^{(j)}(a) > 0$, $j = 1, 2$ *for sufficiently large* n *depending on* a.

Remark 1.2. As we shall see, the study of the sequence $\{V_n(a)\}$ will allow us to address questions concerning completeness (Propostions 2.4 and 2.5), frame property (Proposition 4.1) or basis property (Theorem 5.5) of the original sequence $\{W_n(a)\}$.

In this paper, we only consider the case when a is real. For a complex, a similar modification can be made to $W_k(a)$, so our method can be used to evaluate the excess of the system $\{e^{W_k(a)t}\}$. But this will not be presented here.

Chapter 2

Completeness of complex exponentials

2.1 Introduction

From the definition of completeness, we know that only the null function is perpendicular to all elements of $\{e^{i\lambda_n t}\}$ if it is complete in $L^p(-\pi, \pi)$. On the other hand, we say that the above sequence is closed in $L^q(-\pi, \pi)$ if every $f \in L^q(-\pi, \pi)$ can be approximated in L^q norm by linear combinations of the functions $e^{i\lambda_n x}$. Duality shows that closure in L^q is equivalent to completeness in L^p if $\frac{1}{p} + \frac{1}{q} = 1$, and $1 < p < \infty$. Especially, when $p = q = 2$, completeness is equivalent to closure in $L^2(-\pi, \pi)$ (see [Levinson, 1940] or [Young, 1980]).

For completeness, Levinson [1940] showed that $\{e^{i\lambda_n t}\}$ is complete if $|\lambda_n - n| \leq \frac{1}{4}$, and incomplete if $\lambda_n = n + (\frac{1}{4} + \epsilon)\text{sign}(n)$, $\lambda_0 = 0$ for any $\epsilon > 0$. So it is not surprising that the two special cases $\{e^{i\lambda_n^+}\}$ and $\{e^{i\lambda_n^-}\}$ have received a lot of attention (see [Redheffer, 1983][Sedletskii, 1977, 1983, 1988][Young 1980, 1984, 1987]), where the

λ_n^+ and λ_n^- are defined by:

$$\lambda_n^+ = \begin{cases} n+\frac{1}{4} & n>0 \\ 0 & n=0 \\ n-\frac{1}{4} & n<0 \end{cases} \tag{2.1}$$

and

$$\lambda_n^- = \begin{cases} n-\frac{1}{4} & n>0 \\ 0 & n=0 \\ n+\frac{1}{4} & n<0. \end{cases} \tag{2.2}$$

To study the completeness of the solution system $\{e^{W_k(a)t}\}$, this chapter begins with two special sequences $\{\rho_n(a)\}$ defined in Section 1.4 which are analogous to the above $\{\lambda^+\}$ and $\{\lambda^-\}$, and then returns to the solution system itself.

2.2 Two special sequences

As mentioned in the last chapter, Theorem 1.2 gives a necessary and sufficient condition for the incompleteness of exponential systems, but for the convenience of applications, Levinson [1940] gave a more practical result which can be expressed in the following general form.

Given a sequence $\{\lambda_n\}$ of complex numbers, let $n_\lambda(t)$ denote the number of points λ_k in the disc $|z| \leq t$ and let

$$N(R) = \int_0^R \frac{n_\lambda(t)}{t} dt.$$

Theorem 2.1. *If the sequence $\{\lambda_n\}$ of complex numbers is such that*

$$\limsup_{R\to\infty}\{N(R)-\frac{2\gamma}{\pi}R+\frac{1}{p}\ln R\}>-\infty,$$

then the system $\{e^{i\lambda_n t}\}$ is complete in $L^p(-\gamma,\gamma)$ $(1<p<\infty)$. Especially if λ_n satisfy:

$$|\lambda_n|\le|n|+\frac{1}{2p}, n\in Z,$$

then the system $\{e^{i\lambda_n t}\}$ is complete in $L^p(-\pi,\pi)$. The constant $\frac{1}{2p}$ is the best possible.

Remark 2.1. Although the constant $\frac{1}{2p}$ is best possible, the next condition is obtained via the "shining light" density of Beurling and Malliavin (see [Koosis, p70]).

Theorem 2.2 [Sedleckii, 1977]. *Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two real sequences and $0<\gamma<\infty$. Then the excess $E_2(\lambda)$ is equal to the excess $E_2(\mu)$ in $L^2(-\gamma,\gamma)$ only if one of the following conditions holds:*

1) for some $0<s<\infty$, $\sum|\lambda_n-\mu_n|^s<\infty$,

2) $|\lambda_n-\mu_n|\le\alpha_n$, $\alpha_n\to 0$, as $|n|\to\infty$ and $\sum'\frac{\alpha_n}{|n|}<\infty$.

Now we will extend the result above to the case of complex sequences:

Theorem 2.3. *Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two sequences of distinct complex numbers satisfying*

$$|\lambda_n-\mu_n|\le\phi(|n|) \tag{2.3}$$

where $\phi(|n|)$ is nonincreasing and tends to 0, and $\sum'\frac{\phi(|n|)}{|n|}<\infty$. Then the excesses of the two exponential system $\{e^{i\lambda_n t}\}$ and $\{e^{i\mu_n t}\}$ satisfy that $E_2(\lambda)=E_2(\mu)$.

The proof of this theorem will be given in the next section.

For a particular regularly distributed sequence, a stronger result can be obtained:

Theorem 2.4 [Redheffer and Young, 1983]. *Let λ_1 be an arbitrary positive number and let $\lambda_n = n + \frac{1}{2q} + \frac{\beta}{\log n}$ for $n \geq 2$ where $\beta \geq 0$, $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the set $\{1, e^{\pm i\lambda_n x}\}$ is complete in $L^p(-\pi, \pi)$ if $\beta \leq \min(\frac{1}{4}, \frac{1}{2q})$ and not if $\beta > \max(\frac{1}{4}, \frac{1}{2q})$.*

As shown in [Redheffer and Young, 1983], the fact that such a regular distribution of $\{\lambda_n\}$ can lead to completeness without satisfying Levinson's condition (Theorem 2.1) is surprising.

The following results are direct consequences of Theorem 2.1:

Corollary 2.1. *If $\{\lambda_n\}$ is either $\{\lambda_n^+\}$ or $\{\lambda_n^-\}$ as defined by* (2.1) *and* (2.2), *then $\{e^{i\lambda_n t}\}$ is a complete system in $L^2(-\pi, \pi)$.*

Proposition 2.1. *Under the definition in Section 1.4, if $a > 0$, then $\{e^{i\rho_n(a)t}, n \in Z\}$ is complete in $L^p(-\pi, \pi)$ for $1 < p < \infty$, and if $a < 0$, then $\{e^{i\rho_n(a)t}, n \in Z\}$ is complete in $L^p(-\pi, \pi)$ for $1 < p \leq 2$.*

Proof: When $a > 0$, from Proposition 1.1, we see that $|\rho_n(a)| \leq |n|$ for large n, and so from Theorem 2.1, $\{e^{i\rho_n(a)t}\}$ is complete in $L^p(-\pi, \pi)$ for $1 < p < \infty$.

A similar proof also holds in the case of $a < 0$.

We note that for the case of $a > 0$ and the case of $a < 0$, the numbers of elements in the exponential system $\{e^{i\rho_n(a)t}\}$ are different. In fact, when $a < 0$, the element $\frac{1}{2\pi}\Im W(-1,a)$ is removed from the set $\{\frac{1}{2\pi}\Im W(k,a)\}$. That means that the exponential system of the imaginary part of Lambert W functions is complete with excess $E_2 \geq 1$ for $a < 0$. This suggests the same might be true for $a > 0$.

Actually, Redheffer[1983,p107] and Young[1980] have noted the difference of (2.1) and (2.2) on their excesses, and proved the exactness of (2.1) and the over-completeness of (2.2), respectively:

Theorem 2.5. *If $\{\lambda_n\}$ is defined by (2.1), then $\{e^{i\lambda_n t}, n \in Z\}$ is exact in $L^2(-\pi,\pi)$, and if $\{\lambda_n\}$ is defined by (2.2), then $\{e^{i\lambda_n t}, n = \pm 1, \pm 2, ...\}$ is exact and complete in $L^2(-\pi,\pi)$.*

Now suppose $\{\rho_n(a)\}$ is defined as in Section 1.4, $\{\lambda_n^+\}$ and $\{\lambda_n^-\}$ is defined as in (2.1) and (2.2), then from Proposition 1.1 we see that for $a < 0$,

$$|\rho_n(a) - \lambda_n^+| = O(\frac{\log|n|}{|n|}),$$

and for $a > 0$,

$$|\rho_n(a) - \lambda_n^-| = O(\frac{\log|n|}{|n|}).$$

Combining with Theorems 2.2 and 2.5 we immediately get that

Proposition 2.2. *If $a < 0$, then $\{e^{i\rho_n(a)t}, n \in Z\}$ is an exact complete sequence in $L^2(-\pi,\pi)$, and if $a > 0$, then $\{e^{i\rho_n(a)t}, n = \pm 1, \pm 2, ...\}$ is exact and complete in $L^2(-\pi,\pi)$.*

Next we consider some basis properties. Note that if $\{e_1, e_2, ...\}$ is a basis for a Banach space X, then every vector x in the space has a unique series expansion of the form

$$x = \sum_{n=1}^{\infty} c_n e_n.$$

It is clear that each coefficient c_n is a linear function of x, so we may write

$$x = \sum_{n=1}^{\infty} f_n(x) e_n.$$

The functionals f_n are called the coefficient functionals associated with the basis $\{e_n\}$.

Based on the above definition, it is proved in [Young, 1984, 1987] that

Theorem 2.6. *If* $\{\lambda_n^+\}$ *and* $\{\lambda_n^-\}$ *are defined by* (2.1) *and* (2.2), *respectively, then neither* $\{e^{i\lambda_n^+ t}, n \in Z\}$ *nor* $\{e^{i\lambda_n^- t}, n \in Z \setminus \{0\}\}$ *is a basis for* $L^2(-\pi, \pi)$.

There is a natural question as to whether $\{e^{i\rho_n(a)t}\}$ is a basis in $L^2(-\pi, \pi)$ when $a < 0$ or $a > 0$? We first give a stability result:

Theorem 2.7. *Suppose* $\{\lambda_n\}$ *and* $\{\mu_n\}$ *are two sequences of distinct real numbers. Suppose* $\{e^{i\lambda_n t}\}$ *is a basis and* $\{e^{i\mu_n t}\}$ *is complete in* $L^2(-\pi, \pi)$. *If* $\{\mu_n\}$ *satisfies*

$$\sum |\lambda_n - \mu_n| < \infty, \tag{2.4}$$

then $\{e^{i\mu_n t}\}$ *is a basis in* $L^2(-\pi, \pi)$.

In order to prove Theorem 2.7, we will need the following two lemmas (see [Young 1980, p23 and p40]).

Lemma 2.1. *If $\{x_n\}$ is a basis for a Banach space X and if $\{f_n\}$ is the associated sequence of coefficient functionals, then there exists a constant M such that*

$$1 \leq \|x_n\|\|f_n\| \leq M \quad (n = 1, 2, \ldots).$$

Lemma 2.2. *Let $\{x_n\}$ be a basis for a Banach space X and let $\{f_n\}$ be the associated sequence of coefficient functionals. If $\{y_n\}$ is complete in X and if*

$$\sum_{n=1}^{\infty} \|x_n - y_n\|\|f_n\| < \infty,$$

then $\{y_n\}$ is a basis for X.

Proof of Theorem 2.7:

Suppose $\{f_n\}$ is the associated sequence of coefficient functionals of $\{e^{i\lambda_n t}\}$. Since λ_n is real, $\|e^{i\lambda_n t}\| = 1$. By Lemma 2.1, $\|f_n\| \leq M$.

Take a real sequence $\{\mu_n\}$ satisfying (2.4), then we have

$$\begin{aligned} \|e^{i\mu_n t} - e^{i\lambda_n t}\| &= \Big(\int_{-\pi}^{\pi} (e^{i\mu_n t} - e^{i\lambda_n t})\overline{(e^{i\mu_n t} - e^{i\lambda_n t})} dt\Big)^{\frac{1}{2}} \\ &= \Big(\int_{-\pi}^{\pi} (1 - e^{i(\mu_n - \lambda_n)t})(1 - e^{-i(\mu_n - \lambda_n)t}) dt\Big)^{\frac{1}{2}} \\ &= 2\Big(\int_{-\pi}^{\pi} (\sin \frac{\mu_n - \lambda_n}{2} t)^2 dt\Big)^{\frac{1}{2}}. \end{aligned}$$

Since

$$|\sin \frac{\mu_n - \lambda_n}{2} t| < \frac{\pi}{2} |\mu_n - \lambda_n|,$$

we have that

$$\begin{aligned} \sum \|e^{i\mu_n t} - e^{i\lambda_n t}\|\|f_n\| &\leq M \sum \|e^{i\mu_n t} - e^{i\lambda_n t}\| \\ &\leq \pi\sqrt{2\pi} M \sum |\mu_n - \lambda_n| < \infty, \end{aligned}$$

by (2.4). Thus Lemma 2.2 guarantees that $\{e^{i\mu_n t}\}$ is a basis in $L^2(-\pi,\pi)$. This ends the proof.

Now suppose $\{\rho_n\}$ is defined as in Section 1.4 and $\{\lambda_n\}$ as in (2.1) or (2.2). Note that $|\rho_n - \lambda_n| \sim \frac{\log n}{n}$, so it does not meet the convergent condition (2.4). However we still have

Proposition 2.3. *Neither the exponential systems $\{e^{i\rho_n(a)t}, n \in Z\}$ when $a < 0$, nor $\{e^{i\rho_n(a)t}, n \in Z \setminus \{0\}\}$ when $a > 0$ is a basis for $L^2(-\pi,\pi)$.*

A sketch of the proof will be presented in Chapter 5.

2.3 Stability properties of completeness

As mentioned in the last section, Sedletskii has shown that given an exponential system, the excess will remain the same upon small variations of its sequence $\{\lambda_n\}$ (Theorem 2.2). If we know that the variations are only vertical displacements, it is possible to relax such conditions. Actually, Elsner [1969] and Young [1976] independently proved that

Theorem 2.8. *Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two sequences of complex numbers which lie in a fixed horizontal strip, and suppose that*

$$\Re(\lambda_n) = \Re(\mu_n).$$

If $\{e^{i\lambda_n t}\}$ is complete in $L^2(-\pi,\pi)$, then $\{e^{i\mu_n t}\}$ is also complete in $L^2(-\pi,\pi)$.

Almost at the same time, Peterson and Redheffer [1977] gave a stronger form:

Theorem 2.8'. *Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two sequences of complex numbers, and suppose that*

$$\Re(\lambda_n) = \Re(\mu_n) \quad \text{and} \quad |\Im\lambda_n - \Im\mu_n| \leq const$$

then the L^2 excesses of the two exponential systems $\{e^{i\lambda_n t}\}$ and $\{e^{i\mu_n t}\}$ are equal, i.e. $E_2(\lambda) = E_2(\mu)$.

Sedletskii[1978] pointed out that the above theorem fails to be true in $L^1(-\pi, \pi)$ or $C(-\pi, \pi)$. Furthermore, in [1985] he constructed in $L^2(-\pi, \pi)$ an example which shows that, in general, the boundedness conditions on the imaginary parts in the above two theorems can not be removed. But for some particular cases he proved the following two theorems:

Theorem 2.9. *Suppose $\{h_n\}$ is a real sequence, then $\{e^{i(n+ih_n)t}\}$ is complete in $C(-\pi, \pi)$ provided*

$$\sum{}' \frac{h_n^2}{n^2} < \infty.$$

Theorem 2.10. *Suppose $\{h_n\}$ is a real sequence, and E_2 is the excess of the exponential system $\{e^{i(n+ih_n)t}\}$ in $L^2(-\pi, \pi)$.*

1) for some $\alpha \in [0, \infty)$ let

$$|h_n| \leq \alpha \log |n| \qquad (|n| \geq n_0).$$

Then $E_2 \leq [\alpha\pi] + 1$. If, in addition, $\{\alpha\pi\} < \frac{1}{2}$, then $E_2 \leq [\alpha\pi]$. Here $[x]$ and $\{x\}$ denote respectively the integral and fractional parts of x.

2) if moreover

$$\sum{}' \frac{h_n^2}{n^2} < \infty \tag{2.5}$$

and

$$|h_n| \geq \alpha \log |n| \qquad (|n| \geq n_0, \alpha \in [0, \infty)),$$

then $E_2 \geq [\alpha\pi]$. *If, in addition,* $\{\alpha\pi\} \geq 1/2$, *then* $E_2 \geq [\alpha\pi] + 1$.

In particular, if condition (2.5) *is fulfilled and* $h_n/\log|n| \to \infty$ *as* $n \to \infty$, *then the system* $\{e^{i(n+ih_n)t}\}_{n=-\infty}^{\infty}$ *has infinite excess in* $L^2(-\pi, \pi)$.

Note that in Theorems 2.9 and 2.10, the trigonometric system may have vertical displacements that tend to infinity. For example, if $\lambda_n = n + \text{sign}(n)\frac{\log|n|}{4\pi}$, and $\lambda_0 = 0$, then $E_2 = [\frac{1}{4}] = 0$. That is, the system $\{e^{i\lambda_n t}\}$ is then complete $L^2(-\pi, \pi)$.

It is natural to ask whether this proposition can be translated to non-harmonic series for which n is replaced by $n + c$ with $c > 0$? Before answering this question, we first introduce a nice result required later for our proof.

Theorem 2.11[Sedletskii, 1985]. *Assume that the points* $\{\lambda_n\}$ *and* $\{\mu_n\}$ *lie in the curvilinear strip* $\{z : |y| \leq \phi(|x|)\}$, *where* $\phi(x)(x \geq 0)$ *is a positive non-decreasing function such that*

$$\sum_{n=1}^{\infty} \phi^2(n)/n^2 < \infty \qquad \textit{and} \qquad \Re\lambda_n = \Re\mu_n \qquad n \in Z.$$

If $|\Im\lambda_n| \leq |\Im\mu_n|$ *for all* $n \in Z$, *then* $E_2(\lambda) \leq E_2(\mu)$.

From the remark following Theorem 2.10, we see that $E_2(\lambda)$ increases with the order of growth of h_n. But Theorem 2.10 does not show the difference of the two excesses, so here we are interested in the stability of $E_2(\lambda)$ in such non-harmonic cases:

Theorem 2.12. *Suppose* $\{\lambda_n\}$ *is defined by* (2.1), *and* $h_n = \frac{1}{2\pi}\log|n|$ *for* $n \neq 0$, *and* 0 *for* $n = 0$. *Then* $\{e^{i(\lambda_n + ih_n)t}, n \in Z\}$ *is exact.*

The above result can be obtained from a misprinted claim of Sedletskii's (we will discuss it later). But instead of a direct evaluation on the Mittag-Leffler function as in his paper, we here give our proof based on some properties of the Lambert W function. First we give two lemmas:

Lemma 2.3. *Let* $U_n = -iW_n(-\frac{1}{e})$. *Then the system* $\{e^{i(U_n - i)t}, n \in Z \setminus \{0, -1\}\}$ *is incomplete in* $C[-\frac{1}{2}, \frac{1}{2}]$ *(or* $L^p(-\frac{1}{2}, \frac{1}{2})$, $1 < p < \infty$*).*

Proof: To prove this lemma, we only need to construct an entire function in the Paley-Wiener space which vanishes at every point $z_n = U_n - i = -iW_n(-\frac{1}{e}) - i$.

Set $f(z) = \int_{-A}^{A} e^{izt} d\omega(t)$ where $\omega(t) = (t + A)^2$. Then $f(z)$ is a nontrivial entire function of exponential type. Substitute t by $t - A$, then

$$\begin{aligned} f(z) &= \int_{-A}^{A} e^{izt} d(t+A)^2 \\ &= \int_{0}^{2A} e^{iz(t-A)} dt^2 \\ &= 2e^{-iAz} \int_{0}^{2A} te^{izt} dt \\ &= \frac{2e}{-z^2} e^{-iAz} \{i(2Az + i)e^{i(2Az+i)} + \frac{1}{e}\}. \end{aligned}$$

Notice that for $f(z) = 0$, $i(2Az + i)$ must be one of the $W_k(-\frac{1}{e})$. So if we take $A = \frac{1}{2}$, then the zeroes $\{z_n\}$ of $f(z)$ come only from $\{-iW_n(-\frac{1}{e}) - i\}_{-\infty}^{\infty}$.

On the other hand, from the discussion in Section 1.4, we see that $W_n(-\frac{1}{e}) = -1$ for $n = 0$ or $n = -1$. So $z_n = 0$ for $n = 0, -1$. By simple computation, we see $f(0) = 1 \neq 0$. Thus all the zeros of $f(z)$ are exactly $\{-iW_n(-\frac{1}{e}) - i, n \in Z \setminus \{0, -1\}\}$.

Thus by Theorem 1.2, $\{e^{i(U_n - i)t}, n \in Z \setminus \{0, -1\}\}$ is incomplete in $C(-\frac{1}{2}, \frac{1}{2})$ (or $L^p(-\frac{1}{2}, \frac{1}{2})$, $1 < p < \infty$).

Lemma 2.4. *(1) Let* $\mu_n = \sigma\lambda_n$ $\sigma > 0$. *If* $\{e^{i\lambda_n t}\}$ *is complete in* $L^p(-\gamma, \gamma), 1 < p < \infty$, *then* $\{e^{i\mu_n t}\}$ *is complete in* $L^p(-\gamma/\sigma, \gamma/\sigma)$, $1 < p < \infty$.

(2) Let $\mu_n = \lambda_n + \tau$. *If* $\{e^{i\lambda_n t}\}$ *is complete in* $L^p(-\gamma, \gamma), 1 < p < \infty$, *then* $\{e^{i\mu_n t}\}$ *is complete in* $L^p(-\gamma - \tau, \gamma - \tau)$, $1 < p < \infty$.

Proof of (1): If $\{e^{i\mu_n t}\}$ is not complete in $L^p(-\frac{\gamma}{\sigma}, \frac{\gamma}{\sigma})$, then there exists an entire function g of the form

$$g(z) = \int_{-\frac{\gamma}{\sigma}}^{\frac{\gamma}{\sigma}} e^{izt} d\omega(t),$$

such that $g(\mu_n) = 0$, where $\omega(t)$ is in $L^q(-\frac{\gamma}{\sigma}, \frac{\gamma}{\sigma})$ and $1/p + 1/q = 1$.

Replace z by σw in the above integral, then

$$\begin{aligned} g_1(w) &= g(\sigma w) = \int_{-\frac{\gamma}{\sigma}}^{\frac{\gamma}{\sigma}} e^{i\sigma zt} d\omega(t) \\ &= \int_{-\gamma}^{\gamma} e^{iws} d\omega_1(s) \end{aligned}$$

where $\omega_1(s) = \omega(s/\sigma)$ is in $L^q(-\gamma, \gamma)$, and $s = \sigma t$. So $g_1(w)$ is an entire function satisfying $g_1(\lambda_n) = g(\mu_n) = 0$. By Theorem 1.2, $\{e^{i\lambda_n t}\}$ is incomplete in $L^p(-\gamma, \gamma)$

which is a contradiction.

The proof of (2) is the same as that for (1) provided that z is replaced by $z + \pi$.

Proof of Theorem 2.12:

Since $\{e^{i\lambda_n t}, n \in Z\}$ is complete, by Theorem 2.11, $\{e^{i(\lambda_n + ih_n)t}, n \in Z\}$ is complete. Thus to prove the theorem, we only need to show that $\{e^{i(\lambda_n + ih_n)t}, n \in Z \setminus \{0\}\}$ is incomplete (see Remark 1.2 and Theorem 1.4).

Suppose $\{U_k\}$ is defined as in Lemma 2.3, then from Lemma 2.3 and Lemma 2.4, we see $\{e^{\frac{1}{2\pi}(U_k - i)t}, k \in Z \setminus \{0, -1\}\}$ is incomplete in $L^2(-\pi, \pi)$. It follows from Theorem 2.8' that $\{e^{\frac{1}{2\pi}U_k t}, k \in Z \setminus \{0, -1\}\} = \{e^{\frac{1}{2\pi}W_k(-\frac{1}{e})t}, k \in Z \setminus \{0, -1\}\}$ is incomplete in $L^2(-\pi, \pi)$.

Following the approach in Section 1.4, we may reindex the sequence of $\{U_k\}$ in the following way: keep each nonnegative index and add one to each negative index since $U_0 = U_{-1}$. Let $\{Q_n\}$ denote the new sequence, then the new exponential system $\{e^{\frac{1}{2\pi}Q_k t}, k \in Z/\{0\}\}$ is incomplete in $L^2(-\pi, \pi)$.

Furthermore if we set $V_n = Q_n/2\pi = \rho_n + i\sigma_n$, then $\{V_n\}$ and $\{\rho_n\}$ coincide with $\{V_n(-\frac{1}{e})\}$ and $\{\rho_n(-\frac{1}{e})\}$ respectively, as defined in Section 1.4. In addition, by Lemma 1.2

$$|\sigma_n - \frac{1}{2\pi}\log|n|| \leq \text{const}.$$

Thus again from Theorem 2.8', $\{e^{i(\rho_n+i\frac{1}{2\pi}\log|n|)t}, n \neq 0\}$ is incomplete in $L^2(-\pi,\pi)$.

Since $\rho_n = \rho_n(-\frac{1}{e})$, and λ_n is defined by (2.1), then by Proposition 1.1 we have

$$|\rho_n - \lambda_n| = O(\frac{\log n}{n}).$$

It follows that the two sequences $\{\rho_n + \frac{i}{2\pi}\log|n|\}$ and $\{\lambda_n + \frac{i}{2\pi}\log|n|\}$ satisfy the condition in Theorem 2.3. Thus the two exponential systems $\{e^{i(\rho_n+\frac{i}{2\pi}\log|n|)t}\}$ and $\{e^{i(\lambda_n+\frac{i}{2\pi}\log|n|)t}\}$ have the same excess, and thus $E_2(\rho) = E_2(\lambda)$. It follows that $\{e^{i(\lambda_n+\frac{i}{2\pi}\log|n|)t}, n \in Z \setminus \{0\}\}$ is incomplete. This ends the proof of Theorem 2.12.

From Theorem 2.11 and Theorem 2.12, the following result is obvious:

Corollary 2.2. *Suppose λ_n is defined by* (2.1), *$\mu_n = \lambda_n + i\sigma_n$ with $|\sigma_n| \leq \frac{1}{2\pi}\log|n|$ for $n \neq 0$, and $\mu_0 = 0$. Then $E_2(\lambda) = E_2(\mu)$.*

By a similar argument as that for Theorem 2.12, we have

Proposition 2.4. *Suppose $\{V_n(a)\}$ is defined as in Section 1.4 with $a < 0$, then $\{e^{iV_n(a)t}, n \in Z\}$ is exact in $L^2(-\pi,\pi)$, i.e. $E_2(V) = 0$.*

Proof of Theorem 2.3: To prove the theorem, we only need to show that $\{\mu_n\}$ is not a complete sequence if $\{\lambda_n\}$ is assumed to be incomplete in a interval $(-\gamma,\gamma)$ (see Remark 1.2 and Theorem 1.4). Without loss of generality, assume $\gamma = \pi$.

From Theorem 1.2, there is an entire function $F(z)$ in the Paley-Wiener space such that $\{\lambda_n\}$ is a subset of its zeros. Let $\{\tilde{\lambda}_n\} = \{\lambda_n\} \cup \{\gamma_n\}$ be the collection of all the zeros of F. We can assume all the elements $\tilde{\lambda}_n$ to be distinct. Otherwise, we may consider $F_1(z) = \frac{1}{z-\tilde{\lambda}_n}F(z)$ instead of $F(z)$ for a multiple zero $\tilde{\lambda}_n$. For each γ_n, we choose a γ_n' such that the γ_n' are distinct, $\gamma_n' \notin \{\mu_n\}$ and $|\gamma_n - \gamma_n'| < \frac{1}{n^2}$.

Set $\{\tilde{\mu_n}\} = \{\mu_n\} \cup \{\gamma_n'\}$. It follows easily from the hypotheses that $|\tilde{\lambda_n} - \tilde{\mu_n}| < \tilde{\phi}(|n|)$ where $\tilde{\phi}$ is non-increasing and satisfies $\sum' \frac{\tilde{\phi}(|n|)}{|n|} < \infty$. Note that if the number of elements in $\{\gamma_n\}$ is finite, we add $\{\gamma_n\}$ and $\{\gamma_n'\}$ to $\{\lambda_n\}$ and $\{\mu_n\}$, respectively, without changing the original question; if the number of elements in $\{\gamma_n\}$ is infinite, and if we can show $E_2(\tilde{\mu}) \leq E_2(\tilde{\lambda})$, then both $E_2(\lambda)$ and $E_2(\mu)$ are $-\infty$.

So without loss of generality, we assume that $\{\lambda_n\}$ is all of the zero set of $F(z)$. Then by Theorem 1.3, we have $\lim_{n\to\infty} \frac{|\lambda_n|}{n} = \alpha$.

From Hadamard's factorization theorem, $F(z)$ can be written as

$$F(z) = e^{az} \prod{}'(1 - \frac{z}{\lambda_n})e^{\frac{z}{\lambda_n}}.$$

Since $|\lambda_n - \mu_n| \leq \phi(n)$ with $\sum' \frac{\phi(n)}{|n|} < \infty$, we have that

$$\sum{}' \frac{\lambda_n - \mu_n}{\lambda_n \mu_n} = b$$

for some constant b.

Now set

$$F^*(z) = e^{(a-b)z} \prod{}'(1 - \frac{z}{\mu_n})e^{\frac{z}{\mu_n}}.$$

Then the canonical product $F^*(z)$ is an entire function, and satisfies that

$$\begin{aligned} F^*(z) &= e^{(a-b)z}\prod{}'(1-\frac{z}{\mu_n})e^{\frac{z}{\lambda_n}}\prod{}' e^{\frac{\lambda_n-\mu_n}{\lambda_n\mu_n}z} \\ &= e^{az}\prod{}'(1-\frac{z}{\mu_n})e^{\frac{z}{\lambda_n}}. \end{aligned}$$

From Lemma 1.1 with $h = 0$, to reach our goal we only need to show that $|F^*(x)| \leq \text{const}|F(x)|$, $x \in R$.

Let $\lambda_n = \rho_n + i\sigma_n$. With the hypothesis of $\sum \frac{\phi(|n|)}{|n|} < \infty$, Theorem 2.8' allow us to change the imaginary part of μ_n to agree with that of λ_n. Also we can assume $\sigma_n \geq 1$ for all n. So we may set $\mu_n = \lambda_n + \tau_n$ where τ_n is real. It follows that for $x \in R$

$$\begin{aligned} |1-\frac{x}{\mu_n}|^2 &= \frac{(\rho_n - x + \tau_n)^2 + \sigma_n^2}{(\rho_n - x)^2 + \sigma_n^2}|\frac{\lambda_n}{\mu_n}|^2|1-\frac{x}{\lambda_n}|^2 \\ &= (1+\eta_n)|1+\frac{\lambda_n-\mu_n}{\mu_n}|^2|1-\frac{x}{\lambda_n}|^2 \end{aligned}$$

where $\eta_n = |2\tau_n(\rho_n - x) + \tau_n^2|/\{(\rho_n - x)^2 + \sigma_n^2\}$. Note that when $0 \leq u \leq \frac{1}{2}$, we have

$$\log(1+u) = \sum{}'\frac{u^k}{k} \leq 2u.$$

So $\sum \log(1+u_n)$ is convergent if $\sum u_n$ is convergent.

Since $\lim_{n\to\infty} \frac{\lambda_n}{n} = \alpha$ and $|\lambda_n - \mu_n| \to 0$, we have that $\mu_n \sim \alpha n$. From the hypothesis on ϕ, $\sum \log|1+\frac{\lambda_n-\mu_n}{\mu_n}|$ is convergent.

From (1) in Theorem 1.3, we see that $\lim_{n\to\infty} \frac{\rho_n}{n} = \lim_{n\to\infty} \frac{\lambda_n}{n} = \alpha$. Note that

$$|\eta_n| \leq \text{const}\frac{\tau_n}{|\rho_n - x| + 1}$$

Set $N_1 = [\frac{x}{2\alpha}]$, and $N_0 = [\frac{x}{\alpha}]$. We consider the following three cases of η_n:

1) when $1 \leq n < N_1$, then $|x - \rho_n| \geq \rho_n = O(n)$

$$\sum_{n=1}^{N_1-1} |\eta_n| \leq \sum_{n=1}^{\infty} \frac{\phi(n)}{1+n}$$

which is uniformly bounded.

2) when $N_1 \leq n < N_0$, then

$$\sum_{n=N_1}^{N_0} |\eta_n| \leq \text{const} \int_{\alpha N_1}^{\alpha N_0} \frac{\phi(t)}{1+(x-t)} dt \leq O(\phi(N_1) \log N_1).$$

Since $\sum \frac{\phi(n)}{n} < \infty$ and $\phi(n) \to 0$, then $\phi(n) \log n \to 0$ as $n \to \infty$. Thus $\sum_{n=N_1}^{N_0} |\eta_n|$ is uniformly bounded.

3) when $n \geq N_0$, set $\rho_n - x = u_{n-N_0}$. Then we have

$$u_k \sim (\rho_{k+N_0} - x) \sim \alpha(k + N_0) - x \sim \alpha k.$$

So

$$\sum_{n=N_1}^{N_0} |\eta_n| \leq \sum_{k=1}^{\infty} \frac{\phi(k+N_0)}{1+u_k} \leq \sum_{k=1}^{\infty} \frac{\phi(k)}{1+u_k}.$$

Thus we finally get $|F^{*}(x)| \leq \text{const}|F(x)|$ which ends the proof.

2.4 A note on an excess formula

After the proof of a special case, Sedletskii [1978] claims the following formula:

Sedletskii's claim: *Suppose* $\lambda_n = n - \beta sign(n) + i\alpha \log n$, $n \in Z \setminus \{0\}$, *where* $\alpha \geq 0$, β *real. Define* $\phi(z)$ *by*

$$\phi(z) = z \prod{}'(1 - \frac{z}{\lambda_n}).$$

Then $|\phi(z)| = O(|z|^{(\alpha+2\beta)\pi})$ *on* $\Im z = -1$. *Consequently, the excess* E_2 *in* $L^2(-\pi,\pi)$ *of the system* $\{e^{i\lambda_n t}\}$ *satisfies that*

1) $E_2 = [(\alpha+2\beta)\pi]$ *if* $\{(\alpha+2\beta)\pi\} < \frac{1}{2}$*; or*

2) $E_2 = [(\alpha+2\beta)\pi] + 1$ *if* $\{(\alpha+2\beta)\pi\} \geq \frac{1}{2}$.

Unfortunately, the statement above is incorrect. For example, consider the system $\{e^{i(n+\frac{1}{4}\text{sign}(n)+\frac{1}{2\pi}\log|n|)t}\}$. Then $\alpha = \frac{1}{2\pi}$, and $\beta = -\frac{1}{4}$. So we have $(\alpha+2\beta)\pi = \frac{1}{2} - \frac{\pi}{2} = -1.07 = -2 + 0.93$, According to the claim, $E_2 = -1$ which is in contradiction with Theorem 2.12.

So after rechecking the details of the proof, they actually show that:

Remark 2.2: Under the assumption of Sedletskii's claim, we have that $|\phi(z)| = O(|z|^{\alpha\pi+2\beta})$ on $\Im z = -1$, and

1) $E_2 = [\alpha\pi + 2\beta]$ if $\{\alpha\pi + 2\beta\} < \frac{1}{2}$; or

2) $E_2 = [\alpha\pi + 2\beta] + 1$ if $\{\alpha\pi + 2\beta\} \geq \frac{1}{2}$.

Corollary 2.3. *Suppose* $\{\lambda_n\}$ *is defined by* (2.2), $\mu_n = \lambda_n + it_n$ *with* $t_n = \frac{1}{2\pi}\log|n|$, *and* $t_0 = 0$. *Then* $E_2(\mu) = [\alpha\pi + 2\beta] = 1$.

Set $\lambda_n = n + (\frac{1}{4}+\epsilon)\text{sign}(n)$, $\lambda_0 = 0$. Levinson show that the corresponding system $\{e^{i\lambda_n t}\}_{n=-\infty}^{\infty}$ is incomplete. But it is not true when λ_n is replaced by $\lambda_n + i\alpha\log|n|$. Actually, we have

Corollary 2.4. *Suppose* λ_n *is defined as above. Then the excess* E_2 *of* $\{e^{i(\lambda_n+\frac{1}{2\pi}\log|n|)t}\}$ *satisfies* $E_2 = 0$.

Corollary 2.4 and Levinson's example mentioned above thus gives a simple example where all of the two sequences $\{\lambda_n\}$ and $\{\mu_n\}$ are situated in the curvilinear strip $\{z : |y| \leq \log|x|\}$ and $\Re\mu_n = \Re\lambda_n$ for all n, but $E_2(\lambda) \neq E_2(\mu)$.

From Corollary 2.3, Theorem 2.3 and Theorem 2.8' we get that

Proposition 2.5. *Suppose $\{V_n(a)\}_{n=-\infty}^{\infty}$ is defined as in Section 1.4 with $a > 0$, then $E_2(V) = 1$.*

Next if we set $f(z) = \int_{-\pi}^{\pi} 2(t+\pi)e^{izt}dt$, then from the proof of Lemma 2.3, $f(z)$ can be written in the form

$$f(z) = -\frac{2e}{z^2}e^{-i\pi z}\{i(2\pi z + i)e^{i(2\pi z+i)} + \frac{1}{e}\} \tag{2.6}$$

which is related to Lambert W functions. With the following proposition about the function f, we can give a proof of Sedletskii's claim similar to that of Sedletskii [1978] but it avoids the long discussion of a function of Mittag-Leffler type.

Proposition 2.6. *Suppose an entire function $f(z)$ is defined by* (2.6), *then it can be written as*

$$f(z) = Be^{ikz} \prod_{n\in Z\setminus\{0\}} (1 - \frac{z}{V_n(-\frac{1}{e}) - \frac{i}{2\pi}}),$$

where k and B are real and V_n is defined in Section 1.4.

Furthermore, if set $G(z) = zf(z)$, and set

$$G_n(z) = \frac{G(z)}{G'(V_n(-\frac{1}{e}) - \frac{i}{2\pi})(z - V_n(-\frac{1}{e}) + \frac{i}{2\pi})}$$

and set

$$g_n(t) = \lim_{A\to\infty} \int_{-A}^{A} G_n(x)e^{ixt}dx,$$

then $\{g_n\}_{-\infty}^{\infty}$ *is a sequence in* $L^2(-\pi,\pi)$ *biorthogonal to* $\{e^{i(V_n(-\frac{1}{e})-\frac{i}{2\pi})t}\}_{-\infty}^{\infty}$.

Proof: let $f(z)$ be defined by (2.6), then from the proof of Lemma 2.3. we see that all the zeros of $f(z)$ are $\{-\frac{i}{2\pi}(W_n(-\frac{1}{e})+1), n \in Z \setminus \{0,-1\}\}$ which is equal to $\{V_n(-\frac{1}{e}) - \frac{i}{2\pi}, n \in Z \setminus \{0\}\}$.

Recall that $V_n(-\frac{1}{e}) = \rho_n + i\sigma_n$ as in Section 1.4, and that the sequence has the symmetric property that

$$\overline{V_n(-\frac{1}{e})} = -V_{-n}(-\frac{1}{e}).$$

So set $\sigma_n = V_n(-\frac{1}{e}) - \frac{i}{2\pi}$, then we have that

$$\overline{(1-\frac{x}{\sigma_n})(1-\frac{x}{\sigma_{-n}})} = (1-\frac{-x}{\sigma_{-n}})(1-\frac{-x}{\sigma_n}). \tag{2.7}$$

Since $f(0) \neq 0$, by Hadamard's factorization theorem, we can write $f(z)$ as

$$f(z) = Be^{cz}\prod_{n=1}^{\infty}(1-\frac{z}{\sigma_n})(1-\frac{z}{\sigma_{-n}}) = Be^{cz}g(z).$$

From the equation (2.6), it is easy to verify that $f(x)$ satisfies $\overline{f(x)} = f(-x)$. Combining with the fact that $\overline{g(x)} = g(-x)$, we get that $\overline{cx} = -cx$ and $\bar{B} = B$. Thus $c = ik$ with k real and B is real.

Set $G(z) = zf(z)$ and

$$G_n(z) = \frac{G(z)}{G'(\sigma_n)(z-\sigma_n)}.$$

Since $f \in P$, we have $G_n \in P$ which satisfies that $\int_{-\infty}^{\infty} G_n(x)K_m(x)dx = G_n(\sigma_m) = \delta_{mn}$ where $K_m(x) = \frac{\sin\pi(x-\sigma_m)}{\pi(x-\sigma_m)}$ is the reproducing function.

Suppose

$$g_n(t) = \lim_{A\to\infty} \int_{-A}^{A} G_n(x)e^{ixt}dx$$

is the inverse Fourier transform of $G_n(z)$. Since the complex Fourier transform is an isometric isomorphism from $L^2(-\pi,\pi)$ onto all of P and since $K_m(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\sigma_m t}e^{-izt}dt$ is the Fourier transform of $e^{i\sigma_m t}$, it is easy to see that its inverse Fourier transform $\{g_n\}$ is biorthogonal to $\{e^{i\sigma_n t}\}$.

2.5 Difference of excesses in complex domain

In the last section, we have discussed and compared the excesses of two complex sequences having the same real parts. In this section, we consider some cases when they have the same imaginary parts.

In 1999, Fujii, Nakamura and Redheffer proved the following result:

Theorem 2.13. *Let $\{\mu_n\}$ be a complex sequence such that $|\mu_n - n| \le c$ for $-\infty < n < \infty$. Let $\lambda_0 = \mu_0$ and*

$$\lambda_n = \mu_n + a, \qquad \lambda_{-n} = \mu_{-n} - b, \qquad n > 0,$$

where $a \ge 0$ and $b \ge 0$ are constants. Then $E_2(\lambda) \le E_2(\mu)$ on the interval $(-\pi,\pi)$.

As was the case with Theorem 2.8, the condition $|\mu_n - n| \le c$ is too strong. Indeed, we will now extend the result above to complex sequences such that $|\Re\mu_n - n| \le c$. Details are explained in the following theorem:

Theorem 2.14. *Assume that $\{\lambda_n\}$ and $\{\mu_n\}$ are two complex sequences whose points lie in the curvilinear strip $\{z = x + iy : |y| \leq \phi(|x|)\}$, where $\phi(x)(x \geq 0)$ is a positive non-decreasing function such that*

$$\sum \frac{\phi^2(n)}{n^2} < \infty.$$

Assume that $\lambda_0 = \mu_0$ and

$$\lambda_n = \mu_n + a, \qquad \lambda_{-n} = \mu_{-n} - b, \qquad n > 0,$$

where $a \geq 0$, $b \geq 0$ are constants. If $|\Re\mu_n - n| \leq c$ for all integers n, then $E_2(\lambda) \leq E_2(\mu)$ on the interval $(-\pi, \pi)$.

Proof of Theorem 2.14:

A simple induction argument using Theorem 1.4 shows that any finite number of terms can be replaced without altering the completeness as long as no repetition occurs. Therefore, we assume without loss of generality that $\mu_0 = \lambda_0 = 0$ and

$$\Re\mu_n < 0 \quad \text{for } n < 0, \qquad \Re\mu_n \geq 0 \quad \text{for } n > 0.$$

This may increase the value of c, but it does no harm to our proof.

Now suppose $\{\mu_n\}_{-\infty}^{\infty}$ is exact. To finish our proof, we only need to show that $\{\lambda_n\}_{n\in Z\setminus\{0\}}$ is incomplete.

Set $\mu_n = \beta_n + i\sigma_n$ and $\lambda_n = \alpha_n + i\sigma_n$ where β_n, α_n and σ_n are real. Then

$$\alpha_n = \beta_n + a, \qquad \alpha_{-n} = \beta_{-n} - b, \qquad n > 0.$$

From the conditions of $|\beta_n - n| \le c$ and $\sum' \frac{\sigma_n^2}{n^2} < \infty$, we see that

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\frac{1}{\mu_n}+\frac{1}{\mu_{-n}}\right| &= \sum_{n=1}^{\infty}\left|\frac{(\mu_n-n)+(\mu_{-n}+n)}{\mu_n\mu_{-n}}\right| \\ &\le \sum_{n=1}^{\infty}\left(\frac{2c}{|\mu_n||\mu_{-n}|}+\frac{2\sigma_n}{|\mu_n||\mu_{-n}|}\right) < \infty.\end{aligned}$$

Consequently, we can assume $\sum_{n=1}^{\infty}(\frac{1}{\mu_{-n}}+\frac{1}{\mu_n}) = b$. It follows from Theorem 1.2 that there is an entire function $F_1(z)$ corresponding to the sequence $\{\mu_n\}_{n\in Z\setminus\{0\}}$ such that $F_1(x) \in L^2(-\infty,\infty)$. By Hadamard's factorization theorem it follows that

$$\begin{aligned}F_1(z) &= e^{(a-b)z}\prod_{n=1}^{\infty}(1-\frac{z}{\mu_{-n}})(1-\frac{z}{\mu_n})e^{(\frac{1}{\mu_{-n}}+\frac{1}{\mu_n})z} \\ &= e^{az}\prod_{n=1}^{\infty}(1-\frac{z}{\mu_{-n}})(1-\frac{z}{\mu_n})\end{aligned}$$

for some constant a.

By a similar argument as above, we set

$$G_1(z) = e^{az}\prod_{n=1}^{\infty}(1-\frac{z}{\beta_{-n}})(1-\frac{z}{\beta_n}),$$

$$G_2(z) = e^{az}\prod_{n=1}^{\infty}(1-\frac{z}{\alpha_{-n}})(1-\frac{z}{\alpha_n}),$$

$$F_2(z) = e^{az}\prod_{n=1}^{\infty}(1-\frac{z}{\lambda_{-n}})(1-\frac{z}{\lambda_n}).$$

Then all three functions $G_1(z)$, $G_2(z)$ and $F_2(z)$ are entire. From Lemma 1.1, it suffices to show that

$$|F_2(x-i)| \le C|F_1(x-i)| \tag{2.8}$$

on the x-axis for some constant C.

From the hypotheses of the theorem, we have that the series $\sum' \frac{\sigma_n^2}{n^2} < \infty$. Since $\frac{\sigma_n^2}{\alpha_n^2} = O(\frac{\sigma_n^2}{n^2})$, it follows that $\sum' \frac{\sigma_n^2}{\alpha_n^2}$ converges. So

$$\prod{}'\left|\frac{\alpha_n}{\lambda_n}\right| = e^{-\frac{1}{2}\sum'\log(1+\frac{\sigma_n^2}{\alpha_n^2})}$$

converges to a non-zero constant A. Consequently,

$$\begin{aligned}
\left|\frac{F_2(z)}{G_2(z)}\right| &= \prod_{n=1}^{\infty}\left|\frac{(1-\frac{z}{\lambda_{-n}})(1-\frac{z}{\lambda_n})}{(1-\frac{z}{\alpha_{-n}})(1-\frac{z}{\alpha_n})}\right| \\
&= \prod_{n=1}^{\infty}\left|\frac{\alpha_{-n}}{\lambda_{-n}}\frac{\alpha_n}{\lambda_n}\frac{(z-\lambda_{-n})(z-\lambda_n)}{(z-\alpha_{-n})(z-\alpha_n)}\right| \\
&= A|\phi_-(z)||\phi_+(z)|
\end{aligned}$$

where $\phi_-(z) = \prod_{n=1}^{\infty}\frac{z-\lambda_{-n}}{z-\alpha_{-n}}$ and $\phi_+(z) = \prod_{n=1}^{\infty}\frac{z-\lambda_n}{z-\alpha_n}$.

Note that

$$\begin{aligned}
|\phi_-(x-i)| &= \prod_{n=1}^{\infty}\left|\frac{(x-\alpha_{-n})-(\sigma_{-n}+1)i}{(x-\alpha_{-n})-i}\right| \\
&= \prod_{n=1}^{\infty}\left|1+\frac{\sigma_{-n}^2+2\sigma_{-n}}{(x-\alpha_{-n})^2+1}\right| \\
&= e^{\frac{1}{2}\sum\log|1+\frac{\sigma_{-n}^2+2\sigma_{-n}}{(x-\alpha_{-n})^2+1}|}
\end{aligned}$$

which converges. Furthermore $|\phi_-(x-i)| \to 1$ as $x \to +\infty$.

Consequently,

$$\left|\frac{F_2(x-i)}{G_2(x-i)}\right|/|\phi_+(x-i)| \to A \quad \text{as} \quad x \to +\infty.$$

Similarly, we get that $|\phi_+(x-i)| \to 1$ as $x \to -\infty$, and

$$\left|\frac{F_2(x-i)}{G_2(x-i)}\right|/|\phi_-(x-i)| \to A \quad \text{as} \quad x \to -\infty.$$

Next consider $F_1(z)/G_1(z)$. Similar to above discussion, we have

$$\prod_{n=1}^{\infty}\left|\frac{(1-\frac{z}{\mu_{-n}})(1-\frac{z}{\mu_n})}{(1-\frac{z}{\beta_{-n}})(1-\frac{z}{\beta_n})}\right| = B|\psi_-(z)||\psi_+(z)|$$

where

$$B = \prod_{n=1}^{\infty} |\frac{\beta_{-n}\beta_n}{\mu_{-n}\mu_n}|, \quad \phi_-(z) = \prod_{n=1}^{\infty} \frac{z-\mu_{-n}}{z-\beta_{-n}} \text{ and } \phi_+(z) = \prod_{n=1}^{\infty} \frac{z-\mu_n}{z-\beta_n}$$

.

By the same argument as above, we get that

$$|\psi_-(x-i)| \to 1 \quad \text{as} \quad x \to +\infty;$$

$$|\psi_+(x-i)| \to 1 \quad \text{as} \quad x \to -\infty.$$

Consequently

$$|\frac{F_1(x-i)}{G_1(x-i)}|/|\psi_+(x-i)| \to B \quad \text{as} \quad x \to +\infty$$

$$|\frac{F_1(x-i)}{G_1(x-i)}|/|\psi_-(x-i)| \to B \quad \text{as} \quad x \to -\infty.$$

Note that when $n > 0$, $\lambda_n = \mu_n + a$, and when $n < 0$, $\lambda_n = \mu_n - b$. So we have that

$$\begin{aligned} \phi_+(x-i) &= \prod_{n=1}^{\infty} \frac{x-i-\lambda_n}{x-i-\alpha_n} \\ &= \prod_{n=1}^{\infty} \frac{(x-i-a)-\mu_n}{(x-i-a)-\beta_n} \\ &= \psi_+(x-a-i) \end{aligned}$$

and that

$$\phi_-(x-i) = \psi_-(x+b-i).$$

To prove (2.8) for all $x \in R$, we consider the three cases: $x < -M$, $x > M$ and $-M \leq x \leq M$ for some sufficiently large M.

First for $x < -M$, we have that

$$\begin{aligned} |F_2(x-i)| &= \Big|\frac{F_2(x-i)}{G_2(x-i)}\Big||G_2(x-i)| \\ &\leq 2A|\phi_-(x-i)||G_2(x-i)| \\ &= 2A|\psi_-(x-i+b)||G_2(x-i)| \\ &\leq \frac{4A}{B}\Big|\frac{F_1(x-i+b)}{G_1(x-i+b)}\Big||G_2(x-i)|. \end{aligned}$$

Next it is sufficient to show that $|\frac{G_2(x-i)}{G_1(x-i+b)}|$ is uniformly bounded. Note that

$$\begin{aligned} \frac{G_2(x-i)}{G_1(x+b-i)} &= \prod_{n=1}^{\infty} \frac{(1-\frac{x-i}{\alpha_{-n}})(1-\frac{x-i}{\alpha_n})}{(1-\frac{x-i+b}{\beta_{-n}})(1-\frac{x-i+b}{\beta_n})} \\ &= \prod_{n=1}^{\infty} \frac{\beta_{-n}\beta_n}{\alpha_{-n}\alpha_n} \frac{x-i-\alpha_n}{x-i-\beta_n+b} \\ &= \prod_{n=1}^{\infty} L(n,x), \end{aligned}$$

where

$$\begin{aligned} L(n,x) &= \frac{[\alpha_{-n}\alpha_n + (b\alpha_n - a\alpha_{-n}) - ab][x-i-\alpha_n]}{\alpha_{-n}\alpha_n(x-i-\alpha_n+a+b)} \\ &= \frac{\alpha_{-n}\alpha_n(x-i-\alpha_n) + (b\alpha_n - a\alpha_{-n} - ab)(x-i-\alpha_n)}{\alpha_{-n}\alpha_n(x-i-\alpha_n+a+b)} \\ &= 1 + \frac{\{-(a+b)\alpha_n\alpha_{-n} + (b\alpha_n - a\alpha_{-n} - ab)(-\alpha_n)\}}{\alpha_{-n}\alpha_n(x-i-\alpha_n+a+b)} \\ &+ \frac{(x-i)(b\alpha_n - a\alpha_{-n} - ab)}{\alpha_{-n}\alpha_n(x-i-\alpha_n+a+b)}. \end{aligned}$$

Since $\alpha_n = \beta_n + O(1) = n + O(1)$ for all $n \in Z$, by a simple calculation, we have that for sufficiently large $M > 0$ and $x < -M$, there are constants $K_1 > 0$, and $K_2 > 0$ such that

$$|L(n,x)| \leq \Big|1 + \frac{K_1 n + K_2 n x}{n^2(n-x)}\Big| \leq \Big|1 + \frac{K_1}{n^2}\Big|$$

Thus when $x < -M$, we have that

$$\left|\frac{G_2(x-i)}{G_1(x-i+b)}\right| < e^{\sum_{n=1}^{\infty} \log|1+\frac{K}{n^2}|}$$

which is uniformly bounded. So there exists a constant $C_1(M)$ such that $|F_2(x-i)| \leq C_1(M)|F_1(x-i)|$ for any x satisfying $x < -M$.

Similarly we can show that $|F_2(x-i)| \leq C_2(M)|F_1(x-i)|$ for any x satisfying $x > M$.

From Theorem 2.8, we can assume that $\sigma_n \neq -1$ for all $n \in Z$. Then it is obvious that $|F_2(x-i)| \leq C_3(M)|F_1(x-i)|$ for any x satisfying $-M \leq x \leq M$. Thus $|F_2(x-i)|$ is uniformly bounded by $|F_1(x-i)|$ for any $x \in R$. This ends the proof.

Chapter 3

The series expansion of complex exponentials

3.1 Introduction

In the last chapter, we have discussed under various conditions on λ_n the completeness of $\{e^{i\lambda_n x}\}$ in $L^2(-\pi,\pi)$, i.e. if $f(x) \in L^2(-\pi,\pi)$, and if $\int_{\pi}^{\pi} f(t)e^{i\lambda_n t}dt = 0$, then $f(x)$ is a null function.

In general, such results do not imply that $f(x)$ can be represented by a series $\sum a_n e^{i\lambda_n x}$. It only implies that for given any ϵ, it is possible to find a polynomial in $\{e^{i\lambda_n x}\}$, $P_\epsilon(x)$, such that

$$\int_{-\pi}^{\pi} |f(x) - P_\epsilon(x)|^2 dx < \epsilon.$$

Therefore it is of interest to find conditions under which it is possible to get a series representation for $f(x)$ in terms of $\{e^{i\lambda_n x}\}$ analogous to the Fourier series. Under the condition $|\lambda_n - n| \leq D < \frac{1}{\pi^2}$, such series were studied by Paley and Wiener, and Levinson extended the result to $D < \frac{1}{4}$. Furthermore, Levinson showed that his result is sharp in the sense that if $D = \frac{1}{4}$, then the conclusions no longer hold in general.

For the special sequence $\{\lambda_n\}$ where $\lambda_n = n + \frac{1}{4}$, $n > 0$, $\lambda_{-n} = -\lambda_n$, and $\lambda_0 = 0$, Young showed that the conclusion holds in such a extreme $\{\lambda_n\}$. In this chapter we will discuss a case under which the above extreme case of $\{\lambda_n\}$ are perturbed, i.e. we try to get a series representation of $f \in L^2(-\pi, \pi)$ in terms of $\{e^{i\rho_n t}\}$. The main result is the following:

Theorem 3.1. *If the $\{\rho_n\}$ are given as in Section 1.4 with $a < 0$, then each function f in $L^2(-\pi, \pi)$ has a unique non-harmonic Fourier expansion*

$$f(t) \sim \sum_{-\infty}^{\infty} c_n e^{i\rho_n t}$$

which is equiconvergent with its ordinary Fourier series uniformly on each closed subinterval of $(-\pi, \pi)$. Specifically, the system $\{e^{i\rho_n t}\}$ possesses a unique biorthogonal set $\{g_n(x)\}$ such that the series

$$\sum_{-\infty}^{\infty} \{ \frac{e^{inx}}{2\pi} \int_{-\pi}^{\pi} f(\xi) e^{-in\xi} d\xi - e^{i\rho_n x} \int_{-\pi}^{\pi} f(\xi) g_n(\xi) d\xi \}$$

converges uniformly to zero on each closed subinterval of $(-\pi, \pi)$.

For a variety of other equiconvergence results on the complex zeros λ_n of a special class of entire functions, please refer to Sedleckii[1970, 1972, 1975] or Verblunsky[1956, 1961]. For the norm convergence in L^p, one may refer to Benzinge.

Since the Fourier series of an L^2 function converge to the function pointwise almost everywhere (see for example [Young, 1980]), we have that

Corollary 3.1. *Under the hypotheses of Theorem 3.1,*

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{i\rho_n t}$$

almost everywhere on $(-\pi, \pi)$.

3.2 Asymptotic estimation of canonical products

Suppose $\{\lambda_n\}$ is a real sequence. Set

$$Q(z) = \prod{}'(1 - \frac{z}{\lambda_n}). \tag{3.1}$$

In 1934, Paley and Wiener proved that if $|\lambda_n - n| \leq h$ where $h > 0$ is constant, then the above function $Q(x)$ satisfies

$$K_1|x|^{-4h}| \leq |xQ(x)| \leq K_2|x|^{4h}$$

for $x \in R$, $|x| \geq 1$, provided in the case of the first inequality that $|x - \lambda_n| \geq \delta_0$ for all n, where K_1, K_2 and δ_0 are positive constants.

Later, Redheffer pointed out that the above result can be made more precise:

Theorem 3.2[Redheffer, 1954]. *Suppose the real sequence* $\{\lambda_n\}$ *satisfy* $|\lambda_n - n| \leq h$ *for some* $h > 0$ *and suppose* Q *is given by* (3.1), *then for* $x \in R$, $xQ(x) = o(x^{4h})$ *as* $x \to \infty$. *If* $\delta(x) \to 0$ *as* $x \to \infty$, *there exists* $h > 0$ *and a sequence* λ_n *with* $|\lambda_n - n| \leq h$ *such that* $x_jQ(x_j) > \delta(x_j)x_j^{4h}$ *for a sequence* x_j *with* $x_j \to \infty$.

In 1983, with the aid of the Γ function, Young showed that if $\lambda_n = n + \frac{1}{4}$ for all positive integers n, $\lambda_{-n} = -\lambda_n$, and $\lambda_0 = 0$, then $|xQ(x)| = O(\frac{1}{\sqrt{x}})$ as $x \to \infty$. Also in 1983, Redheffer proved that $|xQ(x)| = O(x^{-\frac{1}{2}} \log^{-2\beta} x)$ where $\lambda_n = n + \frac{1}{4} + \frac{\beta}{\log n}$.

Here we give an estimate of $xQ(x)$ for more general perturbation of n.

Theorem 3.3. *Let $\lambda_n = n + \epsilon_n$, $\lambda_{-n} = -\lambda_n$, and $0 \le h_1 \le \epsilon_n \le h_2 \le \frac{1}{2}$ for all positive integers n. Suppose Q is defined by* (3.1)*, then there exist positive constants δ_0, K_1 and K_2 such that*

$$K_1|x|^{-(3h_2-h_1)} \le |xQ(x)| \le K_2|x|^{-(3h_1-h_2)}$$

for sufficiently large x, provided in the first inequality that $|x-\lambda_n| \ge \delta_0$ for all λ_n.

Proof: From the symmetry of $Q(x)$, we assume $x > 0$. Let $\Lambda(r)$ be the number of λ_n on $(0,r)$ and note that $\Lambda(n) = n$ on $(\lambda_n, \lambda_{n+1})$. Then

$$\begin{aligned}
\log|Q(x)| &= \sum_{n=1}^{\infty} \log|1 - \frac{x^2}{\lambda_n^2}| \\
&= \sum_{n=1}^{\infty} n(\log|1 - \frac{x^2}{\lambda_n^2}| - \log|1 - \frac{x^2}{\lambda_{n+1}^2}|) \\
&= -\lim_{\epsilon\to 0}\Big(\int_1^{x-\epsilon} + \int_{x+\epsilon}^{\infty}\Big)\Lambda(r)\frac{d}{dr}(\log|1 - \frac{x^2}{r^2}|)dr \\
&= \lim_{\epsilon\to 0}(\int_1^{x-\epsilon} + \int_{x+\epsilon}^{\infty})\Lambda(r)K(x,r)dr
\end{aligned}$$

where

$$K(x,r) = \frac{d}{dr}\log\frac{r^2}{|x^2-r^2|} = \frac{2x^2}{r(x^2-r^2)}.$$

Suppose $\lambda_n < x < \lambda_{n+1}$ for some n, and take

$$\delta_1 = \frac{1}{4}\min\{|x-\lambda_n|, |x-\lambda_{n+1}\},$$

then for any $0 < \delta < \delta_1$, we have

$$\int_1^{x-\delta}(\Lambda(r)-r)K(x,r)dr = \int_1^{\lambda_n}(\Lambda(r)-r)K(x,r)dr + \int_{\lambda_n}^{x-\delta}(n-r)K(x,r)dr$$

and

$$\int_{x+\delta}^{\infty}(\Lambda(r)-r)K(x,r)dr = \int_{x+\delta}^{\lambda_{n+1}}(n-r)K(x,r)dr + \int_{\lambda_{n+1}}^{\infty}(\Lambda(r)-r)K(x,r)dr.$$

Note that

$$\begin{aligned}
& (\int_{\lambda_n}^{x-\delta} + \int_{x+\delta}^{\lambda_{n+1}})(n-r)K(x,r)dr \\
= & n(\int_{\lambda_n}^{x-\delta} + \int_{x+\delta}^{\lambda_{n+1}})K(x,r)dr - (\int_{\lambda_n}^{x-\delta} + \int_{x+\delta}^{\lambda_{n+1}})\frac{2x^2}{x^2-r^2}dr \\
= & n(\int_{\lambda_n}^{x-\delta} + \int_{x+\delta}^{\lambda_{n+1}})(\frac{2}{r} + \frac{1}{x-r} - \frac{1}{x+r})dr \\
- & x(\int_{\lambda_n}^{x-\delta} + \int_{x+\delta}^{\lambda_{n+1}})(\frac{1}{x-r} + \frac{1}{x+r})dr \\
= & n\{(\log\frac{\lambda_{n+1}}{\lambda_n} + \log|\frac{x-\delta}{x+\delta}|) - \log|\frac{x-\lambda_{n+1}}{x-\lambda_n}| \\
- & (\log|\frac{x+\lambda_{n+1}}{x+\lambda_n}| + \log|\frac{2x-\delta}{2x+\delta}|)\} \\
+ & x\{(\log|\frac{x-\lambda_{n+1}}{x-\lambda_n}| - (\log|\frac{x+\lambda_{n+1}}{x+\lambda_n}| + \log|\frac{2x-\delta}{2x+\delta}|)\} \\
= & nS_1 - xS_2.
\end{aligned}$$

Note that when x is sufficiently large, we have

$$nS_1 \leq -n\log|\frac{x-\lambda_{n+1}}{x-\lambda_n}| + C$$

and

$$xS_2 \leq x\log|\frac{x-\lambda_{n+1}}{x-\lambda_n}| + C$$

where the constant C may be different.

Then we have

$$(\int_{\lambda_n}^{x-\epsilon} + \int_{x+\epsilon}^{\lambda_{n+1}})(n-r)K(x,r)dr \leq (x-n)\log|\frac{x-\lambda_{n+1}}{x-\lambda_n}| + C.$$

Since $\int_0^\infty \frac{2x^2}{x^2-r^2}dr = 0$, it follows that

$$\begin{aligned}
\log|Q(x)| &= \lim_{\epsilon\to 0}(\int_1^{x-\epsilon} + \int_{x+\epsilon}^{\infty})(\Lambda(r)-r)K(x,r)dr - \int_0^1 \frac{2x^2}{x^2-r^2}dr \\
&\leq (\int_1^{\lambda_n} + \int_{\lambda_{n+1}}^{\infty})(\Lambda(r)-r)K(x,r)dr + (x-n)\log|\frac{x-\lambda_{n+1}}{x-\lambda_n}| + C.
\end{aligned}$$

From the definition of $\{\lambda_n\}$, we see that

$$-1 - h_2 \leq \Lambda(r) - r \leq -h_1.$$

Since $\Lambda(r) - r$ is almost periodic in $[1, \infty)$, its graph looks like a sawtooth, so we have

$$\begin{aligned}\log|Q(x)| &\leq \left(\int_1^{\lambda_n} K(x,r)dr\right)\left(-h_1 - \frac{1}{2}\right) \\ &\quad + \left(\int_{\lambda_{n+1}}^{\infty} K(x,r)dr\right)\left(-1 - h_2 + \frac{1}{2}\right) \\ &\quad + (x - n)\log\left|\frac{x - \lambda_{n+1}}{x - \lambda_n}\right| + C.\end{aligned}$$

Since

$$\int_1^{\lambda_n} K(x,r)dr = 3\log x + C,$$

$$\int_{\lambda_{n+1}}^{\infty} K(x,r)dr = -\log x + \log|x - \lambda_{n+1}| + C,$$

and since $\log|Q(x)| \to -\infty$ as $x \to \lambda_n$, we always have

$$\begin{aligned}\log|Q(x)| &\leq 3\left(-h_1 - \frac{1}{2}\right)\log x + \left(-h_1 - \frac{1}{2}\right)(-\log x) + C \\ &= (-3h_1 + h_2 - 1)\log x + C.\end{aligned}$$

So for a sufficiently large $|x|$, there is a constant K_2 such that

$$|xQ(x)| \leq K_2 x^{-(3h_1 - h_2)}.$$

Similarly, if $|x - \lambda_n| \geq \epsilon$ for all n, then

$$\begin{aligned}\log|Q(x)| &\geq (3\log x)\left(-1 - h_2 + \frac{1}{2}\right) + (-\log x)\left(-h_1 - \frac{1}{2}\right) \\ &= -\left(3h_2 + \frac{3}{2} - h_1 - \frac{1}{2}\right)\log x \\ &= -(3h_2 - h_1 + 1)\log x.\end{aligned}$$

Thus there exists a constant K_1 such that

$$|xQ(x)| \geq K_1 x^{-(3h_2-h_1)}.$$

This ends the proof.

When $h_1 = h_2 = \frac{1}{4}$, Young's estimation follows from the above results.

Now for $\lambda_n = \rho_n$ with ρ_n defined as in Section 1.4 with $a < 0$, let us consider the corresponding $xQ(x)$:

Set $h_1 = \frac{1}{4} - \epsilon$, and $h_2 = \frac{1}{4}$, then after replacing finitely many terms of ρ_n by $n + \frac{1}{4}$, all ρ_n satisfy the condition in Theorem 3.3. Since replacing finitely many terms in $\{\rho_n\}$ does not change the limit of $|xQ(x)|$, so we have the following estimation:

$$K_1 |x|^{-\frac{1}{2}-\epsilon} \leq |xQ(x)| \leq |x|^{-\frac{1}{2}+\epsilon},$$

where ϵ is any positive constant.

It is possible to obtain a slightly more precise estimate:

Lemma 3.1. *Suppose $\lambda_n = \rho_n$, where $\rho_n = \rho_n(a)$ is defined in Section 1.4 with $a < 0$, and Q is defined as before, then there exist positive constants δ_1, K_1 and K_2 such that*

$$\frac{K_1}{\sqrt{x}} \leq |xQ(x)| \leq \frac{K_2}{\sqrt{x}}$$

as $x \to \infty$, $x \in R$, provided in the first inequality that $|x - \rho_n| > \delta_1$.

Proof: Since $\rho_n(a) = n + \frac{1}{4} + O(\frac{\log n}{n})$ for $n > 0$, we see that the number $\Lambda_\rho(r)$ of ρ_n in $(0, r)$ should satisfy

$$\Lambda_\rho(r) \le \Lambda_{n+\frac{1}{4}}(r) \le r - \frac{1}{4}.$$

On the other hand, since $2\frac{\log r}{r} \ge \frac{\log n}{n}$ for $1 \le n \le r \le n+1$, then we have

$$\Lambda_\rho(r) \ge r - 1 - \frac{1}{4} + O(\frac{\log r}{r}).$$

From the proof of Theorem 3.3, we only need to show that

$$\int_1^\infty K(x,r)\frac{\log r}{r}dr < \infty.$$

Using partial fractions to expand $\frac{2x^2}{r^2(x^2-r^2)}$, then we see that

$$\int_1^\infty \frac{2x^2}{r(x^2-r^2)}\frac{\log r}{r}dr = 2\int_1^\infty \frac{\log r}{x^2-r^2}dr + O(1).$$

Now set $\rho_x = \{z : |z-x| = L, 0 \le \arg z \le \frac{\pi}{2}\}$, then $\int_{\rho_x} \frac{\log z}{x^2-z^2}dz = O(\frac{1}{x}) = O(1)$ as $L \to \infty$, then by integration over a contour in the first quadrant, we can replace integration over the positive real axis by integration over the positive imaginary axis and we obtain that

$$\int_1^\infty \frac{2x^2}{r^2(x^2-r^2)}\log r dr = 2\int_1^\infty \frac{\log u}{x^2+u^2}du + O(1) = O(1).$$

This completes the proof.

Lemma 3.2. *Suppose* $\{\lambda_n = n + \text{sign}(n)\frac{1}{4}\}_{n\neq 0}$, *and* $Q_1(z) = \prod'(1 - \frac{z}{\lambda_n})$. *Set* $G_1(z) = zQ_1(z)$, *then for* $x \in R$,

$$|G_1(x)| \le K|x|^{-\frac{1}{2}} \prod_{k=N}^{N+2} |\lambda_k - x|,$$

where $N := N(x) = \max\{n \in \mathbf{N}; n + \frac{1}{4} \le |x|\}$, *and* K *is constant independent of* N, *and* x.

Proof: Set $N = \max\{n \in \mathbf{N}, n + \frac{1}{4} \le |x|\}$, then

$$N < N + \frac{1}{4} \le |x| \le N + \frac{5}{4} < N + 2.$$

We write the product $G_1(x)$ in three parts:

$$\begin{aligned} |G_1(x)| &= |x| \prod_{k=1}^{N-1} |1 - \frac{x}{\lambda_k}||1 - \frac{x}{\lambda_{-k}}| \prod_{k=N}^{N+2} |\frac{\lambda_k - x}{\lambda_k}||1 - \frac{x}{\lambda_{-k}}| \\ &\times \prod_{k=N+3}^{\infty} |1 - \frac{x}{\lambda_k}||1 - \frac{x}{\lambda_{-k}}| \\ &= |x|(\prod_{k=N}^{N+2} |\lambda_k - x|)H(x), \end{aligned}$$

where $H(x)$ is defined by

$$H(x) = \prod_{k=1}^{N-1}(k + \frac{1}{4} - x) \prod_{k=1}^{N+2} \frac{k + \frac{1}{4} + x}{(k + \frac{1}{4})^2} \prod_{k=N+3}^{\infty} \frac{(k + \frac{1}{4} - x)(k + \frac{1}{4} + x)}{(k + \frac{1}{4})^2}.$$

From the functional equation $\Gamma(1+z) = z\Gamma(z)$ (see for example, [Artin,1964]), we know

$$\prod_{k=1}^{N-1}(k + \frac{1}{4} - x) = \frac{\Gamma(N + \frac{1}{4} - x)}{\Gamma(1 + \frac{1}{4} - x)},$$

$$\prod_{k=1}^{N+2} \frac{(k + \frac{1}{4} + x)}{(k + \frac{1}{4})^2} = \frac{\Gamma(N + 3 + \frac{1}{4} + x)\Gamma^2(1 + \frac{1}{4})}{\Gamma(1 + \frac{1}{4} + x)\Gamma^2(N + 3 + \frac{1}{4})},$$

and

$$\begin{aligned} \prod_{k=N+3}^{M} \frac{(k + \frac{1}{4} - x)(k + \frac{1}{4} + x)}{(k + \frac{1}{4})^2} &= \frac{\Gamma(M + 1 + \frac{1}{4} - x)\Gamma(M + 1 + \frac{1}{4} + x)}{\Gamma(N + 3 + \frac{1}{4} - x)\Gamma(N + 3 + \frac{1}{4} + x)} \\ & \frac{\Gamma^2(N + 3 + \frac{1}{4})}{\Gamma^2(M + 1 + \frac{1}{4})}. \end{aligned}$$

It follows that

$$H(x) = \frac{\Gamma(N+\frac{1}{4}-x)}{\Gamma(1+\frac{1}{4}-x)} \frac{\Gamma(N+3+\frac{1}{4}+x)\Gamma^2(1+\frac{1}{4})}{\Gamma(1+\frac{1}{4}+x)\Gamma^2(N+3+\frac{1}{4})} \frac{\Gamma^2(N+3+\frac{1}{4})}{\Gamma(N+3+\frac{1}{4}-x)\Gamma(N+3+\frac{1}{4}+x)} \lim_{M\to\infty} \frac{\Gamma(M+1+\frac{1}{4}-x)\Gamma(M+1+\frac{1}{4}+x)}{\Gamma^2(M+1+\frac{1}{4})}.$$

Since the Gamma function $\Gamma(x)$ satisfies

$$\lim_{M\to\infty} \frac{\Gamma(M+a_1)\Gamma(M+a_2)}{\Gamma(M+b_1)\Gamma(M+b_2)} = 1$$

provided $a_1 + a_2 = b_1 + b_2$ (see [Artin, 1964]), we get

$$\begin{aligned} H(x) &= \frac{\Gamma^2(1+\frac{1}{4})\Gamma(N+\frac{1}{4}-x)}{\Gamma(1+\frac{1}{4}-x)\Gamma(1+\frac{1}{4}+x)\Gamma(N+3+\frac{1}{4}-x)} \\ &= \frac{1}{\pi}\Gamma^2(1+\frac{1}{4}) \frac{\sin(x-\frac{1}{4})}{\prod_{k=0}^{2}(N+k+\frac{1}{4}-x)} \frac{\Gamma(x-\frac{1}{4})}{\Gamma(1+\frac{1}{4}+x)} \end{aligned}$$

where the last equality follows from (see [Artin, 1964])

$$\Gamma(1-(x-\frac{1}{4}))\Gamma(x-\frac{1}{4}) = \frac{\pi}{\sin\pi(x-\frac{1}{4})}$$

and

$$\Gamma(N+\frac{1}{4}-x) = \frac{\Gamma(N+3+\frac{1}{4}-x)}{\prod_{k=0}^{2}(N+k+\frac{1}{4}-x)}.$$

Since

$$\left|\frac{\sin\pi(x-\frac{1}{4})}{\prod_{k=0}^{2}(N+k+\frac{1}{4}-x)}\right| = \left|\frac{\sin\pi(N+\frac{1}{4}-x)}{\prod_{k=0}^{2}(N+k+\frac{1}{4}-x)}\right|,$$

all the possible singularities of this function have disappeared. So it is bounded.

On the other hand, since $\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim |z|^{\alpha-\beta}$ (see [Artin, 1964]), we have that

$$\frac{\Gamma(x-\frac{1}{4})}{\Gamma(1+\frac{1}{4}+x)} = |x|^{-1-\frac{1}{2}}.$$

Thus $|G_1(x)| \leq K\prod_{k=N}^{N+2}|\lambda_k - x||x|^{-\frac{1}{2}}$, which completes the proof of Lemma 3.2.

Remark 3.1. Actually from the proof of Lemma 3.2, we have got that

$$K_1|x|^{-\frac{1}{2}}\prod_{k=N}^{N+2}|\lambda_k - x| \leq |G_1(x)| \leq K_2|x|^{-\frac{1}{2}}\prod_{k=N}^{N+2}|\lambda_k - x|$$

where K_1 and K_2 are positive and N is the same as in Lemma 3.2.

Lemma 3.3. *Suppose that in Lemma 3.2, $\{\lambda_n\}$ is replaced by $\{\rho_n(a)\}$ which is defined in Section 1.4 with $a < 0$, and $Q(z) = \prod'(1 - \frac{z}{\rho_n})$. Set $G(z) = zQ(z)$, then we have that for all $x \in R$,*

$$|G(x)| \leq K|x|^{-\frac{1}{2}}\prod_{k=n_0-1}^{n_0+1}|\rho_k - x|,$$

where K is a constant and n_0 is an integer dependent on x.

Proof: Suppose $Q(z) = \prod'(1 - \frac{z}{\rho_n})$, and $Q_1(z)$ is defined as in Lemma 3.2, then both of $Q(x)$ and $Q_1(x)$ are even functions. So we only need to consider the case of $x > 0$.

Since replacing finitely many terms of $\{\rho_n\}$ does not change the behaviour of $G(x)$ as x tends to infinity, we assume $\rho_n < \lambda_n$ for all positive n. Let Δ_n be the interval (ρ_n, λ_n) for all $n \geq 1$. Suppose $\Lambda_\lambda(t)$ and $\Lambda_\rho(t)$ are the numbers of $\{\lambda_n\}$ and $\{\rho_n\}$ in the interval $(1, t)$. Then we have

$$\Lambda_\rho(t) - \Lambda_\lambda(t) = \begin{cases} 1 & t \in \cup_{n=1}^{\infty}\Delta_n \\ 0 & t \in (1,\infty) - \cup_{n=1}^{\infty}\Delta_n. \end{cases}$$

From the proof of Theorem 3.3, we see

$$\log|Q(x)| - \log|Q_1(x)| = \int_2^{\infty}(\Lambda_\rho(r) - \Lambda_\lambda(r)K(x,r)dr + C \quad (3.2)$$

$$= \sum_{n=2}^{\infty}\int_{\Delta_n} K(x,r)dr + C \quad (3.3)$$

$$= \sum_{n=2}^{\infty}\int_{\rho_n}^{\lambda_n}(\frac{2}{r} - \frac{1}{x+r} + \frac{1}{x-r})dr + C. \quad (3.4)$$

Since

$$\sum_{n=2}^{\infty}\int_{\rho_n}^{\lambda_n}(\frac{2}{r} - \frac{1}{x+r})dr \leq 2\sum_{n=2}^{\infty}\int_{\rho_n}^{\lambda_n}\frac{1}{r}dr \leq 2\sum_{n=2}^{\infty}\frac{1}{\rho_n}|\Delta_n|,$$

and $|\Delta_n| = \lambda_n - \rho_n = O(\frac{\log n}{n})$, and $\rho_n \sim n$, the first two terms of the right side of the equation 3.2 are bounded by a constant which is independent of x. Thus

$$\log|\frac{Q(x)}{Q_1(x)}| \leq \sum_{n=2}^{\infty}\int_{\rho_n}^{\lambda_n}\frac{1}{x-r}dr + C.$$

Now suppose x satisfy $\rho_{n_0} \leq x < \rho_{n_0+1}$ for some integer n_0, then

$$\sum_{n=2}^{\infty}\int_{\rho_n}^{\lambda_n}\frac{1}{x-r}dr \leq \sum_{k=-1}^{1}\log|\frac{x-\rho_{n_0+k}}{x-\lambda_{n_0+k}}| + (\sum_{n=2}^{n_0-2}\int_{\rho_n}^{\lambda_n} + \sum_{n=n_0+2}^{\infty}\int_{\rho_n}^{\lambda_n})\frac{1}{|x-r|}dr.$$

Recall that $|\Delta_n| = O(\frac{\log \rho_n}{\rho_n})$, so we get that

$$\sum_{n_0+2}^{\infty}\int_{\rho_n}^{\lambda_n}\frac{1}{|x-r|}dr \leq O(\sum_{n=n_0+2}^{\infty}\frac{1}{\rho_n - x}\frac{\log\rho_n}{\rho_n})$$

$$\leq O(\sum_{n=n_0+2}^{\infty}\frac{1}{\rho_n - x}\frac{\log(\rho_n - x)}{(\rho_n - x)})$$

$$= O(\sum_{k=1}^{\infty}\frac{\log u_k}{u_k^2}),$$

where $u_k = \rho_{k+n_0+1} - x$. Since $\frac{log u_k}{u_k^2} \sim \frac{\log k}{k^2}$, it is uniformly bounded by a constant independent of x.

Note that

$$\begin{aligned}\sum_{n=2}^{n_0-2}\int_{\rho_n}^{\lambda_n}\frac{1}{|x-r|}dr &\le \sum_{n=2}^{n_0-2}\frac{1}{x-\rho_n}|\Delta_n| \\ &\le \sum_{n=2}^{n_0-2}\frac{\log\rho_n}{\rho_n(x-\rho_n)} \\ &= \frac{1}{x}\sum_{n=2}^{n_0-2}(\frac{1}{\rho_n}+\frac{1}{x-\rho_n})\log\rho_n.\end{aligned}$$

Since

$$\sum_{n=2}^{n_0-2}\frac{\log\rho_n}{\rho_n} < \int_{n=1}^{\rho_{n_0-2}}\frac{\log r}{r}dr = O(\log^2\rho_{n_0}) = O(\log^2 x),$$

and since

$$\sum_{n=2}^{n_0-2}\frac{\log\rho_n}{x-\rho_n} < \log x\int_1^{x-1}\frac{dr}{x-r} < \log x\int_1^{x-1}\frac{du}{u} = O(\log^2 x),$$

we get that

$$\sum_{n=2}^{n_0-2}\int_{\rho_n}^{\lambda_n}\frac{1}{x-r}dr < \frac{\log^2 x}{x} \to 0$$

as $x\to\infty$. Thus

$$\log\frac{Q(x)}{Q_1(x)} = \sum_{n=2}^{\infty}\int_{\rho_n}^{\lambda_n}\frac{1}{x-r}dr + C < \sum_{k=-1}^{1}\log|\frac{x-\rho_{n_0+k}}{x-\lambda_{n_0+k}}| + C.$$

Set $G(x)=xQ(x)$ and $G_1(x)=xQ_1(x)$, then the above deduction yields that

$$|G(x)| \le K|G_1(x)|\prod_{k=-1}^{1}|\frac{x-\rho_{n_0+k}}{x-\lambda_{n_0+k}}|.$$

Recall that N in Lemma 3.2 and n_0 defined above satisfy the conditions

$$\rho_{n_0} \le x < \rho_{n_0+1} \quad and \quad \lambda_N \le x < \lambda_{N+1},$$

so if $x\in[\rho_{n_0},\lambda_{n_0})$, then $n_0 = N+1$. In this case, Lemma 3.2 guarantees that

$$|G(x)| \le K|G_1(x)| \prod_{k=0}^{2} |\frac{x-\rho_{N+k}}{x-\lambda_{N+k}}| \le Kx^{-\frac{1}{2}} \prod_{k=-1}^{1} |x-\rho_{n_0+k}|.$$

So the lemma is true in this case. If $x \in [\lambda_{n_0}, \rho_{n_0+1})$, then $n_0 = N$. Similarly we have

$$\begin{aligned} |G(x)| &\le Kx^{-\frac{1}{2}} \prod_{k=0}^{2} |x-\lambda_{N+k}| \prod_{k=-1}^{1} |\frac{x-\rho_{N+k}}{x-\lambda_{N+k}}| \\ &\le Kx^{-\frac{1}{2}} \prod_{k=-1}^{1} |x-\rho_{n_0+k}||\frac{x-\lambda_{N+2}}{x-\lambda_{N-1}}| \\ &\le 2x^{-\frac{1}{2}} \prod_{k=-1}^{1} |x-\rho_{n_0+k}|. \end{aligned}$$

This completes the proof of the lemma.

Remark 3.2. By changing the roles of $Q(x)$ and $Q_1(x)$ in the above proof, we can show that

$$K_3|x|^{-\frac{1}{2}} \prod_{k=n_0-1}^{n_0+1} |\rho_k - x| \le |G(x)| \le K_4|x|^{-\frac{1}{2}} \prod_{k=n_0-1}^{n_0+1} |\rho_k - x|$$

where K_3 and K_4 are positive and n_0 is the same as in Lemma 3.3.

Next we use a different evaluation method to prove a lemma which plays a key role in Young's proof.

Lemma 3.4. *Under the assumption of Lemma 3.3, the following estimation holds:*

$$\int_{-\infty}^{\frac{n+1}{2}} |\frac{G(x)}{x-\rho_n}|^2 dx = O(\frac{\log n}{n^2})$$

and

$$\int_{-\infty}^{\infty} |\frac{G(x)}{x-\rho_n}|^2 dx = O(\frac{1}{n})$$

when $n \to \infty$.

Proof: Assuming n is a large and positive number, we write

$$\begin{aligned} I &= \int_{-\infty}^{\infty} |\frac{G(x)}{x-\rho_n}|^2 dx \\ &= (\int_{-\infty}^{-\frac{n+1}{2}} + \int_{-\frac{n+1}{2}}^{\frac{n+1}{2}} + \int_{\frac{n+1}{2}}^{\rho_n-\frac{1}{2}} + \int_{\rho_n-\frac{1}{2}}^{\rho_n+\frac{1}{2}} + \int_{\rho_n+\frac{1}{2}}^{\infty})|\frac{G(x)}{x-\rho_n}|^2 dx \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

Then from Lemma 3.1 and Lemma 3.3, we see that

$$\begin{aligned} I_1 &= \int_{-\infty}^{-\frac{n+1}{2}} |\frac{G(x)}{x-\rho_n}|^2 dx \le \int_{-\infty}^{-\frac{n+1}{2}} |\frac{1}{x(x-\rho_n)^2}| dx \\ &= O(\frac{1}{n^2}), \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_{-\frac{n+1}{2}}^{\frac{n+1}{2}} |\frac{G(x)}{x-\rho_n}|^2 dx = (\int_{-\frac{n+1}{2}}^{-1} + \int_{-1}^{1} + \int_{1}^{\frac{n+1}{2}})|\frac{G(x)}{x-\rho_n}|^2 dx \\ &\le \frac{4}{\rho_n^2}(\int_{-\frac{n+1}{2}}^{-1} + \int_{1}^{\frac{n+1}{2}})|\frac{1}{x}| dx + O(\frac{1}{n^2}) \\ &= O(\frac{\log n}{n^2}). \end{aligned}$$

Thus

$$\int_{-\infty}^{\frac{n+1}{2}} |\frac{G(x)}{x-\rho_n}|^2 dx = I_1 + I_2 = O(\frac{\log n}{n^2}).$$

Similarly, we get

$$\begin{aligned} I_3 &= \int_{\frac{n+1}{2}}^{\rho_n-\frac{1}{2}} |\frac{G(x)}{x-\rho_n}|^2 dx \le \frac{K_3}{n} \int_{\frac{n+1}{2}}^{\rho_n-\frac{1}{2}} |\frac{1}{(x-\rho_n)^2}| dx \\ &= O(\frac{1}{n}), \end{aligned}$$

and

$$I_5 = \int_{\rho_n+\frac{1}{2}}^{\infty} |\frac{G(x)}{x-\rho_n}|^2 dx = O(\frac{1}{n}).$$

Now consider $x \in (\rho_n - \frac{1}{2}, \rho_n + \frac{1}{2})$. From the definition of n_0 in Lemma 3.3, we also have

$$\rho_{n_0} \leq x < \rho_{n_0+1},$$

then $\rho_{n_0} \leq \rho_n + \frac{1}{2}$ and $\rho_{n_0+1} > \rho_n - \frac{1}{2}$.. It follows that

$$n - 1 \leq n_0 \leq n + 1.$$

Thus $n \in \{n_0 - 1, n_0, n_0 + 1\}$. Combining with Lemma 3.3, we get

$$\begin{aligned} I_4 &= \int_{\rho_n-\frac{1}{2}}^{\rho_n+\frac{1}{2}} |\frac{G(x)}{x-\rho_n}|^2 dx \\ &\leq K \int_{\rho_n-\frac{1}{2}}^{\rho_n+\frac{1}{2}} \frac{1}{|x|} |\frac{\prod_{k=-1}^{1}(\rho_{n_0+k} - x)}{x-\rho_n}|^2 dx \\ &= O(\frac{1}{n}). \end{aligned}$$

Therefore $I = O(\frac{1}{n})$ as $n \to \infty$, which ends the proof.

For complex variable z, the value $|zQ(z)|$ was estimated by Levinson[Lemma 16.1], who proved that if $\{\lambda_n\}$ satisfies $|\lambda_n - n| \leq \delta < \frac{1}{4}$, and $Q(z)$ is defined as above, then

$$K_1|y|(1+|z|)^{-1-4\delta} e^{\pi|y|} \leq |zQ(z)| \leq K_2(1+|z|)^{4\delta} e^{\pi|y|}.$$

Young[1983] noted that the above is true for $\delta = \frac{1}{4}$. Since $|\rho_n(a) - n| \leq \frac{1}{4}$ for large n, we have the following lemma:

Lemma 3.5. *If λ_n is replaced by $\rho_n(a)$ in Section 1.4 for $a < 0$, and $G(z) = zQ(z)$, and $z = x + iy$, then there exist positive constants K_1, K_2 and C such that*

$$K_1|y|(1+|z|)^{-2}e^{\pi|y|} \le |G(z)| \le K_2(1+|z|)e^{\pi|y|}$$

and $|G(\frac{1}{2}+iy)| \ge C$.

3.3 Integral expression of canonical products

Let $\lambda_n = n + \frac{1}{4}$, $\lambda_{-n} = -\lambda_n$, and $\lambda_0 = 0$. Redheffer and Young[2] pointed out that the system $\{e^{i\lambda_n t}\}$ is orthogonal to the system $\{g_n/G'(\lambda_n)\}$ where

$$g_n(t) = -ie^{-i\lambda_n t}\int_{-\pi}^{t} f(x)e^{i\lambda_n x}dx,$$

with $f(t) = (\cos\frac{t}{2})^{-\frac{1}{2}}\sin\frac{t}{2}$ and $G(z) = \int_{-\pi}^{\pi} f(t)e^{izt}dt$. Since $\rho_n - n$ is different from $\lambda_n - n$ in that the former is dependent on n, we cannot use Levinson's method to find such an explicit formula. Before continuing on this topic, we first give the following result:

Theorem 3.4. *Let $\{\mu_n\}$ be a real symmetric sequence satisfying $|\mu_n - n - \frac{1}{4}| \le |\rho_n - n - \frac{1}{4}|$ where $\{\rho_n(a)\}$ is defined in Section 1.4 with $a < 0$, and let*

$$Q(z) = \prod_{n=1}^{\infty}(1 - \frac{z^2}{\mu_n^2}).$$

Then $Q(z)$ is expressible in the form

$$Q(z) = \int_{-\pi}^{\pi}\phi(t)e^{izt}dt$$

with $\phi(t)$ in $L^2(-\pi,\pi)$ satisfying $\phi(t) = \phi(-t)$ a.e. in $(-\pi,\pi)$.

Proof: We already know from Proposition 2.2 that if $a < 0$, $\{e^{i\mu_n t}, n = \pm 1, \pm 2, ...\}$ is incomplete. Then by Theorem 1.2 there is a nontrival entire function $f_1(z)$ of exponential type π, vanishing at each μ_n, and expressible in the form

$$f_1(z) = \int_{-\pi}^{\pi} \phi(t) e^{izt} dt$$

with $\phi(t)$ in $L^2(-\pi, \pi)$. We will show that Q is a multiple of f_1.

First we see $f_1(0) \neq 0$. Otherwise, such ϕ would be orthogonal to all $\{e^{i\mu_n t}, n \in Z\}$ which is complete in $L^2(-\pi, \pi)$. It would follow that $f_1 = \phi = 0$.

Now set $f_2(z) = \frac{f_1(z)}{Q(z)}$. Let $n(r), n_1(r), n_2(r)$ be the number of zeros of Q, f_1, and f_2 in the disk $|z| \leq r$.

From Proposition 1.1, we see that $n(r) = 2[r - \frac{1}{4}] + O(\frac{\ln r}{r})$, and thus

$$\begin{aligned} \int_0^r \frac{n(t)}{t} dt &\geq 2\{\int_0^r \frac{t - \frac{1}{4}}{t} dt - \int_0^r \frac{t - \frac{1}{4} - [t - \frac{1}{4}] - \frac{1}{2}}{t} dt\} \\ &- \int_0^r \frac{1 - O(\frac{\log t}{t})}{t} dt \\ &= 2r - \frac{3}{2} \log r + O(1). \end{aligned}$$

On the other hand, since f_1 is of exponential type π,

$$|f_1(re^{i\theta})| = O(e^{\pi r |\sin\theta|}).$$

By Jensen's formula, we have

$$\begin{aligned} \int_0^r \frac{n_1(t)}{t} dt &= \frac{1}{2\pi} \int_0^{2\pi} \log |f_1(re^{i\theta})| d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \pi r |\sin\theta| d\theta + O(1) \\ &\leq 2r + O(1) \end{aligned}$$

Thus

$$\begin{aligned}\int_0^r \frac{n_2(t)}{t}dt &= \int_0^r \frac{n_1(t)}{t}dt - \int_0^r \frac{n(t)}{t}dt \\ &\leq \frac{3}{2}\log r + O(1)\end{aligned}$$

which implies that $n_2(r) \leq 1$, so $f_2(z)$ has at most one zero, say at $z = z_0$. Since $Q(z)$ is at most of order one, by Hadamard's theory,

$$f_1(z) = f_1(0)(1 - \frac{z}{z_0})^p e^{Az} Q(z) \quad 0 \leq p \leq 1.$$

Now to finish the proof, we only need to show that f_1 is even on R, since it will then follows that $A = p = 0$.

Set

$$F(z) = \int_{-\pi}^{\pi} (\frac{\phi(t) + \phi(-t)}{2}) e^{izt} dt.$$

Since $\int_{-\pi}^{\pi} \phi(t)dt = \int_{-\pi}^{\pi} \phi(-t)dt$, we have that $F(0) = f_1(0)$. By the symmetry of $\{\mu_n\}$, we also have $F(\mu_n) = 0$ for $n = \pm 1, \pm 2, \ldots$. It follows that $f_1(z) - F(z) = \int_{-\pi}^{\pi} (\frac{\phi(t)-\phi(-t)}{2}) e^{izt} dt$ vanishes at all of $\{\mu_n\}_{n=-\infty}^{\infty}$. Since $\{e^{i\mu_n t}\}_{n=-\infty}^{\infty}$ is complete (by Corollary 2.1 and Theorem 2.2) in $L^2(-\pi, \pi)$, then we have that $\phi(t) = \phi(-t)$ a.e on $(-\pi, \pi)$.

Furthermore it guarantees that

$$f_1(-z) = \int_{-\pi}^{\pi} \phi(t) e^{-izt} dt = \int_{-\pi}^{\pi} \phi(-s) e^{izs} ds = f_1(z).$$

Since both of Q and f_1 are even, we have $A = p = 0$ which completes the proof.

From Theorem 3.2, we see that Q is in the Paley-Wiener space with $\| Q \| = \| \phi \|$. This gives a short proof of a part of Lemma 3.5

Corollary 3.1. *Under the assumption of Lemma 3.3, there exists a constant $M > 0$ such that, for all $z = x + iy$,*

$$|Q(z)| \leq Me^{\pi|y|}.$$

Proof: Since $Q(z)$ is in the Paley-Wiener space and satisfies $\int_{-\infty}^{\infty} |Q(x)|^2 dx = \int_{-\pi}^{\pi} |\phi(t)|^2 dt < \infty$, we see that $Q(x)$ is bounded on the real axis, namely by M. Then the corollary follows from Young[1980, p.82].

Now set $G(z) = zQ(z)$ and define $G_n(z)$ by

$$G_n(z) = \frac{G(z)}{G'(\rho_n)(z - \rho_n)} = \frac{1}{G'_n(\rho_n)} \frac{z}{z - \rho_n} Q(z),$$

then $G_n(z)$ also belongs to the Paley-Wiener space. Next take the integral as a limit in the mean of the L^2 space, then we define

$$g_n(x) = \int_{-\infty}^{\infty} G_n(t) e^{-ixt} dt.$$

Lemma 3.6. *The sequence $\{g_n(t)\}$ defined as above is a biorthogonal system of $\{e^{i\rho_n t}, n \in Z\}$ in $L^2(-\pi, \pi)$, and $g_n(t)$ vanishes almost everywhere outside $(-\pi, \pi)$. Furthermore $\{g_n(t)\}$ is not a Riesz basis.*

Proof: As in the proof of the Paley-Wiener theorem (see Young[1980, pp.101-103]), we obtain $g_n(t) = 0$ almost everywhere outside $(-\pi, \pi)$. By the Fourier inversion formula, we have that

$$G_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(t)e^{ixt}dx.$$

So

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(t)e^{i\rho_m t}dt = G_n(\rho_m) = \delta_{nm}$$

for $n = 0, \pm 1, \pm 2, \ldots$ Thus $\{g_n(t)\}$ is the unique biorthogonal sequence dual to $\{e^{i\rho_n t}\}$. Furthermore $\{g_n(t)\}$ is not a Riesz basis since $\{e^{i\rho_n t}\}$ is not a basis (Proposition 2.3).

Note that $\{e^{i\rho_n t}\}$ is complete and exact, and thus any f in the Paley-Wiener space is uniquely determined by its values at $\{\rho_n\}$. So

$$f(z) = \sum_{n=-\infty}^{\infty} f(\rho_n)\frac{G(z)}{G'(\rho_n)(z-\rho_n)}$$

is valid for every function f in the Paley-Wiener space.

3.4 Proof of Theorem 3.1

For the entirety of the proof, we only summarize the ideas and unchanged calculations in Levinson's proof and Young's proof, but set forth the necessary modification explicitly.

Let C denote a rectangular path in the complex ξ plane with vertices at $(N + \frac{1}{2} + iM, -N - \frac{1}{2} + iM, -N - \frac{1}{2} - iM, N + \frac{1}{2} - iM)$. $G(z)$ is defined by

$$G(z) = z\prod_{1}^{\infty}(1 - \frac{z}{\rho_n})(1 - \frac{z}{\rho_{-n}})$$

where ρ_n is defined in Section 1.4 with $a < 0$. By Lemma 3.6. $\{g_n(t)\}$ is a biorthogonal system of $\{e^{i\rho_n t}\}$. Then using residues, Levinson shows that

$$\begin{aligned}&\lim_{A\to\infty}\frac{1}{4\pi^2 i}\int_{-A}^{A}G(u)e^{-iuy}du\int_C\frac{e^{i\xi x}}{G(\xi)(u-\xi)}d\xi\\ =\ &\sum_{-N}^{N}g_n(y)e^{i\rho_n x}-\frac{\sin(N+\frac{1}{2})(x-y)}{\pi(x-y)}.\end{aligned}$$

Suppose $f\in L^2(-\pi,\pi)$, then

$$\begin{aligned}&\sum_{-N}^{N}e^{i\rho_n x}\int_{-\pi}^{\pi}f(y)g_n(y)dy-\frac{1}{\pi}\int_{-\pi}^{\pi}f(y)\frac{\sin(N+\frac{1}{2})(x-y)}{(x-y)}dy\qquad(3.5)\\ =\ &\frac{1}{4\pi^2 i}\int_{-\pi}^{\pi}f(y)dy\lim_{A\to\infty}\int_{-A}^{A}G(u)e^{-iuy}du\int_C\frac{e^{i\xi x}}{G(\xi)(u-\xi)}d\xi.\end{aligned}$$

Now by well-known results from the theory of Fourier series, to show

$$\sum_{-\infty}^{\infty}\{\frac{e^{inx}}{2\pi}\int_{-\pi}^{\pi}f(\xi)e^{-in\xi}d\xi-e^{i\rho_n x}\int_{-\pi}^{\pi}f(\xi)g_n(\xi)d\xi\}=0,$$

we only need to show the left side of equation (3.5) tends to zero as $N\to\infty$. Next we consider the right side of that equation:

Let $I_1(x)$ denote the absolute value of that part of the right side of (3.5) for which ξ varies over the upper horizontal side of the rectangle C. Then

$$I_1(x)=\Big|\int_{-\pi}^{\pi}f(y)dy\lim_{A\to\infty}\int_{-A}^{A}G(u)e^{-iuy}du\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}}\frac{e^{i(\xi+iM)x}}{G(\xi+iM)(u-\xi-iM)}d\xi\Big|.$$

Since $G(x)=O(\frac{1}{\sqrt{x}})$, we have $\frac{G(u)}{(u+i)^2}\in L^1(-\infty,\infty)$. Appealing to Levinson's argument, the order of integration can be changed, i.e.

$$I_1(x)=\Big|\int_{-\infty}^{\infty}G(u)\hat{f}(u)du\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}}\frac{e^{i(\xi+iM)x}}{G(\xi+iM)(u-\xi-iM)}d\xi\Big|$$

where $\hat{f}(x)/\sqrt{2\pi}$ is the Fourier transform of $f(y)$. Clearly

$$|\frac{G(u)}{u-\xi-iM}| < C_1 N|\frac{G(u)}{u}|$$

for $M > 1$ and $|\xi| < 2N$. So

$$I_1(x) \leq C_1 N e^{M|x|} \int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \frac{d\xi}{|G(\xi+iM)|} \int_{-\infty}^{\infty} |\frac{G(u)\hat{f}(u)}{u}|du.$$

Since Theorem 3.4 guarantees that $\frac{G(z)}{z}$ is in the Paley-Wiener space, by Hölder's inequality, we have

$$\begin{aligned} \int_{-\infty}^{\infty} |\frac{G(u)\hat{f}(u)}{u}|du &\leq (\int_{-\infty}^{\infty} |\frac{G(u)}{u}|^2 du)^{\frac{1}{2}} (\int_{-\infty}^{\infty} |\hat{f}(u)|^2 du)^{\frac{1}{2}} && (3.6) \\ &= C_2. && (3.7) \end{aligned}$$

By Lemma 3.5, $|G(\xi+iM)| \geq BM(M^2+N^2)^{-1}e^{\pi M}$ for $|\xi| \leq N+\frac{1}{2}$. Thus for any x in $[-\pi+\delta, \pi-\delta]$, $I_1(x) \leq C_3 N^2(M^2+N^2)e^{-M(\pi-|x|)}$ which tends to zero as $M \to \infty$.

Next let $I_2(x)$ denote the absolute value of that part of the right side of equation (3.5) for which ξ varies over the right vertical side of the rectangle C. Then

$$I_2(x) = \Big| \int_{-\pi}^{\pi} f(y)dy \lim_{A\to\infty} \int_{-A}^{A} G(u)e^{-iuy}du \int_{-M}^{M} \frac{e^{-i\eta x}}{G(N+\frac{1}{2}+i\eta)(u-N-\frac{1}{2}-i\eta)}d\eta \Big|.$$

By interchanging the order of integration, we find that

$$I_2(x) = \Big| \int_{-\infty}^{\infty} G(u)\hat{f}(u)du \int_{-M}^{M} \frac{e^{-i\eta x}}{G(N+\frac{1}{2}+iM)(u-N-\frac{1}{2}-i\eta)}d\eta \Big|.$$

Then with a change of variable, Levinson shows

$$I_2(x) = \Big| \int_{-\infty}^{\infty} \hat{f}(u+N+\frac{1}{2})du \int_{-M}^{M} \frac{G(u+N+\frac{1}{2})e^{-i\eta x}}{G(i\eta+N+\frac{1}{2})(u-i\eta)}d\eta \Big|.$$

Let $\mu_n = -N + \rho_{N+n}$. then

$$\left|\frac{G(u+N+\frac{1}{2})}{G(i\eta+N+\frac{1}{2})}\right| = \left(\frac{u+\frac{1}{2}-\mu_0}{i\eta+\frac{1}{2}-\mu_0}\right)\prod_{n=1}^{\infty}\frac{u+\frac{1}{2}-\mu_n}{i\eta+\frac{1}{2}-\mu_n}\frac{u+\frac{1}{2}-\mu_{-n}}{i\eta+\frac{1}{2}-\mu_{-n}}.$$

Set

$$G_N(u) = u\prod_{n=1}^{\infty}(1-\frac{u}{\mu_n})(1-\frac{u}{\mu_{-n}}).$$

Using the above two equations, together with Lemma 3.5, Levinson shows that

$$\begin{aligned} I_2(x) &\leq \frac{C_5}{\pi-|x|^2}\left(\int_{-\infty}^{-\frac{N}{2}}\left|\frac{G_N(u+\frac{1}{2})}{u+\frac{1}{2}-\mu_0}\right|^2 du\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}|\hat{f}(u)|^2 du\right)^{\frac{1}{2}} \\ &+ \frac{C_6}{\pi-|x|^2}\left(\int_{-\infty}^{\infty}\left|\frac{G_N(u+\frac{1}{2})}{u+\frac{1}{2}-\mu_0}\right|^2 du\right)^{\frac{1}{2}}\left(\int_{\frac{N}{2}}^{\infty}|\hat{f}(u+\frac{1}{2})|^2 du\right)^{\frac{1}{2}} \\ &+ C_7\left(\int_{\frac{N}{2}}^{\infty}|\hat{f}(u+\frac{1}{2})|^2 du\right)^{\frac{1}{2}}. \end{aligned}$$

Following Young's method, from the definition of μ_n, we see that

$$\prod_{n=1}^{\infty}(1-\frac{u}{\mu_n})(1-\frac{u}{\mu_{-n}}) = \left(\frac{1-\frac{u}{\mu_{-N}}}{1-\frac{u}{\mu_0}}\right)\prod_{n=1}^{\infty}(1-\frac{u}{\rho_n-N})(1-\frac{u}{\rho_{-n}-N}).$$

Note that $\rho_0 = 0$, $\rho_n = -\rho_{-n}$, $\mu_{-N} = -N$, and $\mu_0 = \rho_N - N \to \frac{1}{4}$, so

$$\frac{1-\frac{u}{\mu_{-N}}}{1-\frac{u}{\mu_0}} = \frac{(\rho_N-N)(N+u)}{N(\rho_N-N-u)},$$

then

$$\begin{aligned} \prod_{n=1}^{\infty}(1-\frac{u}{\rho_n-N})(1-\frac{u}{\rho_{-n}-N}) &= \prod_{n=1}^{\infty}\frac{(1-\frac{u+N}{\rho_n-N})(1-\frac{u+N}{\rho_{-n}})}{(1-\frac{N}{\rho_n})(1-\frac{N}{\rho_{-N}})} \\ &= \frac{NG(u+N)}{(u+N)G(N)} \end{aligned}$$

It follows that

$$G_N(u) = -(\rho_N-N)\frac{G(u+N)}{G(N)}.$$

So

$$\int_{-\infty}^{\infty}|\frac{G_N(u+\frac{1}{2})}{u+\frac{1}{2}-\mu_0}|^2du = (\frac{\rho_N-N}{G(N)})^2\int_{-\infty}^{\infty}|\frac{G(u+N+\frac{1}{2})^2}{u+\frac{1}{2}+N-\rho_N}|^2du$$
$$= (\frac{\rho_N-N}{G(N)})^2\int_{-\infty}^{\infty}|\frac{G(x)}{x-\rho_N}|^2dx.$$

Since by Lemma 3.1, $|\frac{\rho_N-N}{G(N)}|^2 = O(N)$ as $N \to \infty$, and by Lemma 3.4, $\int_{-\infty}^{\infty}|\frac{G(x)}{x-\rho_N}|^2dx = O(\frac{1}{N})$, we see that $\int_{-\infty}^{\infty}|\frac{G_N(u+\frac{1}{2})}{u+\frac{1}{2}-\mu_0}|^2du$ is bounded as $N \to \infty$.

Similarly, again by Lemma 3.4,

$$\int_{-\infty}^{-\frac{N}{2}}\left|\frac{G_N(u+\frac{1}{2})}{u+\frac{1}{2}-\mu_0}\right|^2du = \frac{K}{G(N)}\int_{-\infty}^{-\frac{N}{2}}\left|\frac{G(u+\frac{1}{2}+N)}{u+\frac{1}{2}+N-\rho_N}\right|^2du$$
$$= \frac{K}{G(N)}\int_{-\infty}^{\frac{N+1}{2}}\left|\frac{G(u)}{x+-\rho_N}\right|^2du \to 0.$$

Combining this with the fact that $\hat{f} \in L^2$, we get

$$I_2(x) \to 0 \qquad as \qquad N \to \infty \tag{3.8}$$

whenever $|x| \leq \pi - \delta$, $\delta > 0$.

If we denote by $I_3(x)$ and $I_4(x)$ the values of that part of the right side of (3.5) on the other two sides of the rectangle C, respectively, then clearly (3.6) remains valid when $I_1(x)$ is replaced by $I_3(x)$, and (3.8) remains valid when $I_2(x)$ is replaced by $I_4(x)$.

Now for any given $\delta > 0$, we can have $I_2(x) + I_4(x) < \delta$ whenever N is sufficiently large. Fixing N in this way, we may choose M sufficiently large such that $I_1(x) + I_3(x) < \delta$. This shows that the right side of (3.5) tends to zero as $N \to \infty$.

Chapter 4

Exponential Frames

4.1 Introduction

In the previous chapters, we have discussed the completeness and pointwise convergence of nonharmonic series. In this chapter, we will connect our previous discussion with the concept of "frames". From the definition (see Section 1.3), it is clear that a frame is a complete set. But the converse is not true. We will show that typical examples are provided by $\{\rho_n\}$ and $\{V_n\}$ which are defined in Section 1.4. Furthermore, the difference between the stability of completeness and the stability of frames will be illustrated by these examples.

One aspect of sampling theory is to compare in different ways the values of g at the sampling points with the values of g on a line. For example, Polya first proved that

$$\limsup_{\substack{n\to\infty \\ n\in Z}} n^{-1}\log|f(n)| = \limsup_{\substack{r\to\infty \\ r\in R}} r^{-1}\log|f(r)| = h(0)$$

for all entire functions f of exponential type $\gamma < \pi$, where $h(\theta)$ is the Phragmén-Lindelöf indicator function.

Here Polya's theorem compares the exponential rate of growth of the values of a function at integers and its values on the real line. Furthermore, Cartwright pointed out that $|f(x)|$ is bounded for $x \geq 0$ if $|f(n)| \leq M$ for all nonnegative integer n. This time the assumption on the rate of growth is that it is bounded by a constant. Finally, from the point of view of convergence, Parseval's identity can be seen as a theorem where the comparison between the two kinds of values are based on a rate of growth comparable to $1/x$.

We will see that the integers $\{n\}$ in all of the above three theorems can be replaced by some sequence $\{\lambda_n\}$ with λ_n close to n. But how close must they be?

4.2 Preliminary results about frames

Bernstein, and Duffin and Schaeffer [1945] extended Polya's theorem and Cartwright's theorem respectively to the following forms:

Theorem 4.1 [Bernstein]. *Let $f(z)$ be an entire function of exponential type $\gamma < \pi$. Then we have*

$$\limsup_{n\to\infty} |\lambda_n|^{-1} \log |f(\lambda_n)| = \limsup_{r\to\infty} r^{-1} \log |f(r)| = h(0)$$

with $n \in N$, $r \in R$, provided that $\{\lambda_n\}_{n=1}^{\infty}$ is a complex sequence such that $n/\lambda_n \to 1$ as $n \to \infty$, and for some $\delta > 0$

$$|\lambda_n - \lambda_m| \geq \delta |n - m|, \quad n \neq m.$$

Theorem 4.2 [Duffin and Schaeffer, 1945]. *Let* $\{\lambda_n\}_{n=1}^{\infty}$ *be a complex sequence of uniform density 1. If* $f(z)$ *is an entire function of exponential type* $\gamma < \pi$ *such that* $|f(\lambda_n)| \leq 1$, *then for* $z = x + iy$

$$|f(z)| \leq M(L, \delta, \gamma)e^{\gamma|y|}.$$

The condition on the sequence $\{\lambda_n\}$ in Theorem 4.2 can not simply be weakened to $|\lambda_n - n| = o(n)$ as in Theorem 4.1. The deviation $\phi(n)$ of $\{\lambda_n\}$ (see Definition 1.6) is related to the form that the comparison can take. The following result can in some way illustrate this:

Theorem 4.3 [Boas, 1954]. *Let* $\{\lambda_n\}_{n=1}^{\infty}$ *be a complex sequence satisfying*

$$|\lambda_n - \lambda_m| \geq \mu|n - m| \quad \textit{and} \quad |\lambda_n - n| \leq \frac{\epsilon(n)}{\log(n/\epsilon(n))}$$

for some $\mu > 0$, *where* $\epsilon(x)$ *is increasing and tending to* ∞, *but* $\epsilon(x) = o(x)$. *Suppose* $f(z)$ *is an entire function of exponential type* $\gamma < \pi$. *Then*

$$\limsup_{x\to\infty} \frac{\log|f(x)|}{\epsilon(x)} < \infty \quad \textit{if} \quad \limsup_{n\to\infty} \frac{\log|f(\lambda_n)|}{\epsilon(\lambda_n)} < \infty.$$

Based on the above observation, Boas asked whether the condition $|\lambda_n - n| < L$ in Theorem 4.2 can be weakened to $|\lambda_n - n| < \epsilon(n)$ with some $\epsilon(n)$ which becomes infinite? This question remains open.

In [1952], Duffin extended Parseval's identity to the following form:

Theorem 4.4 [Duffin and Schaeffer]. *Let* $\{\lambda_n\}_{n=-\infty}^{\infty}$ *be a complex sequence of uniform density* d, *and let* $0 < \gamma < \pi d$. *If* $f(z)$ *is an entire function of exponential type* γ *such that* $f(x) \in L^2(-\infty, \infty)$, *then*

$$A \leq \frac{\sum_n |f(\lambda_n)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx} \leq B,$$

where the positive constants A *and* B *depend only on* γ *and* $\{\lambda_n\}$.

From Theorem 1.1, we see that an entire function of exponential type in $L^2(-\infty, \infty)$ can be expressed as an integral of a function in $L^2(-\pi, \pi)$, so Theorem 4.4 can be written in the following form which is the beginning of the theory of frames.

Theorem 4.4' [Duffin and Schaefer]. *If* $\{\lambda_n\}_{n=-\infty}^{\infty}$ *is a sequence of uniform density* d, *then the set of functions* $\{\exp(i\lambda_n t)\}$ *is a frame over the interval* $(-\gamma, \gamma)$ *where* $0 < \gamma < \pi d$.

Similar to the question raised by Boas, it is natural to ask whether the condition $|\lambda_n - n| < L$ in the above theorem can be replaced by $|\lambda_n - n| < \epsilon(|n|)$ with $\epsilon(|n|) \to \infty$. We will be able to answer (in the negative) this question (see Theorem 4.6 or Proposition 4.1). First, we state a result for the case of a Riesz basis:

Theorem 4.5 [Young, 1980, p.181]. *If the system of exponentials* $\{e^{i\lambda_n t}\}_{n=-\infty}^{\infty}$ *is a Riesz basis for* $L^2(-\pi, \pi)$, *then the points* λ_n *lie in a strip parallel to the real axis and are separated.*

The above proposition is not only for a Riesz basis, it also holds for general exponential frames:

Theorem 4.6. *If the system of exponentials* $\{e^{i\lambda_n t}\}$ *is a frame for* $L^2(-\pi,\pi)$, *then the points* λ_n *lie in a strip parallel to the real axis.*

Proof: Suppose $\{e^{i\lambda_n t}\}$ is a frame for $L^2(-\pi,\pi)$. Then it is a Bessel sequence, that is $\sum_n |\int_{-\pi}^{\pi} f(t)e^{i\lambda_n t}dt|^2 < \infty$ for all $f \in L^2(-\pi,\pi)$. Then from Bari's theorem[Young, 1980, p.155], there is a constant M such that

$$\|\sum_n c_n e^{i\lambda_n t}\|^2 \leq M \sum |c_n|^2$$

holds for every finite sequence of scalars $\{c_n\}$.

Take $c_n = 1$ when $n = N$ and 0 otherwise, then

$$\|\sum c_n e^{i\lambda_n t}\|^2 = \int_{-\pi}^{\pi} e^{-2\Im\lambda_N t}dt = \frac{1}{2\Im\lambda_N}[e^{2\Im\lambda_N\pi} - e^{-2\Im\lambda_N\pi}].$$

So if $\sup_n |\Im\lambda_n| = \infty$, then $\sup_N \|\sum c_n e^{i\lambda_n t}\|^2 = \infty$ which is in contradiction with Bari's theorem.

Proposition 4.1. *Suppose* $V_n(a)$ *is defined as in Section 1.4. Then the exponential system* $\{e^{iV_n(a)t}\}$ *is not a frame.*

Note that even though the sequence $\{V_n(a)\}$ in the above proposition does not satisfy the sampling conditions for the lower rate of growth in Theorem 4.4, it does satisfy the conditions in Theorem 4.3 with a function of (rapid) growth $\epsilon(x) = (\log x)^2$. We thus have:

Proposition 4.2. *Suppose* $\{V_n(a)\}$ *is defined as in Section 1.4 with* $n > 0$ *and* $f(z)$ *is an entire function of exponential type* $\gamma < \pi$. *Then we have*

$$\limsup_{x\to\infty} \frac{\log|f(x)|}{(\log x)^2} < \infty \quad \textit{if} \quad \limsup_{n\to\infty} \frac{\log|f(V_n(a))|}{(\log n)^2} < \infty.$$

Proof: Taking $\epsilon(x) = (\log x)^2$, then $\epsilon(x)$ satisfies the hypotheses of Theorem 4.3. From Proposition 1.1, we see that $|V_n(a) - V_m(a)|$ is greater than $\frac{1}{2}|n-m|$, and for sufficiently large n

$$|V_n(a) - n| \le C_1 \log n \le C_1 \frac{\epsilon(n)}{\log(n/\epsilon(n))}.$$

So Proposition 4.2 follows from Theorem 4.3.

4.3 The stability of frames

In this section we will discuss the stability of frames. In opposition to complete sequences, frames have a uniform constant bound on the variation of their sampling sequence.

Theorem 4.7 [Duffin and Schaeffer]. *Let $\{e^{i\lambda_n t}\}$ be a frame over $(-\gamma,\gamma)$. Then there is a $\delta_1 > 0$ such that $\{e^{i\mu_n t}\}$ is a frame over the same interval whenever $|\mu_n - \lambda_n| \le \delta_1$.*

For an application of this theorem, we first introduce the following definition:

Definition 4.1. *An exact complete sequence $\{\mu_n\}$ is said to be* **extreme** *if there exists a complex sequence $\{\lambda_n\}$ such that $|\mu_n - \lambda_n| \to 0$ as $n \to \infty$, and λ_n satisfies that $\{\lambda_n + \epsilon sign(n)\}$ is incomplete for any $\epsilon > 0$.*

Corollary 4.1. *An extreme sequence is not a frame sequence.*

Proof: Suppose $\{\mu_n\}$ is an exteme sequence and suppose it is a frame. Then by Theorem 4.7 there is a constant $\delta_1 > 0$ such that $\{e^{i\sigma_n t}\}$ is a frame for $L^2(-\pi, \pi)$ provided $|\sigma_n - \mu_n| < \delta_1$ for all n.

On the other hand, from the extreme property of $\{\mu_n\}$ there is a sequence $\{\lambda_n\}$ such that for a given integer $N > 0$,

$$|\mu_n - \lambda_n| \leq \frac{\delta_1}{2} \quad \text{when} \quad |n| \geq N$$

and $\{\lambda_n + \epsilon \text{sign}(n)\}$ is incomplete for any $\epsilon > 0$.

Now take $\alpha_n = \lambda_n + \frac{\delta_1}{2}\text{sign}(n)$ when $|n| \geq N$ and $\alpha_n = \mu_n$ when $|n| < N$. Then by Theorem 4.7, $\{\alpha_n\}$ is a frame, and thus complete. Replace $\{\alpha_n\}_{|n|<N}$ by $\{\lambda_n + \frac{\delta_1}{2}\text{sign}(n)\}_{|n|<N}$, and denote the new sequence by $\{\tilde{\alpha_n}\}$. Then the system $\{e^{i\tilde{\alpha_n}t}\}$ is still complete (see Theorem 1.4). But this is in contradiction with the assumption on $\{\lambda_n\}$.

Corollary 4.2. *After adding (or removing) a finite number of elements from an extreme sequence $\{\alpha_n\}$, the newly formed sequence $\{\tilde{\alpha_n}\}$ is still not a frame.*

Proof: Suppose $\{\alpha_n\}$ is an extreme sequence. We claim that $\{\beta_1, ..., \beta_k, \alpha_n\}_{n\in N}$ is not a frame. Actually from Definition 4.1 and Corollary 4.1, we know that $\{\alpha_n\}$ is complete and not a frame. Thus Theorem 1.5 guarantees that $\{\beta_1, \alpha_n\}_{n\in N}$ is not a frame but complete. Continuing in this way, we complete our proof.

From Theorem 2.5, we have that $\{\lambda_n^-\}_{n\neq 0}$ and $\{\lambda_n^+\}_{n\in Z}$ are exact. Furthermore by the formula in Remark 2.2 with $\alpha = 0$, it can be shown that $\{\lambda_n^- + \epsilon\text{sign}(n)\}_{n\in Z\setminus\{0\}}$ and $\{\lambda_n^+ + \epsilon\text{sign}(n)\}_{n\in Z}$ are incomplete. Then Corollary 4.1 and 4.2 gives that:

Proposition 4.3. *Suppose $\rho_n(a)$ is defined as in Section 1.4 for $a > 0$ or $a < 0$. Then the exponential system $\{e^{i\rho_n(a)t}\}$ is not a frame in $L^2(-\pi,\pi)$.*

Next assume $\{\lambda_n\}$ is a (complex) sequence of uniform density d. From Theorem 4.4', we see that if one removes any finite number of elements from $\{\lambda_n\}$, the remaining sequence is still a frame sequence for $L^2(-\gamma,\gamma)$, where $0 < \gamma < \pi d$. Indeed, this only changes the bound L where $|\lambda_n - n/d| < L$ to a larger one. This means that the excess of the frame system $\{e^{i\lambda_n t}\}$ for $L^2(-\gamma,\gamma)$ is infinite. Note that a Riesz basis is an exact frame (see [Young, 1980]), so $\{e^{i\lambda_n t}\}$ is too overcomplete to form a Riesz basis for $L^2(-\gamma,\gamma)$.

Now we turn to the case of $\gamma = \pi$. Paley and Wiener's result shows that the system $\{e^{i\lambda_n t}\}$ is a Riesz basis for $L^2(-\pi,\pi)$ provided $|\lambda_n - n| < \delta = \frac{1}{\pi^2}$. The constant L was improved by Duffin and Eachus to $L = \frac{\ln 2}{\pi} = 0.2206\ldots$ Finally, Kadec proved that:

Kadec's theorem. *If $\{\lambda_n\}$ satisfies $|\lambda_n - n| \leq \delta < \frac{1}{4}$ with $n \in Z$, then $\{e^{i\lambda_n t}\}$ is a Riesz basis for $L^2(-\pi,\pi)$.*

In 1983, by studying the biorthogonal sequence, Redheffer and Young proved that $\{e^{i\lambda_n t}\}$ is not a Riesz basis for $L^2(-\pi,\pi)$ if λ_n is defined by (2.1). Furthermore they pointed out that Kadec's condition above cannot be loosen to $|\lambda_n - n| < \frac{1}{4}$.

Since a Riesz basis is a frame, then from Corollary 4.1 we have

Proposition 4.4. *Suppose* λ_n *is defined by* (2.1) *or* (2.2). *Then* $\{e^{i\mu_n t}\}$ *is not a Riesz basis provided* $\mu_n - \lambda_n \to 0$ *as* $n \to \infty$.

The above proposition indicates that the constant δ in Kadec's theorem can not be replaced by any function $\delta(n)$ tending to $\frac{1}{4}$.

4.4 Explicit bounds for horizontal displacement

In the applications of frames (wavelet theory, irregular sampling as well as our goal in delay-differential equations), it is very important to have good estimates for the optimal frame bounds. The reason is that they play a decisive role for the speed of convergence for some reconstruction algorithms. The constant L allows us to obtain lower and upper bounds of a frame.

Note that in the last section, Kadec's theorem discussed the stability of a trigonometric system to remain a Riesz basis under small pertubations. For exponential frames, a similar result also holds true. Balan [1997] and Christensen [1999] independently proved the following result:

Theorem 4.8. *Suppose* $\{e^{i\lambda_n t}\}$ *is a frame for* $L^2(-\gamma, \gamma)$ *with bounds* A, B, *where*

$\{\lambda_n\}$ *are real. Set*

$$L(\gamma) = \frac{\pi}{4\gamma} - \frac{1}{\gamma}\arcsin(\frac{1}{\sqrt{2}}(1 - \sqrt{\frac{A}{B}})).$$

If the real sequence $\{\mu_n\}$ *satisfies* $|\mu_n - \lambda_n| \leq \delta < L(\gamma)$, *then* $\{e^{i\mu_n t}\}$ *is a frame for* $L^2(-\gamma, \gamma)$ *with bounds:*

$$A(1 - \sqrt{\frac{A}{B}}(1 - \cos\gamma\delta + \sin\gamma\delta))^2, \quad B(2 - \cos\gamma\delta + \sin\gamma\delta)^2.$$

Since $L(\gamma) > L_0(\gamma) = \frac{1}{\gamma}\ln(1 + \sqrt{\frac{A}{B}})$, Balan's result above is an improvement of the earlier result of Duffin and Eachus in which the variation of the sequence $\{\lambda_n\}$ is bounded by $L_0(\gamma)$. It also extends Kadec's theorem to tight frames. We next make some modifications to Theorem 4.8 for some sequences which are "nicely" distributed.

Theorem 4.9. *Suppose* $\{\lambda_n\}$ *is a frame sequence of real numbers for* $L^2(-\pi, \pi)$ *with bounds* A, B. *Let* $\{\rho_n\}$ *be a real sequence satisfying* $0 < \theta \leq |\rho_n - \lambda_n| \leq \delta \leq \frac{1}{4}$, *and let* $\sigma > 0$ *satisfy* $(1 + \sigma)\frac{\sin\pi\theta}{\pi\theta} < 1$. *Then* $\{e^{i\rho_n t}\}$ *is a frame over* $L^2(-\pi, \pi)$ *with bounds*

$$A\Big(1 - \frac{\sigma + \sqrt{\frac{A}{B}}(1 - (1 + \sigma)(\cos\pi\delta - \sin\pi\delta))}{1 + \sigma}\Big)^2$$

and

$$B\Big(1 + \frac{\sigma + (1 - (1 + \sigma)(\cos\pi\delta - \sin\pi\delta))}{1 - \sigma}\Big)^2$$

provided δ *satisfies*

$$\delta < \frac{1}{4} - \frac{1}{\pi}\arcsin\Big(\frac{1}{(1 + \sigma)\sqrt{2}}(1 - \sqrt{\frac{A}{B}})\Big).$$

As an example, if we take $\frac{A}{B} = 0.76$ and $\gamma = \pi$, then Theorem 4.8 guarantees the stability of the frame with $L = 0.2211$, that is, if $|\mu_n - \lambda_n| \leq 0.2211$, then $\{e^{i\mu_n t}\}$ is a frame if $\{e^{i\lambda_n t}\}$ is. By Theorem 4.9, we have that $\{e^{i\mu_n t}\}$ is a frame whenever $\{e^{i\lambda_n t}\}$ is and $\frac{1}{4.5} \leq |\mu_n - \lambda_n| \leq 0.2234$. Note that since $\frac{1}{4.5} = 0.222...$, this case was not covered by Theorem 4.8.

The proof of Theorem 4.9 will be given shortly. But first let us state a theorem on the perturbation of general frames. In 1995, Christensen proved:

Theorem 4.10. *Let $\{f_i\}_{i\in I}$ be a frame for a Hilbert space H with bounds A and B. Then any family $\{g_i\}_{i\in I}$ of elements in H is a frame for H with bounds $A(1-\sqrt{\frac{R}{A}})^2$ and $B(1+\sqrt{\frac{R}{B}})^2$ provided*

$$R := \sum_{i\in I} \|f_i - g_i\|^2 < A.$$

It is interesting that the perturbation condition only depends on the lower bound A. Later, Cazassa and Christensen improved the above theorem and obtained:

Theorem 4.10'. *Let $\{f_i\}_{i=1}^{\infty}$ be a frame for a Hilbert space H with bounds A, B. Let $\{g_i\}_{i=1}^{\infty}$ be a sequence H. Assume there exist nonnegative constants μ_1, μ_2, and μ such that $max(\mu_1 + \frac{\mu}{\sqrt{A}}, \mu_2) < 1$, and*

$$\| \sum_{i=1}^{n} c_i(f_i - g_i) \| \leq \mu_1 \| \sum_{i=1}^{n} c_i f_i \| + \mu_2 \| \sum_{i=1}^{n} c_i g_i \| + \mu(\sum_{i=1}^{n} |c_i|^2)^{\frac{1}{2}}$$

for all $c_1, c_2, ..., c_n$. Then $\{g_i\}_{i=1}^{\infty}$ is a frame with bounds

$$A(1 - \frac{\mu_1 + \mu_2 + \frac{\mu}{\sqrt{A}}}{1+\mu_2})^2, \qquad B(1 + \frac{\mu_1 + \mu_2 + \frac{\mu}{\sqrt{B}}}{1-\mu_2})^2.$$

Proof of Theorem 4.9:

Let $n \in N$, and $c_k \in C$ (k=1,2,..., n) be arbitrary. Set $\delta_k = \rho_k - \lambda_k$, and set

$$U = \| \sum_{k=1}^{n} c_k (e^{i\rho_k x} - e^{i\lambda_k x}) \| .$$

The conditions on δ and σ imply that $\sigma \in [0, 1)$. So

$$U \leq \| \sum_{k=1}^{n} c_k e^{i\lambda_k x}(1 - (1+\sigma)e^{i\delta_k x})\| + \sigma \| \sum_{k=1}^{n} c_k e^{i\rho_k x}\|.$$

Following Kadec's proof, we expand $1 - (1+\sigma)e^{i\delta_k x}$ in Fourier series to obtain:

$$\begin{aligned} 1 - (1+\sigma)e^{i\delta_k x} &= (1 - (1+\sigma)\frac{\sin \pi\delta_k}{\pi\delta_k}) \\ &+ (1+\sigma)\sum_{\tau=1}^{\infty} \frac{(-1)^\tau 2\delta_k \sin \pi\delta_k}{\pi(\tau^2 - \delta_k^2)} \cos(\tau x) \\ &+ (1+\sigma) i \sum_{\tau=1}^{\infty} \frac{(-1)^\tau 2\delta_k \cos \pi\delta_k}{\pi((\tau - \frac{1}{2})^2 - \delta_k^2)} \sin((\tau - \frac{1}{2})x). \end{aligned}$$

Since $\| \cos(\tau x)\phi(x)\| \leq \|\phi\|$ and $\| \sin((\tau - \frac{1}{2})x)\phi(x)\| \leq \|\phi\|$, then

$$\begin{aligned} U &\leq \| \sum_{k=1}^{n} (1 - (1+\sigma)\frac{\sin \pi\delta_k}{\pi\delta_k}) c_k e^{i\lambda_k x} \| \\ &+ (1+\sigma)\sum_{\tau=1}^{\infty} \| \sum_{k=1}^{n} \frac{2\delta_k \sin \pi\delta_k}{\pi(\tau^2 - \delta_k^2)} c_k e^{i\lambda_k x}\| \\ &+ (1+\sigma)\sum_{\tau=1}^{\infty} \| \sum_{k=1}^{n} \frac{2\delta_k \cos \pi\delta_k}{\pi((\tau - \frac{1}{2})^2 - \delta_k^2)} c_k e^{i\lambda_k x}\| + \sigma \| \sum_{k=1}^{n} c_k e^{i\rho_k x}\|. \end{aligned}$$

Note that

$$\| \sum_{k=1}^{n} a_k c_k e^{i\lambda_k x}\| \leq \sqrt{B}(\sum_{k=1}^{n} \|a_k c_k\|^2)^{\frac{1}{2}} \leq \sqrt{B} \sup |a_k| (\sum_{k=1}^{n} \|c_k\|^2)^{\frac{1}{2}}.$$

Since σ satisfies $1 + \sigma < \frac{\pi\theta}{\sin \pi\theta}$, then we have

$$|1 - (1+\sigma)\frac{\sin \pi\delta_k}{\pi\delta_k}| \leq 1 - (1+\sigma)\frac{\sin \pi\delta}{\pi\delta},$$

$$|\frac{2\delta_k \sin \pi\delta_k}{\pi(\tau^2 - \delta_k^2)}| \leq \frac{2\delta \sin \pi\delta}{\pi(\tau^2 - \delta^2)}.$$

and

$$|\frac{2\delta_k \cos \pi\delta_k}{\pi((\tau - \frac{1}{2})^2 - \delta_k^2)}| \leq \frac{2\delta \cos \pi\delta}{\pi((\tau - \frac{1}{2})^2 - \delta^2)}.$$

So combining with the estimation in [Kadec], we get

$$\begin{aligned} U &\leq \sqrt{B}\Big\{1 - (1+\sigma)\frac{\sin \pi\delta}{\pi\delta} + (1+\sigma)\sum_{\tau=1}^{\infty} \frac{2\delta \sin \pi\delta}{\pi(\tau^2 - \delta^2)} \\ &+ (1+\sigma)\sum_{\tau=1}^{\infty} \frac{2\delta \cos \pi\delta}{\pi((\tau - \frac{1}{2})^2 - \delta^2)}\Big\}(\sum_{k=1}^{n} \|c_k\|^2)^{\frac{1}{2}} + \sigma\|\sum_{k=1}^{n} c_k e^{i\rho_k x}\| \\ &= \sqrt{B}\Big\{1 - (1+\sigma)\frac{\sin \pi\delta}{\pi\delta} + (1+\sigma)\sin \pi\delta(\frac{1}{\pi\delta} - \cot \pi\delta) \\ &+ (1+\sigma)\cos \pi\delta \tan \pi\delta\Big\}(\sum_{k=1}^{n} \|c_k\|^2)^{\frac{1}{2}} + \sigma\|\sum_{k=1}^{n} c_k e^{i\rho_k x}\| \\ &= \sqrt{B}\{1 + (1+\sigma)(\sin \pi\delta - \cos \pi\delta)\}(\sum_{k=1}^{n} \|c_k\|^2)^{\frac{1}{2}} + \sigma\|\sum_{k=1}^{n} c_k e^{i\rho_k x}\|. \end{aligned}$$

Note that this implies that

$$1 + (1+\sigma)(\sin \pi\delta - \cos \pi\delta) > 0.$$

Now taking Theorem 4.10' with $\mu_1 = 0$, $\mu_2 = \sigma$, and $\mu = \sqrt{B}\{1 + (1+\sigma)(\sin \pi\delta - \cos \pi\delta)\}$, we see that for $\{e^{i\rho_k x}\}$ to be a frame over $L^2(-\pi, \pi)$, we only require $\mu < \sqrt{A}$. That means

$$\sin \pi\delta - \cos \pi\delta < \frac{1}{1+\sigma}(\sqrt{\frac{A}{B}} - 1).$$

Thus $\delta < L = \frac{1}{4} - \frac{1}{\pi}\arcsin(\frac{1}{(1+\sigma)\sqrt{2}}(1-\sqrt{\frac{A}{B}}))$. The bounds on the frame now follows directly from Theorem 4.10'. This completes the proof.

Now we know that $\{e^{\pm i(n+\frac{1}{4})}\}$ is not a frame over $L^2(-\pi,\pi)$, but it is a frame over $L^2(-\gamma,\gamma)$ when $\gamma < \pi$. To compute the frame bounds in this case, we have to find a basic sequence $\{\lambda_n\}$ for applying Theorem 4.8 and 4.9, and a reasonable sequence is all the integers. To figure out the upper and lower bounds of such an integer frame for $L^2(-\gamma,\gamma)$ with $\gamma < \pi$, one may refer to Plancherel and Polya's work [1937].

Since $\int_{-\infty}^{\infty}|f(x+c)|^2dx = \int_{-\infty}^{\infty}|f(x)|^2dx$ for any real constant c, we have that the bounds actually depend on the distance between any two α_n. The following theorem illustrates this:

Theorem 4.11[Plancherel and Polya]. *If f is an entire function of exponential type γ, then for any real increasing sequence $\{\lambda_n\}$ such that $\lambda_{n+1}-\lambda_n \geq \delta$ for some $\delta > 0$, we have*

$$\sum_{n=-\infty}^{\infty}|f(\lambda_n)|^2 \leq \frac{4(e^{\gamma\delta}-1)}{\pi\gamma\delta^2}\int_{-\infty}^{\infty}|f(x)|^2dx,$$

and in particular

$$\sum_{n=-\infty}^{\infty}|f(n)|^2 \leq \frac{4(e^{\gamma}-1)}{\pi\gamma}\int_{-\infty}^{\infty}|f(x)|^2dx. \tag{4.1}$$

Though the constant bound in (4.1) is dependent on the type γ, we have the following result:

Theorem 4.12. *For any entire function $f(z)$ of exponential type $\gamma > 0$ satisfying*

$\int_{-\infty}^{\infty}|f(x)|^2dx<\infty$, *there exists a real constant a such that* $g(z)=f(z+a)$ *and*

$$\sum_{n=-\infty}^{\infty}|g(n)|^2\le\frac{4}{\pi}\int_{-\infty}^{\infty}|g(x)|^2dx<\infty.$$

Proof: Let $f(z)$ is an entire function of exponential type $\gamma>0$. Since $|f|^2$ is subharmonic, then for $\delta>0$ and $w\in R$, we have

$$\begin{aligned}|f(w)|^2 &\le \frac{1}{\pi\delta^2}\int\int_{|z-w|<\delta}|f(z)|^2dxdy\\ &\le \frac{1}{\pi\delta^2}\int_{-\delta}^{\delta}\int_{w-\delta}^{w+\delta}|f(x+iy)|^2dxdy.\end{aligned}$$

Suppose k is a positive integer. Let $\delta=\frac{1}{2^k}$ and $w=n+\frac{2j}{2^k}$ for $j=1,...,2^{k-1}$. Then it follows that

$$|f(n+\frac{2j}{2^k})|^2\le\frac{1}{\pi\delta^2}\int_{-\delta}^{\delta}\int_{(n+\frac{2j}{2^k})-\delta}^{(n+\frac{2j}{2^k})+\delta}|f(x+iy)|^2dxdy$$

for $j=1,...,2^{k-1}$.

Set $f_j(z)=f(z+\frac{2j}{2^k})$, then f_j is an entire function of exponential type γ, and we have that

$$\begin{aligned}\sum_{j=1}^{2^{k-1}}\sum_{n=-\infty}^{\infty}|f_j(n)|^2 &\le \frac{1}{\pi\delta^2}\int_{-\delta}^{\delta}\int_{-\infty}^{\infty}|f(x+iy)|^2dxdy\\ &\le \frac{1}{\pi\delta^2}\int_{-\delta}^{\delta}(e^{2\gamma|y|}\int_{-\infty}^{\infty}|f(x)|^2dx)dy\\ &= \frac{e^{2\gamma\delta}-1}{\pi\gamma\delta^2}\int_{-\infty}^{\infty}|f(x)|^2dx.\end{aligned}$$

Choose f_{j_0} from $\{f_j\}$ such that $\sum_{n=-\infty}^{\infty}|f_{j_0}(n)|^2\le\sum_{n=-\infty}^{\infty}|f_j(n)|^2$ for $j=1,...,2^{k-1}$, then

$$2^{k-1}\sum_{n=-\infty}^{\infty}|f_{j_0}(n)|^2\le\frac{2^{2k}}{\pi\gamma}(e^{\frac{\gamma}{2^{k-1}}}-1)\int_{-\infty}^{\infty}|f(x)|^2dx.$$

So

$$\sum_{n=-\infty}^{\infty} |f_{j_0}(n)|^2 \le \frac{4}{\pi}\frac{e^{\frac{\gamma}{2^{k-1}}}-1}{\frac{\gamma}{2^{k-1}}}\int_{-\infty}^{\infty} |f(x)|^2 dx.$$

Since $\frac{e^{\frac{\gamma}{2^{k-1}}}-1}{\frac{\gamma}{2^{k-1}}} \to 1$ as $k \to \infty$, we have obtained that

$$\sum_{n=-\infty}^{\infty} |f_{j_0}(n)|^2 \le \frac{4}{\pi}\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{4}{\pi}\int_{-\infty}^{\infty} |f_{j_0}(x)|^2 dx$$

which completes the proof of Theorem 4.12.

4.5 Explicit bounds for vertical displacement

In the last section, we have discussed the stability of frames when the variation occurs along the real axis and new upper and lower bounds were obtained from the size of these displacements. Here we will be concentrating on displacements in the direction of the imaginary axis.

Theorem 4.13. *Let $\lambda_n = \alpha_n + i\beta_n$ be a sequence of uniform density 1 with α_n, β_n real, $|\beta_n| < \beta$. If $\{e^{i\alpha_n t}\}$ is a frame over an interval $(-\gamma, \gamma)$ with bounds A and B, and $f(z)$ is an entire function of exponential type γ with $0 < \gamma \le \pi$, and $f \in L^2(-\infty, \infty)$, then*

$$Ae^{-2\gamma\beta} \le \frac{\sum_{n=-\infty}^{\infty} |f(\lambda_n)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx} \le B\{e^{-\beta\gamma} + \sqrt{\frac{B}{A}}(1-e^{-\beta\gamma})\}^2 e^{2\gamma\beta}.$$

Before giving the proof of this theorem, we state some lemmas.

Lemma 4.1[Duffin and Schaeffer, 1952]. *If $f(z)$ is an entire function of exponential type γ and $f \in L^2(-\infty, \infty)$, then*

$$\int_{-\infty}^{\infty} |f^{(k)}(x)|^2 \le \gamma^{2k} \int_{-\infty}^{\infty} |f(x)|^2 dx$$

The method to prove the following lemma essentially comes from [Duffin and Schaeffer,1952].

Lemma 4.2. *Let $\{e^{i\lambda_n t}\}$ is a frame over the interval $(-\gamma, \gamma)$ with bounds A and B. Then for any given $\epsilon > 0$, there exists a $\delta > 0$ such that when $|\mu_n - \lambda_n| < \delta$ for all $n \in N$, we have*

$$(1-\epsilon)A < \frac{\sum_n |f(\mu_n)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx} < (1+\epsilon)B$$

for all entire functions $f(z)$ of exponential type γ with $f \in L^2(-\infty, \infty)$.

Proof: By Taylor's series expansion at $z = \lambda_n$, we have

$$f(\mu_n) - f(\lambda_n) = \sum_{k=1}^{\infty} \frac{f^{(k)}(\lambda_n)}{k!}(\mu_n - \lambda_n)^k,$$

and consequently,

$$|f(\mu_n) - f(\lambda_n)|^2 \le \{\sum_{k=1}^{\infty} \frac{|f^{(k)}(\lambda_n)|^2}{k!}\}\{\sum \frac{|\mu_n - \lambda_n|^{2k}}{k!}\}.$$

Given $\epsilon_1 > 0$, suppose $|\mu_n - \lambda_n| < \delta$ where $\delta > 0$ is such that $|\frac{B}{A}(e^{\gamma\delta} - 1)^2| < \epsilon_1$, and choose $\rho = \{\frac{\gamma}{\delta}\}^{1/2}$, then by the above inequality, we get

$$|f(\mu_n) - f(\lambda_n)|^2 \le \{\sum_{k=1}^{\infty} \frac{|f^{(k)}(\lambda_n)|^2}{\rho^{2k} k!}\}\{\sum_{k=1}^{\infty} \frac{(\rho\delta)^{2k}}{k!}\}.$$

Since $f^{(k)}(z)$ is an entire function of type γ, and since $\{e^{i\lambda_n t}\}$ is a frame over the interval $(-\gamma, \gamma)$, we can apply the right hand side inequality of frames with upper bound B. Combining with Lemma 4.1, we get

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |f(\mu_n) - f(\lambda_n)|^2 &\leq \{e^{\gamma\delta} - 1\}\{\sum_{k=1}^{\infty} \frac{1}{\rho^{2k} k!} \sum_{n=-\infty}^{\infty} |f^{(k)}(\lambda_n)|^2\} \\
&\leq \{e^{\gamma\delta} - 1\} \sum_{k=1}^{\infty} \frac{B}{\rho^{2k} k!} \int_{-\infty}^{\infty} |f^{(k)}(x)|^2 dx \\
&\leq \{e^{\gamma\delta} - 1\} B \sum_{k=1}^{\infty} \frac{\gamma^{2k}}{\rho^{2k} k!} \int_{-\infty}^{\infty} |f(x)|^2 dx \\
&= B(e^{\gamma\delta} - 1)(e^{\gamma^2/\rho^2} - 1) \int_{-\infty}^{\infty} |f(x)|^2 dx \\
&\leq \frac{B}{A}(e^{\gamma\delta} - 1)^2 \sum_{n\in N} |f(\lambda_n)|^2 \\
&< \epsilon_1 \sum_{n=-\infty}^{\infty} |f(\lambda_n)|^2.
\end{aligned}$$

By Minkowski's inequality, it follows that

$$\begin{aligned}
(\sum_{n\in N} |f(\mu_n)|^2)^{1/2} &= (\sum_{n\in N} |f(\mu_n) - f(\lambda_n) + f(\lambda_n)|^2)^{1/2} \\
&\leq (\sum_{n\in N} |f(\lambda_n)|^2)^{1/2} + (\sum_{n\in N} |f(\mu_n) - f(\lambda_n)|^2)^{1/2} \\
&\leq (1 + \epsilon_1^{1/2})(\sum_{n\in N} |f(\lambda_n)|^2)^{1/2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\sum_{n\in N} |f(\mu_n)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx} &= \frac{\sum_{n\in N} |f(\mu_n)|^2}{\sum_{n\in N} |f(\lambda_n)|^2} \frac{\sum_{n\in N} |f(\lambda_n)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx} \\
&\leq (1 + \epsilon_1^{1/2})^2 B.
\end{aligned}$$

On the other hand,

$$(\sum_{n\in N} |f(\lambda_n)|^2)^{\frac{1}{2}} = (\sum_{n\in N} |f(\lambda_n) - f(\mu_n) + f(\mu_n)|^2)^{\frac{1}{2}}$$

$$\leq \left(\sum_{n\in N} |f(\lambda_n) - f(\mu_n)|^2\right)^{\frac{1}{2}} + \left(\sum_{n\in N} |f(\mu_n)|^2\right)^{\frac{1}{2}}$$
$$\leq \epsilon_1^{\frac{1}{2}}\left(\sum_{n} |f(\lambda_n)|^2\right)^{\frac{1}{2}} + \left(\sum_{n\in N} |f(\mu_n)|^2\right)^{\frac{1}{2}}.$$

It follows that

$$(1-\epsilon_1^{\frac{1}{2}})^2\left(\sum_{n\in N} |f(\lambda_n)|^2\right) \leq \left(\sum_{n\in N} |f(\mu_n)|^2\right).$$

Therefore

$$\begin{aligned}\frac{\sum_{n\in N} |f(\mu_n)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx} &= \frac{\sum_{n\in N} |f(\mu_n)|^2}{\sum_{n\in N} |f(\lambda_n)|^2} \frac{\sum_{n\in N} |f(\lambda_n)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx} \\ &\geq (1-\epsilon_1^{\frac{1}{2}})^2 A.\end{aligned}$$

It is obvious that the ϵ_1 can be chosen such that both of $(1-\epsilon_1^{\frac{1}{2}})^2 > 1-\epsilon$ and $(1+\epsilon_1^{\frac{1}{2}})^2 < 1+\epsilon$ hold for any given $\epsilon > 0$. Thus the proof of the lemma is completed.

In [Duffin, Lemma 2], if we choose $\rho = (\gamma/M)^{1/2}$, then the lemma can be expressed in the following form:

Lemma 4.3. *Let $\{e^{i\sigma_n t}\}$ be a frame over the interval $(-\gamma, \gamma)$ with bounds A and B. If $\{\mu_n\}$ is a sequence satisfying $|\mu_n - \sigma_n| \leq M$ for some constant M, then any function f in the Paley-Wiener space, we have*

$$\frac{\sum_{n\in N} |f(\mu_n)|^2}{\sum_{n\in N} |f(\sigma_n)|^2} \leq \{1 + \sqrt{\frac{B}{A}}(e^{\gamma M} - 1)\}^2.$$

Proof of Theorem 4.13:

The second inequality follows easily from Lemma 4.3. Next we consider the first one. Suppose $f(z)$ is in the Paley-Wiener space. By a reflection and a translation, Duffin and Schaeffer [1952] (or see [Young, 1980, pp.192-195])constructed a new function f_1 and a new sequence $\lambda_n^{(1)} = \alpha_n + i\beta_n^{(1)}$ with $|\beta_n^{(1)}| \leq \beta/2$, such that

$$e^{-\beta\gamma}\frac{\sum_n |f_1(\lambda_n^{(1)})|^2}{\int_{-\infty}^{\infty} |f_1(x)|^2 dx} \leq \frac{\sum_n |f(\lambda_n)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx}.$$

Now for any given $\epsilon > 0$, choose $\delta > 0$ as to satisfy Lemma 4.2. Suppose K_0 is sufficiently large such that

$$|\lambda_n^{(K_0)} - \alpha_n| = |\beta_n^{(K_0)}| \leq |\beta/2^{K_0}| < \delta.$$

Repeat the above process K_0 times, then Lemma 4.2 guarantees that

$$\frac{\sum_{n\in N} |f_{K_0}(\lambda_n^{(K_0)})|^2}{\int_{-\infty}^{\infty} |f_{K_0}(x)|^2 dx} \geq (1-\epsilon)A.$$

Therefore, after combining these K_0 steps, we get that

$$\begin{aligned} \frac{\sum_{n\in N} |f(\lambda_n)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx} &\geq e^{-\gamma(\beta+\beta/2+\ldots+\frac{\beta}{2^{K_0-1}})}\frac{\sum_{n\in N} |f_{K_0}(\lambda_n^{(K_0)})|^2}{\int_{-\infty}^{\infty} |f_{K_0}(x)|^2 dx} \\ &\geq (1-\epsilon)Ae^{-2\beta\gamma}. \end{aligned}$$

Since ϵ is arbitary, the proof is complete.

Theorem 4.14. *Under the assumption of Theorem 4.13, if* $\gamma = \pi$, *and* $|\lambda_n - n| < L$ *for some constant* L, *then we have*

$$Ae^{-2\beta\pi} \leq \frac{\sum_{n=-\infty}^{\infty} |f(\lambda_n)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx} \leq e^{2L\pi}$$

for all entire functions of exponential type π *belonging to* $L^2(-\infty,\infty)$.

Proof: Actually, it suffices to prove the second inequality. In Lemma 4.3, we set $\gamma = \pi$, $\sigma_n = n$ and $\mu_n = \lambda_n$, then it follows from Parseval's identity that $A = B = 1$. The conclusion of Lemma 4.3 can thus be written as

$$\frac{\sum_{n=-\infty}^{\infty} |f(\lambda_n)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx} \leq e^{2L\pi}.$$

Corollary 4.3. *Suppose* $\{\lambda_n = n + i\beta_n\}$ *is a sequence satisfying* $|\beta_n| < \beta = L$, *then* $\{e^{i\lambda_n t}\}$ *is a frame over* $(-\pi,\pi)$ *with lower bound* $e^{-2\pi L}$ *and upper bound* $e^{2\pi L}$, *respectively.*

Remark 4.1. In Corollary 4.3, the upper and lower bounds cannot be replaced by $c_1 e^{2\gamma L}$ ($c_1 < 1$) and $c_2 e^{-2\gamma L}$ ($c_2 > 1$) respectively. It is obvious that $c_1 e^{2\gamma L} \to c_1 < 1$ and $c_2 e^{-2\gamma L} \to c_2 > 1$ as $L \to 0$. But when $L \to 0$, $\lambda_n \to n$, Theorem 4.8 and Theorem 4.13 imply that the upper and lower bounds B_L and A_L satisfy $B_L \to 1$ and $A_L \to 1$. It follows that $c_1 = c_2 = 1$.

Remark 4.2. The two exponents $-2\gamma\beta$ and $2\gamma\beta$ in Theorem 4.13 can not be improved, i.e. $e^{-2\gamma\beta}$ and $e^{2\gamma\beta}$ can not be replaced by $e^{-2(\gamma-\epsilon)\beta}$ and $e^{2(\gamma-\epsilon)\beta}$. Two examples are given in the next section.

4.6 Two examples

In this section, two examples are given to show that the exponents of the upper and lower bounds are precise.

Let $y = \cosh a(\pi - x)$, $0 \le x \le 2\pi$, then its Fourier expansion is

$$y = \frac{2}{\pi}\sinh a\pi\left[\frac{1}{2a} + \sum_{n=1}^{\infty}\frac{a}{a^2+n^2}\cos nx\right].$$

It follows that

$$\sum_{n=1}^{\infty}\frac{a}{a^2+n^2}\cos nx = \frac{\pi}{2}\frac{\cosh a(\pi - x)}{\sinh a\pi} - \frac{1}{2a}.$$

Since $\cos nx$ is even, we may extend n to the negative infinity, and get that

$$\sum_{n=-\infty}^{\infty}\frac{\cos nx}{a^2+n^2} = \frac{\pi}{a}\frac{\cosh a(\pi - x)}{\sinh a\pi}.$$

Now set $x = 0$ and $a = \beta$, then

$$\sum_{n=-\infty}^{\infty}\frac{1}{\beta^2+n^2} = \frac{\pi}{\beta}\frac{e^{\pi\beta}+e^{-\pi\beta}}{e^{\pi\beta}-e^{\pi\beta}}. \tag{4.2}$$

Next take $x = 2\gamma \le 2\pi$, and $a = \beta$, then

$$\sum_{n=-\infty}^{\infty}\frac{\cos 2\gamma n}{\beta^2+n^2} = \frac{\pi}{\beta}\frac{e^{\beta(\pi-2\gamma)}+e^{-\beta(\pi-2\gamma)}}{e^{\pi\beta}-e^{-\pi\beta}}. \tag{4.3}$$

We will employ (4.2) and (4.3) in the following two examples.

Example 4.1. Suppose $g_1(t) = e^{it}$, and $f_1(z) = (\frac{1}{2\pi})^{1/2} \int_{-\gamma}^{\gamma} g_1(t)e^{izt}dt$. Then

$$f_1(z) = (\frac{1}{2\pi})^{1/2}\frac{e^{\gamma(1+z)i} - e^{-\gamma(1+z)i}}{(1+z)i}.$$

Take $z = \lambda_n = n + i\beta$, we get that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |f_1(\lambda_n)|^2 &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\frac{1}{|1+\lambda_n|^2}|e^{\gamma(1+\lambda_n)i} - e^{-\gamma(1+\lambda_n)i}|^2 \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\frac{1}{(1+n)^2+\beta^2}|e^{-\gamma\beta+(1+n)\gamma i} - e^{\gamma\beta-(1+n)\gamma i}|^2 \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\frac{1}{(1+n)^2+\beta^2}(e^{2\gamma\beta} + e^{-2\gamma\beta} - 2\cos(2\gamma(1+n))) \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\frac{1}{n^2+\beta^2}(e^{2\gamma\beta} + e^{-2\gamma\beta} - 2\cos(2\gamma n)) \\
&= \frac{1}{2\pi}[(e^{2\gamma\beta} + e^{-2\gamma\beta})\sum_{n=-\infty}^{\infty}\frac{1}{n^2+\beta^2} - 2\sum_{n=-\infty}^{\infty}\frac{\cos 2\gamma n}{n^2+\beta^2}].
\end{aligned}$$

Combining the above equalities with (4.2) and (4.3), we get that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |f_1(\lambda_n)|^2 &= \frac{1}{2\pi}[\frac{\pi}{\beta}\frac{(e^{2\gamma\beta} + e^{-2\gamma\beta})(e^{\pi\beta} + e^{-\pi beta})}{e^{\pi\beta} - e^{-\pi\beta}} - \frac{2\pi}{\beta}\frac{e^{(\pi-2\gamma)\beta} + e^{-(\pi-2\gamma)\beta}}{e^{\pi\beta} - e^{-\pi\beta}}] \\
&= \frac{1}{2\beta}\frac{e^{2\gamma\beta+\pi\beta} - e^{2\gamma\beta-\pi\beta} - e^{-2\gamma\beta+\pi\beta} + e^{-2\gamma\beta-\pi\beta}}{e^{\pi\beta} - e^{-\pi\beta}} \\
&= \frac{e^{2\gamma\beta} - e^{-2\gamma\beta}}{2\beta}.
\end{aligned}$$

By Plancherel's theorem,

$$\int_{-\infty}^{\infty} |f_1(x)|^2 dx = \int_{-\gamma}^{\gamma} |g_1(t)|^2 dt = 2\gamma,$$

so we get that

$$\frac{\sum_{-\infty}^{\infty} |f_1(\lambda_n)|^2}{\int_{-\infty}^{\infty} |f_1(x)|^2 dx} = \frac{e^{2\gamma\beta} - e^{-2\gamma\beta}}{4\gamma\beta} = B_\beta.$$

So if in the upper bound of Corollary 4.3, γ is replaced by $\gamma - \epsilon$ for any $\epsilon > 0$, then the above B_β should satisfy $B_\beta \le e^{2(\lambda-\epsilon)\beta}$ for any sufficiently large β. But this is obviously impossible.

Example 4.2. Suppose $g_2(t) = e^{s+it}$ $(s > 0)$, and

$$f_2(z) = (\frac{1}{2\pi})^{1/2} \int_{-\gamma}^{\gamma} g_2(t) e^{izt} dt.$$

Then

$$f_2(z) = (\frac{1}{2\pi})^{1/2} \frac{e^{\gamma(s+(1+z)i)} - e^{-\gamma(s+(1+z)i)}}{s+(1+z)i}.$$

From $z = \lambda_n = n + i\beta$, then it follows that

$$\sum_{-\infty}^{\infty} |f_2(\lambda_n)|^2 = (\frac{1}{2\pi})^{1/2} \frac{|e^{-\gamma(\beta-s)+(1+n)\gamma i} - e^{\gamma(\beta-s)-(1+n)\gamma i}|^2}{|-(\beta-s)+(1+n)i|^2}.$$

Actually the above equality is the same as that in Example 4.1 except that β is replaced by $\beta - s$, so we obtain that

$$\sum_{-\infty}^{\infty} |f_2(\lambda_n)|^2 = \frac{e^{2\gamma(\beta-s)} - e^{-2\gamma(\beta-s)}}{2(\beta-s)}.$$

It follows that $\sum_{-\infty}^{\infty} |f_2(\lambda_n)|^2$ tends to 2γ when s tends to β. On the other hand,

$$\begin{aligned} \int_{-\infty}^{\infty} |f_2(x)|^2 dx &= \int_{-\gamma}^{\gamma} |g_2(t)|^2 dt = \int_{-\gamma}^{\gamma} e^{2st} dt \\ &= \frac{1}{2s}\{e^{2s\gamma} - e^{-2s\gamma}\}. \end{aligned}$$

So it follows that

$$\frac{\sum_{-\infty}^{\infty} |f_2(\lambda_n)|^2}{\int_{-\infty}^{\infty} |f_2(x)|^2 dx} \to (\frac{4\gamma\beta}{1 - e^{-4\gamma\beta}}) e^{-2\gamma\beta}.$$

Since the right hand side is less than $e^{-2(\gamma-\epsilon)\beta}$ for any given ϵ provided β is sufficiently large, the lower bound $e^{-2\gamma\beta}$ can not be replaced by $e^{-2(\gamma-\epsilon)\beta}$ for any $\epsilon > 0$.

4.7 Two reconstruction methods from frames

As we mentioned before, the concept of frames was first introduced in the paper of Duffin and Schauffer in 1952, but for the next 30 years, there were only a few papers discussing its properties. The concept of frames did not become prosperous until the 1980's when people found they played a role in wavelets analysis, where a wavelet frame can be used to reconstruct a function.

In this section, we introduce two important methods to reconstruct a function, and show how the methods are related to frame bounds. Also we will discuss the possibility of their applications to delay-differential equations in the next chapter.

Suppose that $\{\phi_j\}$ is a frame in a separable Hilbert space H. Define the frame operator S by

$$Sf = \sum_J \langle f, \varphi_j \rangle \varphi_j.$$

Then S is a bounded invertible operator. It is also self-adjoint (see Young 1980, p185).

Now if $\{\phi_j\}$ is a tight frame with constant A, it can be seen from [Hernandez, p334] that

$$f = \frac{1}{A} \sum_{j \in J} \langle f, \varphi_j \rangle \varphi_j$$

converges in H.

For a general frame $\{\varphi_j\}$, we define $\langle g, f \rangle_\# = \langle S^{-1} g, f \rangle$ for $f, g \in H$, then as shown in [Hernandez, p400],

$$\sum_j |\langle f, \varphi_j \rangle_\#|^2 = \|f\|_\#^2.$$

and $AI \le S \le BI$.

So $\{\varphi_j : j \in J\}$ is a tight frame with frame constant 1 if we use the inner product $\langle \cdot, \cdot \rangle_{\#}$. Thus for all $f \in H$, we have

$$f = \sum_{j \in J} \langle f, \varphi_j \rangle_{\#} \varphi_j = \sum_{j \in J} \langle f, \tilde{\varphi}_j \rangle \varphi_j$$

or

$$f = \sum_{J} \langle f, \varphi_j \rangle \tilde{\varphi}_j,$$

where $\tilde{\varphi}_j = S^{-1}\varphi_j$.

But to reconstruct the function f, sometimes it is difficult to compute the elements $\tilde{\varphi}_j$ of this dual frame. So we turn to Duffin's approach which reconstructs the function up to a small error.

Suppose B is close to A. Since $AI \le S \le BI$, we may assume $S \sim (\frac{A+B}{2})I$, then $S^{-1} \sim \frac{2}{A+B}I$. Hence,

$$\tilde{\varphi}_j = S^{-1}\varphi_j \sim \frac{2}{A+B}\varphi_j.$$

Let us write

$$f = \frac{2}{A+B}\sum_{J} \langle f, \varphi_j \rangle \varphi_j + R_1 f$$

where $R_1 = I - \frac{2}{A+B}S$. Then it follows that

$$-\frac{B-A}{A+B}I \le R_1 \le \frac{B-A}{A+B}I$$

and

$$\|R_1\| \le \frac{B-A}{A+B} = \frac{r}{r+2}$$

where $r = \frac{B-A}{A}$. Iterating the above procedure, we get

$$R_1 f = \frac{2}{A+B}\sum_J \langle R_1 f, \varphi_j\rangle \varphi_j + R_1(R_1 f)$$

where $\|R_1(R_1 f)\| \le (\frac{r}{r+2})^2 \|f\|_2$. So, after k iterations, the approximation error is smaller than $(\frac{r}{r+2})^k \|f\|_2$ in $L^2(R)$.

Note that Duffin's method is good only for tight frames or nearly tight frames. But in many cases the upper and lower bounds are not close to each other (refer to Theorem 4.8, 4.9 and 4.13). In these situations, a projection method can be employed that is due to Christensen(1993):

Suppose I is a countable index set and $\{I_n\}_{n=1}^{\infty}$ is a family of finite subsets of I such that $I_1 \subseteq I_2 \subseteq ... \subseteq I_n \nearrow I$. Given a family $\{\phi_i\}_{i\in I} \subseteq H$ we define

$$H_n := \text{span}\{\phi_i\}_{i\in I_n}.$$

Note that $\{\phi_i\}_{i\in I_n}$ is a frame for H_n. The frame operator corresponding to $\{\phi_i\}_{i\in I_n}$ is

$$S_n : H_n \to H_n, \quad S_n f = \sum_{i\in I_n} \langle f, \phi_i\rangle \phi_i. \tag{4.4}$$

The the orthogonal projection on H_n then satisfies that

$$P_n f = \sum_{i\in I_n} \langle f, S_n^{-1}\phi_i\rangle \phi_i, \quad f \in H,$$

which will tend to f as $n \to \infty$.

We say that the projection method works if $\langle f, S_n^{-1}\phi_i\rangle \to c_i$ as $n \to \infty$. Christensen proved that

Theorem 4.15

1). *Let $\{\phi_i\}$ be a frame in a Hilbert space H. Then the projection method works if and only if $\|S_n^{-1}\phi_j\| \leq c_j$ for any frame $\{\phi_i\}$ where c_j is constant.*

2). *Also the projection method works for any Schauder basis $\{\phi_i\}_{i=1}^{\infty}$.*

It is interesting that the coefficients obtained from Theorem 4.15 can be used to determine whether a given system $\{\phi_i\}$ is a basis. If for some i, the coefficients $c_i^n = \langle f, S_n^{-1}\phi_i\rangle$ do not tend to some constant c_i, then $\{\phi_i\}$ is not a basis. This fact will be employed in the next chapter.

Furthermore, we will employ this method to approximate the solution of delay-differential equations.

Chapter 5

Approximate solutions of delay-differential equations

5.1 Introduction

This part is to study the approximate solution of some differential-difference equations with constant coefficients. The method developed here represents the approximate solution as a finite sum of non-harmonic exponentials. We could choose the coefficients of this expansion to be the exact residues obtained, for example, by Laplace transform [Wright, 1949], and this would be optimal for large t. But in many applications, it is more desirable to have a good approximate solution for times comparable with a few multiples of the delay times ω_i (see equation (5.1)). In this case, it can be shown that choosing the coefficients of the expansion by some other methods, as in this chapter, can produce better results on this time scale, and reduce unwanted Gibbs-phenomenon-like oscillations.

In order to explain these results, we first introduce the appropriate background material and notation about these kinds of equations and then discuss some reconstruction properties of their solutions. We would also collect some possible approximation methods and implement them with a concrete example in Maple. Finally, a theoretical proof shows that the solution system of a delay-differential equation does not form a basis in L^2.

5.2 Differential-difference equations

The general linear differential-difference equation with constant coefficients and delay is of the form

$$\sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}y^{(j)}(t+\omega_i) = f(t) \tag{5.1}$$

where m and n are positive integers, where $0 = \omega_0 < \omega_1 < ... < \omega_m$, and where $f(t)$ is defined in some interval of the real t-axis. The characteristic function $h(z)$ of (5.1) is defined by

$$h(z) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}z^j e^{\omega_i z}. \tag{5.2}$$

The constants m and n are called the differential and difference order of the equation, respectively.

In the computation of the solution of the equations mentioned above, the location of the zeros of the characteristic functions plays a key role. A typical characteristic function has been thoroughly discussed in Chapter 1. Here we review some general characteristic functions and the asymptotic properties of their roots.

Suppose a characteristic function $h(s)$ is

$$h(s) = \sum_{j=0}^{n} p_j(s) e^{\beta_j s}, \qquad 0 = \beta_0 < \beta_1 < \dots < \beta_n \tag{5.3}$$

where $p_j(s)$ is a polynomial of degree m_j and $\beta_j \in R$. Then we can write

$$h(s) = \sum_{j=0}^{n} q_j [1 + \epsilon(s)] s^{m_j} e^{\beta_j s}$$

where $q_j \neq 0$ $(j = 0, \dots, n)$ are constants and $\epsilon(s) = O(|s|^{-1})$ as $|s| \to \infty$.

Theorem 5.1[R. Bellman, p.409]. *Suppose $h(z)$ is defined by (5.3), and the m_j are integers such that $m_j = m\beta_j$, $j = 0, 1, \dots, n$ for some real constant m.*

(1) If $m = 0$, there exist positive numbers c_1 and c_2 such that all zeros of large modulus lie a strip $|\Re(s)| \leq c_1$. In any rectangle $|\Re(s)| \leq c_1$, $|\Im(s) - a| \leq b$, in which $|s|$ remains sufficiently large, and on the boundary of which $h(s)$ has no zeros, the number $n(R)$, of zeros of $h(s)$ satisfies the inequalities

$$-n + (b/\pi)(\beta_n - \beta_0) \leq n(R) \leq n + (b/\pi)(\beta_n - \beta_0).$$

(2) If $m \neq 0$, the zeros of $h(z)$ are asymptotic to those of the comparison function

$$g_1(s) = \sum_{j=0}^{n} p_j s^{m_j} e^{\beta_j s}.$$

The roots of large modulus have the form

$$\begin{aligned} s &= m(\log|w| - \log|2r\pi m + m \arg w - (\frac{m\pi}{2})|) \\ &+ im(2r\pi + \arg w - \frac{\pi}{2}) + o(1), \end{aligned}$$

as $r \to \infty$, where w is a complex root of the polynomial $\sum_{j=0}^{n} p_j z^{m_j}$.

From the asymptotic properties of these roots, we see that in the case $m = 0$ the solution system is similar to Fourier series in that the real parts of the roots are uniformly bounded and in the case $m \neq 0$ the solution system cannot be a frame or Riesz basis (see Theorem 4.5 and Theorem 4.6). But by the method employed in Chapter 3, the property of completeness can be exploited. However in this thesis, we do not continue on that way.

Next we consider a special case of (5.1). Set $m = n = 1$, and let $\omega \geq 0$. Then (5.1) reduces to

$$a_0 u'(t) + a_1 u'(t-\omega) + b_0 u(t) + b_1 u(t-\omega) = f(t). \tag{5.4}$$

Definition 5.1. *An equation of the form (5.4) is said to be of delay type if* $a_0 \neq 0$ *and* $a_1 = 0$. *It is said to be of neutral type if* $a_0 \neq 0$ *and* $a_1 \neq 0$. *It is said to be of advanced type if* $a_0 = 0$ *and* $a_1 \neq 0$.

Since t usually represents time in application, we are mainly interested in continuing a solution in the direction of increasing t and focus on the equations of delay type.

5.3 Series expansions of delay-differential equations

In the case of differential equations, it is often possible to build up a solution as a sum of simple exponential solutions. A similar case happens in differential-difference equations. Set

$$\mathrm{L}(u) = a_0 u'(t) + b_0 u(t) + b_1 u(t-\omega), \tag{5.5}$$

then $\mathrm{L}(e^{st}) = h(s)e^{st}$ with the characteristic function $h(s)$ given by

$$h(s) = a_0 s + b_0 + b_1 e^{-s\omega}. \tag{5.6}$$

Let the initial condition of (5.5) be $u(t) = g(t)$, $0 \le t \le \omega$. Set

$$p(s) = a_0 g(0) + \int_0^{\omega} [a_0 g'(t_1) + b_0 g(t_1)] e^{-st_1} dt_1$$

$$p_0(s) = a_0 g(0) e^{-\omega s} + b_1 e^{-\omega s} \int_0^{\omega} g(t_1) e^{-st_1} dt_1.$$

Theorem 5.2[R.Bellman p.124]. *Let $u(t)$ be the continuous solution of the equation $L = 0$ with initial condition $g(t)$ where $u(t) = g(t)$, $0 \le t \le \omega$. Assume $a_0 b_1 \ne 0$, $g(t)$ is $C^1[0,\omega]$, and $\{s_r\}$ is the collection of characteristic roots. Then*

$$u(t) = \lim_{i \to \infty} \sum_{|s_r| \le i} p_r(t) e^{s_r t} = \sum_1^{\infty} p_r(t) e^{s_r t} \quad t > 0, \tag{5.7}$$

where $p_r(t)e^{s_r t}$ is the residue of $\frac{p(s)e^{st}}{h(s)}$ at s_r. The series converges uniformly in any finite interval $[t_0, t_0']$, $t_0 > 0$. If all characteristic roots lie in a half-plane $\Re(s) \le c_1 < 0$, the series converges uniformly in $[t_0, \infty]$. If $g(t)$ is merely $C^0(0,\omega)$, the above series expansion holds for $t > \omega$ and $p_r(t)e^{s_r t}$ is the residue of $\frac{p_0(s)e^{st}}{h(s)}$.

Now suppose that we are given an arbitary function $g(t)$ of class C^1 on some interval $[0,\omega]$. As in Chapter 4 we are going to find a series expansion of $g(t)$ in term of the zeros of a function $h(s)$ of the form $h(s) = a_0 s + b_0 + b_1 e^{-\omega s}$, $a_0 b_1 \ne 0$.

To do this, we form the differential-difference equation corresponding to $h(s)$, regarding $g(t)$ as the initial function. This initial function can be continued to a solution $u(t)$ for $t > 0$. Thus we have

Corollary 5.1[R.Bellman]. *Let $g(t)$ be a given function of class $C^1(0,\omega)$, and let*

$h(s)$ be a given function of the form above. Then for $0 < t < \omega$

$$g(t) = \sum_{r=1}^{\infty} p_r(t) e^{s_r t},$$

with uniform convergence in $[t_0, \omega]$ for $t_0 > 0$, where the notation is the same as in Theorem 5.2.

For example, suppose $g(t)$ is $C^1(0, 2\pi)$. We expand it in the form $g(t) = \sum_{k=-\infty}^{\infty} p_k(t) e^{s_k t}$ where the s_k are the roots of $h(s) = s - ae^{-2\pi s}$. The corresponding differential-difference equation is $u'(t) - au(t - 2\pi) = 0$. When $a \neq -\frac{1}{e}$, the zeros s_k of $h(s)$ are simple, and $s_k = \frac{1}{2\pi} W_k(2\pi a)$. So the required expansion is

$$g(t) = \sum_{k=-\infty}^{\infty} a_k e^{s_k t} = \sum_{k=-\infty}^{\infty} (a_k e^{\Re(s_k) t}) e^{i \rho_k t}$$

where ρ_k is defined as in Section 1.4 and

$$a_k = \frac{p(s_k)}{h'(s_k)} = \frac{g(2\pi) e^{-2\pi s_k} + s_k \int_0^{2\pi} g(t) e^{-s_k t} dt}{1 + 2\pi e^{-2\pi s_k}}.$$

In view of Theorem 3.1, we see that in $C^1(0, 2\pi)$ the sequence $\{\rho_n\}$ can be replaced by some sequence $\{\rho_n + ih_n\}$ with $h_n \to \infty$. Actually, Verblunsky has discussed it in L^1.

Theorem 5.3[Verblunsky, 1961]. *Suppose $f \in L^1(0, 1)$, $c \in C$ and the Fourier series of f converges to σ at the point x in the interval $(0, 1)$. Set z_k be the zeros of $ze^z - a$, and*

$$c_k = \frac{z_k}{1 + z_k} \int_0^1 f(t) e^{-z_k t} dt.$$

Then $\sum_{-N}^{N} c_k e^{z_k x} \to \sigma$ provided that $(\log \frac{1}{t})^2 \int_0^t \phi(u) du = o(1)$ as $t \to 0$ where $\phi(t) = f(x + t) - f(x - t)$.

Since the zeros z_k of $ze^z - a$ are the characteristic roots of $u'(t) + cu(t-1) = 0$, and since $f \in C^0(0,1)$ is summable in the term $\{e^{z_r x}\}$ by Theorem 5.3, Corollary 5.1 can be extended to $C^0(0,1)$.

The theorems above suggest that the expansion coefficients can be obtained from the residue of some meromorphic functions. Is the system $\{e^{z_k t}\}$ a Riesz-Fischer sequence? i.e. for any square summable sequence of scalars $\{c_k\}$ does there exist an element ϕ in $L^2(0,1)$ such that

$$\int_0^1 \phi(t) e^{z_k t} dt = c_k, \qquad (n = 1, 2 \ldots)?$$

Proposition 5.1. *Let $\{z_k\}$ be the zeros of $ze^z - a$, $a \in R$. Then the system $\{e^{z_k t}\}$ is not a Riesz-Fischer sequence in $L^2(0,1)$.*

Proof: By a theorem of Boas [1941], who showed that the moment problem $\langle f, f_k \rangle = c_k$ $(k = 1, 2, \ldots)$ is solvable with f in $L^2(0,1)$ for every square summable sequence of scalars $\{c_k\}$ if and only if the inequality

$$m \sum |a_n|^2 \leq \| \sum a_n f_n \|^2 \tag{5.8}$$

is valid for some positive constant m and all finite sequences of scalars $\{a_n\}$, we only need to show that (5.8) fails.

Take $a_n = 1$ if $n = N$ and 0 otherwise, and $f_n(t) = e^{z_n t}$. Then

$$\| \sum a_n f_n \|^2 = \int_0^1 |e^{z_N t}|^2 dt = \frac{1}{2\Re z_N}(e^{2\Re z_N} - 1).$$

Since by Proposition 1.1, $\Re z_N = O(-\log N)$, we get $\| \sum a_n f_n \|^2 \to 0$ which does not satisfy (5.8).

Definition 5.2 *A basis $\{f_n\}$ in a Hilbert space H is a Bessel basis if $\sum_{n=1}^{\infty} c_n f_n$ is convergent in H only if $\sum_{n=1}^{\infty} |c_n|^2 < \infty$.*

Definition 5.3 *A sequence $\{f_n\}$ in a Hilbert space H is said to be a Bessel sequence if*

$$\sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 < \infty$$

for every element $f \in H$.

From [Young, 1980 p37], we see that a basis $\{f_n\}$ is a Bessel basis if and only if there exist a constant A such that

$$A \sum_{i=1}^{n} |c_i|^2 \leq \| \sum_{i=1}^{n} c_i f_i \|^2$$

for arbitrary scalars $c_1, c_2, ..., c_n$ $(n = 1, 2, ...)$, i.e. $\{f_n\}$ is a Riesz-Fischer sequence. So we immediately get that

Corollary 5.2. *Let $\{z_k\}$ be the zeros of $ze^z - a$, $a \in R$. Then the system $\{e^{z_k t}\}$ is not a Bessel basis in $L^2(0, 1)$.*

Proposition 5.2. *Let $\{z_k\}$ be the zeros of $ze^z - a$, $a \in R$. Then the system $\{e^{z_k t}\}$ is a Bessel sequence in $L^2(0, 1)$.*

Proof: Note that $\Re z_k \to -\infty$ as $|k| \to \infty$, so we have

$$\|e^{z_k t}\| = \frac{e^{2\Re z_k} - 1}{2\Re z_k} \to 0$$

as $|k| \to \infty$. Then there exists M such that

$$\| \sum c_n e^{z_n t} \| \leq \sum |c_n|^2 \|e^{z_n t}\| \leq M \sum |c_n|^2.$$

Thus Proposition 5.2 comes from Theorem 3 in [Young 1980, p.155].

From Proposition 5.1 and 5.2, we see that the moment space (see [Young, 1980, p.146]) of $\{e^{z_k t}\}$ is a strict subspace of l^2. It implies that for some coefficient sequences $\{c_i\}$ in l^2, we can't find a solution f in $L^2(0,1)$.

5.4 Finite optimal solutions in $L^2(0,1)$

Now suppose that s_r are the roots of the equation $z - ae^{-z} = 0$. For a nice function $f \in L^2(0,1)$, the above theorems show that $|f(x) - \sum_{i=-N+1}^{N} c_i e^{s_i x}|$ can be small enough for sufficiently large N. But more terms will be required when $a \to 0$ even for a nice function like $f(x) = \sin x$. Since for a numerical solution it is not necessary to keep the coefficients c_i fixed for all N, we are interested here in finding the optimal coefficients c_i^N so that the approximation is best possible at step N. From Propositions 2.4 and 2.5, we know that the system $\{e^{s_k t}\}$ is complete, so such an approximation is definitely possible even for the characteristic equation having one multiple root, that is, when $a = -\frac{1}{e}$.

Now a further question is how to get the optimal coefficients c_i^N? We first give a projection method which is analogous to Theorem 4.16 for computing the frames's coefficients. Since we are only interested in the approximation property (not representation), the strong condition like frames or basis can be reduced:

Theorem 5.4. *Suppose $\{e^{i\lambda_n t}\}$ is complete in $L^2(-\pi,\pi)$, and S_n is defined by*

(4.4). *Then for any f in $L^2(-\pi,\pi)$, we have*

$$\lim_{n\to\infty} \|f - \sum_{j=1}^{n} \langle f, S_n^{-1}\phi_j\rangle \phi_j\| = 0.$$

Proof: Set $\phi_j = e^{i\lambda_j}$, then all the ϕ_j are independent in $L^2(-\pi,\pi)$. Define the subspace $H_n \subset L^2(-\pi,\pi)$ by $H_n := \text{span}\{\phi_j\}_{i=1}^{n}$. Then, from the assumption on $\{\phi_j\}$, we know that $\{\phi_j\}_{j=1}^{n}$ is a basis in H_n. Since it can be obtained from an orthonormal basis by means of a bounded invertible operator, $\{\phi_j\}_{j=1}^{n}$ is a Riesz basis for H_n.

Suppose the operator S_n is defined by (4.4) in H_n. Then from the discussion in Section 4.7, for any $g \in S_n$, we have that

$$g = \sum_{j=1}^{n} \langle g, S_n^{-1}\phi_j\rangle \phi_j.$$

Suppose P_n is the projection operator from H to H_n. Then

$$P_n f = \sum_{j=1}^{n} \langle P_n f, S_n^{-1}\phi_j\rangle \phi_j = \sum_{j=1}^{n} \langle f, S_n^{-1}\phi_j\rangle \phi_j.$$

Since $\|f - P_n f\| \le \|f - \sum_{j=1}^{n} c_j\phi_j\|$ for any scalars c_j, we use the completeness of $\{\phi_j\}$ to get $\|f - P_n f\| \to 0$ as $n \to \infty$. This ends the proof.

The above projection method is easy to implement for the numerical solution of delay-differential equations. There are also other methods: one is by the direct derivative, and another is by the Gram-Schmidt process.

As an illustration of these methods, we consider the equation

$$y'(t) = ay(t-1) \tag{5.9}$$

with the initial condition $g(t) = \sin \pi t$ for $0 \le t \le 1$.

1) We first discuss the pointwise convergence with the residue method:

Since all the characteristic roots z_k of $h(z) = z - ae^{-z}$ are simple except when $a = -\frac{1}{e}$, therefore by Theorem 5.2 we have

$$\begin{aligned} c_k e^{z_k t} &= \text{Residue of } \frac{p(z)e^{zt}}{h(z)} \quad \text{at} \quad z = z_k \\ &= \{\frac{\pi}{1 + ae^{-z_k}} \int_0^1 (\cos \pi s) e^{-z_k s} ds\} e^{z_k t}. \end{aligned}$$

A simple calculation shows that

$$\int_0^1 (\cos \pi s) e^{-z_k s} ds = \frac{z_k(z_k + a)}{a(z_k^2 + \pi^2)}$$

So, for $k \in Z$,

$$c_k = \frac{\pi}{1 + z_k} \frac{z_k(z_k + a)}{a(\pi^2 + z_k^2)}.$$

In File 1, numerical solution of (5.9) shows that the performance of this method is poor when $|a|$ is close to zero.

The approximation method discussed next will be implemented in L^2. Note that that $\{e^{z_k t}\}_{k=-\infty}^{\infty}$ is overcomplete with excess $E_2 = 1$ in $L^2(0,1)$, so removing one term $e^{z_0 t}$ from the system is possible. On the other hand, when $a = -\frac{1}{e}$, there is a double root $z_0 = z_1 = -1$. So in this case, we will remove one term $e^{z_0 t}$ from the system in the following process. Otherwise it may cause singularity problems or extra computations.

2) The projection method:

Since $\{e^{z_k t}\}_{k=1}^{\infty}$ is complete in $L^2(0,1)$, Theorem 5.4 can be applied. If T_N is defined by $T_N f = \sum_{i=1-N}^{N} \langle f, e^{z_i s}\rangle e^{z_i t}$, then the corresponding matrix A can be written as

$$A = (a_{ij})_{N*N} \quad \text{with} \quad a_{ij} = \langle e^{z_i s}, e^{z_j s} \rangle.$$

So T_N^{-1} can be expressed with $B = A^{-1}$. It follows from Theorem 5.4 that

$$S_N(t) = \sum_{i=1}^{N} c_i e^{s_i t} \quad \text{with} \quad c_i = \langle \sin \pi t, T_N^{-1} e^{z_i t} \rangle.$$

where $T_N^{-1} e^{z_i t} = \sum_{j=1-N}^{N} b_{ij} e^{z_j t}$.

3) The derivative method:

Suppose $\vec{b} = (b_1, ..., b_n)$ is a vector in C^n. We reindex the characteristic roots by $s_k = z_{k-n}$. Then set

$$\begin{aligned} M(\vec{b}) &= \int_0^1 (\sum_{i=1}^{n} b_i e^{s_i t} - f(t)) \overline{(\sum_{i=1}^{n} b_i e^{s_i t} - f(t))} dt \\ &= \int_0^1 L\overline{L} dt \end{aligned}$$

To find the minimum value of $M(\vec{b})$, we find the partial derivatives of $M(\vec{b})$ with respect to the real and imaginary parts of b_j and set them equal to zero.

Then for each j, we have

$$\int_0^1 L e^{\bar{s}_j t} dt + \int_0^1 \bar{L} e^{s_j t} dt = 0 \quad \text{and} \quad \int_0^1 L e^{\bar{s}_j t} dt - \int_0^1 \bar{L} e^{s_j t} dt = 0.$$

So it suffices to solve $\int_0^1 L e^{\bar{s}_j t} dt = 0$, that is

$$\int_0^1 \sum_{1}^{n} b_k e^{(\bar{s}_j + s_k)t} = \int_0^1 f(t) e^{\bar{s}_j t} dt,$$

for $j = 1, ..., n$.

Substitute $f(t) = \sin \pi t$, then a simple calculation shows that

$$\sum_{k=1}^{n} \frac{|a|^2 - s_k \bar{s}_j}{s_k \bar{s}_j (s_k + \bar{s}_j)} b_k = \int_0^1 \sin \pi t e^{\bar{s}_j t} dt = \frac{\pi(\bar{a} + \bar{s}_j)}{\bar{s}_j(\bar{s}_j^2 + \bar{a}^2)}$$

for $j = 1, ..., n$.

Set $B = [b_1, ..., b_n]$, $C = [c_1, ..., c_n]$ with $c_j = \frac{\pi(\bar{a}+\bar{s}_j)}{\bar{s}_j(\bar{s}_j^2+\bar{a}^2)}$, and $A = [a_{jk}]_{2n \cdot 2n}$ where

$$a_{jk} = \frac{|a|^2 - s_k\bar{s}_j}{s_k\bar{s}_j(s_k + \bar{s}_j)} = \frac{|a|^2 - z_{k-n}\bar{z}_{j-n}}{z_{k-n}\bar{z}_{j-n}(z_{k-n} + \bar{z}_{j-n})}.$$

Then we get that $B = A^{-1}C$.

The numerical result in File 3 shows that this method provides a good approximation even when $|a|$ is close to zero. But each time N changes, the coefficients b_i change with N.

4) The Gram-Schmidt method:

Since $\{e^{z_k t}\}_{k=-\infty}^{\infty}$ is complete, we may construct recursively an orthonormal system $e_1(t)$, $e_2(t)$,..., $e_{2N}(t)$ by the following method:

Step one: Let $h_1(t) = e^{z_{1-N}t}$, $D_1 = (\int_0^1 |h_1(t)|^2 dt)^{\frac{1}{2}}$, and $e_1(t) = h_1(t)/D_1$.

Step two: For $k = 2, 3, ..., 2N$, let

$$h_k(t) = e^{z_{k-N}t} - \sum_{j=1}^{k}(\int_0^1 e^{z_{k-N}t}\bar{e}_j(t)dt)e_j(t)$$

$D_k = (\int_0^1 |h_k(t)|^2 dt)^{\frac{1}{2}}$ and $e_k(t) = h_k(t)/D_k$.

Step three: $S_{2N}(t) = \sum_{i=1}^{2N}\langle \sin \pi t, e_i(t)\rangle e_i(t)$.

The results in File 4 show that this method has the same effect as that in File 3, and all the coefficients change with N.

The methods mentioned above can be used to compute a finite optimal solution of delay-differential equations. There is an additional question about existence of a complex sequence $\{b_n\}$ such that $f(t) = \sum b_n e^{z_n t}$ in the norm $L^2(0,1)$? i.e. Does the coefficient c_n^N computed by projection methods converge to b_n as N go to infinity? Unfortunately, the numerical result in File 5 suggests the opposite.

5.5 The basis property for a special solution system

In this section, we discuss the basis property of the exponential system $\{e^{i\lambda_n t}\}$ with a complex sequence $\{\lambda_n\}$. In a few papers on this question, a general restriction [Avdonin, 1988] was imposed on the sequence by

$$\sup\{|\Im\lambda_n|\} < \infty.$$

Minkin [1992] explained the essence of the difficulties to remove the restriction.

Related to the above question, we have shown in Chapter 3 that the system $\{e^{\frac{1}{2\pi}W(k,-\frac{1}{e})t}\}_{k\in Z\setminus\{0\}}$ is complete in $L^2(-\pi,\pi)$ where $\sup_k \Re W(k,-\frac{1}{e}) = -\infty$, so it can be used to approximate any function in $L^2(-\pi,\pi)$. In this section, we will show that it is not a basis:

Lemma 5.1. *Suppose $V_n = V_n(-\frac{1}{e})$ is defined in Section 1.4, then the system $\{e^{i(V_n-\frac{1}{2\pi})t}\}_{n\in Z}$ is not a basis for $L^2(-\pi,\pi)$.*

Proof: Recall that the inner product of two functions F and G in the Paley-Wiener space P is defined by

$$(F,G) = \int_{-\infty}^{\infty} F(x)\overline{G(x)}dx$$

and by the virtue of the Paley-Wiener theorem, the complex Fourier transform

$$f(t) \rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-izt}dt$$

is an isometric isomorphism from $L^2(-\pi,\pi)$ onto all of P.

Set $\lambda_n = V_n(-\frac{1}{e}) - \frac{i}{2\pi}$, and assume that $\{e^{i\lambda_n t}\}_{n\in Z}$ is a basis for $L^2(-\pi,\pi)$, so that we can write

$$\cos t = \sum_{n\in Z} c_n e^{i\lambda_n t} \tag{5.10}$$

in the sense of L^2. Furthermore the isomorphism of Fourier transform shows that

$$K_n(z) := \frac{1}{2\pi}\int_{\pi}^{\pi} e^{i\lambda_n t}e^{-izt}dt = \frac{\sin\pi(z-\lambda_n)}{\pi(z-\lambda_n)}$$

forms a basis for P. Let $\{g_n(z)\}$ be biorthogonal to $\{K_n\}$ in P, and write $\cos t = \frac{e^{it}+e^{-it}}{2}$. Applying the Fourier transform to both sides of (5.10), we get that

$$\frac{1}{2}\{\frac{\sin\pi(z-1)}{\pi(z-1)} + \frac{\sin\pi(z+1)}{\pi(z+1)}\} = \sum c_n K_n(z).$$

Remembering that $\frac{\sin\pi(z-\bar{w})}{\pi(z-\bar{w})}$ is the reproducing kernel for P, and after taking the inner product of each side with g_n, we get that

$$c_n = \frac{1}{2}\{\overline{g_n(1)} + \overline{g_n(-1)}\}.$$

Next, we try to find the explicit form for $\{g_n\}$. From Proposition 2.6, we see that

$$F(z) = \int_{-\pi}^{\pi} 2(t+\pi)e^{izt}dt = Be^{ikz} \prod_{n \in Z\setminus\{0\}} (1 - \frac{z}{V_n(-\frac{1}{e}) - \frac{i}{2\pi}})$$

which is obviously in the Paley-Wiener space (by Theorem 1.1).

Next set $G(z) = zF(z)$, and

$$F_n(z) = \frac{G(z)}{G'(\lambda_n)(z-\lambda_n)}.$$

Then from Young [1980, p127], each of the function $\{F_n(z)\}$ is in P. Note that

$$(F_n, K_m) = F_n(\lambda_m) = \delta_{nm},$$

so the system $\{F_n\}$ is also biorthogonal to $\{K_n\}$ in P. But $\{K_n\}$ is complete, so it has a unique biorthogonal sequence. It follows that $F_n = g_n$.

From Proposition 2.6, we know that $\overline{f(1)} = f(-1) = -4\pi i$ and $\overline{f(-1)} = 4\pi i$, so

$$\overline{G(1)} = \overline{G(-1)} = -4\pi i.$$

It follows that

$$\begin{aligned} c_n &= \frac{1}{2}\frac{1}{\overline{G'(\lambda_n)}}\{\frac{\overline{G(1)}}{1-\overline{\lambda_n}} + \frac{\overline{G(-1)}}{-1-\overline{\lambda_n}}\} \\ &= \frac{-2\pi i}{\overline{G'(\lambda_n)}}\{\frac{1}{1-\overline{\lambda_n}} + \frac{1}{-1-\overline{\lambda_n}}\} \\ &= \frac{4\pi i}{\overline{\lambda_n G'(\lambda_n)}}\frac{\overline{\lambda_n}^2}{\overline{\lambda_n}^2 - 1}. \end{aligned}$$

A simple calculation (with the aid of Maple) shows that

$$zG'(z) = \frac{2}{z}[i\pi z e^{i\pi z} - e^{i\pi z} + e^{-i\pi z}] - 2[-2\pi^2 z e^{i\pi z} - i\pi e^{-i\pi z}]. \tag{5.11}$$

Recalling that $\lambda_n = V_n(-\frac{1}{e}) - \frac{i}{2\pi}$, then from Lemma 1.2 we have that

$$\begin{aligned} i\pi\lambda_n &= \frac{1}{2}[-\log|n| - \log 2\pi + o(1)] + i\pi\Re\lambda_n \\ &= \frac{1}{2}\{[-\log|n| - \log 2\pi + o(1)] + 2\pi i[n + \frac{1}{4}\text{sign}(n) + o(1)]\}. \end{aligned}$$

Evaluating (5.11) at $z = \lambda_n$, then for sufficiently large N, when $n > N$, we have the following estimate:

$$\begin{aligned} \lambda_n G'(\lambda_n) &= O(\frac{1}{\sqrt{n}}) - 2[-2\pi^2\lambda_n e^{i\pi\lambda_n} - i\pi e^{-i\pi\lambda_n}] \\ &= -2\pi i\Big\{2(i\pi\lambda_n)e^{i\pi\lambda_n} - e^{-i\pi\lambda_n}\Big\{ + o(1) \\ &= -2\pi i\Big\{((-\log|n| - \log 2\pi + o(1)) + 2\pi i(k + \frac{1}{4}\text{sign}(n) + o(1)) \\ &\quad e^{\frac{1}{2}[-\log|n| - \log 2\pi + o(1)] + i\pi\Re\lambda_n} - e^{-\frac{1}{2}[-\log|n| - \log 2\pi + o(1)] - i\pi\Re\lambda_n}\Big\} \\ &= -2\pi i\Big\{(\epsilon_n^1 + i\theta_n^1\sqrt{2\pi n})e^{i\pi\Re\lambda_n} - \theta_n^2\sqrt{2\pi n}e^{-i\pi\Re\lambda_n}\Big\} \\ &= -(2\pi)^{\frac{3}{2}}\sqrt{n}i\Big\{(\epsilon_n^2 + i\theta_n^1)(\cos\pi\Re\lambda_n + i\sin\pi\Re\lambda_n) \\ &\quad -\theta_n^2(\cos\pi\Re\lambda_n - i\sin\pi\Re\lambda_n)\Big\} \end{aligned}$$

where ϵ_n^i and θ_n^i are real and satisfy that $\epsilon_n^i \to 0$, $\theta_n^i \to 1$, $i = 1, 2$, as $|n| \to \infty$.

Furthermore since $\Re\lambda_n = n + \frac{1}{4} + o(1)$ for large $n > 0$, then

$$\sin\pi\Re\lambda_n = (-1)^n\frac{1}{\sqrt{2}} + o(1) \quad \text{and} \quad \cos\pi\Re\lambda_n = (-1)^n\frac{1}{\sqrt{2}} + o(1)).$$

Thus for sufficient large $n > 0$, we have that

$$\begin{aligned} \lambda_n G'(\lambda_n) &= -i(2\pi)^{\frac{3}{2}}(-1)^n\sqrt{n}[i\theta_n^1(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} + o(1)) \\ &\quad -\theta_n^2(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} + o(1))] \\ &= (2\pi)^{\frac{3}{2}}(-1)^n\sqrt{n}[\frac{\theta_n^1}{\sqrt{2}} + \frac{\theta_n^1}{\sqrt{2}}i + \frac{\theta_n^2}{\sqrt{2}} + \frac{\theta_n^2}{\sqrt{2}}i + o(1)] \\ &= 4\pi^{3/2}(-1)^n\sqrt{n}(1 + i)(1 + o(1)). \end{aligned}$$

It follows that

$$\begin{aligned} \frac{4\pi i}{\overline{\lambda_n G'(\lambda_n)}} &= \frac{(-1)^n i}{\sqrt{\pi n}} \frac{1}{(1-i)(1+o(1))} \\ &= \frac{(-1)^n i}{2\sqrt{\pi n}}(1+i)(1+o(1)) \\ &= \{\frac{(-1)^{n+1}}{2\sqrt{2\pi}} + i\frac{(-1)^n}{2\sqrt{2\pi}}\}(1+o(1)). \end{aligned}$$

So

$$\begin{aligned} c_n &= \frac{4\pi i}{\overline{\lambda_n G'(\lambda_n)}}(1 + \frac{1}{\overline{\lambda_n}^2 - 1}) \\ &= (\frac{(-1)^{n+1}}{2\sqrt{2\pi}} + i\frac{(-1)^n}{2\sqrt{2\pi}})(1+o(1)). \end{aligned}$$

If we set $c_n = a_n + ib_n$ for $n > 0$, then

$$a_n = \frac{(-1)^{n+1}}{2\sqrt{\pi n}}(1+o(1)) \quad \text{and} \quad b_n = \frac{(-1)^n}{2\sqrt{\pi n}}(1+o(1)). \tag{5.12}$$

Recall that $\overline{V_n} = -V_{-n}$ and $\lambda_n = V_n - \frac{i}{2\pi}$, then we have that $\overline{\lambda_n} = -\lambda_{-n}$.

Since

$$G'(z) = \int_{-\pi}^{\pi} e^{izt} d(t+\pi)^2 + z\int_{-\pi}^{\pi} ite^{izt} d(t+\pi)^2,$$

then it is easy to check that

$$\overline{G'(\lambda_n)} = G'(\lambda_{-n}).$$

Thus

$$\begin{aligned} \overline{c_{-n}} &= \frac{-4\pi i}{\lambda_{-n} G'(\lambda_{-n})} \frac{\lambda_{-n}^2}{\lambda_{-n}^2 - 1} \\ &= \frac{-4\pi i}{-\overline{\lambda_n G'(\lambda_n)}} \frac{\overline{\lambda_n}^2}{\overline{\lambda_n}^2 - 1} = c_n. \end{aligned}$$

So $c_{-n} = \overline{c_n} = a_n - ib_n$. Since $\Im V_n = \Im V_{-n}$ and $\Re V_n = -\Re V_{-n}$, it follows that

$$\begin{aligned}
c_n e^{i\lambda_n t} + c_{-n} e^{i\lambda_{-n} t} &= (a_n + ib_n)e^{i(V_n - \frac{i}{2\pi})t} + (a_n - ib_n)e^{i(V_{-n} - \frac{i}{2\pi})t} \\
&= (a_n + ib_n)e^{i(\Re V_n)t - (\Im V_n - \frac{1}{2\pi})t} \\
&\quad + (a_n - ib_n)e^{-i(\Re V_n)t - (\Im V_n - \frac{1}{2\pi})t} \\
&= e^{-(\Im V_n - \frac{1}{2\pi})t}[(a_n + ib_n)(\cos(\Re V_n)t + i\sin(\Re V_n)t) \\
&\quad + (a_n - ib_n)(\cos(\Re V_n)t - i\sin(\Re V_n)t)] \\
&= 2(a_n \cos(\Re V_n)t - b_n \sin(\Re V_n)t)e^{-(\Im V_n - \frac{1}{2\pi})t}
\end{aligned}$$

Using (5.12) to substitute for a_n and b_n, then we obtain

$$\begin{aligned}
c_n e^{i\lambda_n t} + c_{-n} e^{i\lambda_{-n} t} &= \frac{(-1)^{n+1}}{\sqrt{\pi n}}(\cos(\Re V_n)t + \sin(\Re V_n)t)(2\pi n)^{-\frac{t}{2\pi}}(1 + o(1)) \\
&= -\sqrt{2}(2\pi n)^{-\frac{1}{2} - \frac{t}{2\pi}}(\cos(n\pi + \Re V_n t) \\
&\quad + \sin(n\pi + \Re V_n t))(1 + o(1)) \\
&= -2(2\pi n)^{-\frac{1}{2} - \frac{t}{2\pi}} \sin(n\pi + \frac{\pi}{4} + \Re V_n t)(1 + o(1)) \\
&= -2(2\pi n)^{-\frac{1}{2} - \frac{t}{2\pi}} \sin((n + \frac{1}{4})(t + \pi) + o(1))(1 + o(1))
\end{aligned}$$

For sufficiently large $N > 0$, let t satisfy $\frac{\pi}{16} \le (2N+1)(t+\pi) \le \frac{\pi}{8}$, then

$$-\pi + \frac{\pi}{16(2N+1)} \le t \le -\pi + \frac{\pi}{8(2N+1)} < 0$$

and

$$\frac{\pi}{32} \le (n + \frac{1}{4})(\pi + t) \le \frac{\pi}{8}$$

for $N < n \le 2N+1$. Thus when $t \in (-\pi + \frac{\pi}{16(2N+1)}, -\pi + \frac{\pi}{8(2N+1)})$, we have

$$\sin((n + \frac{1}{4})(t + \pi) + o(1)) \ge \sin\frac{\pi}{32} = B_1.$$

Thus

$$\begin{aligned}
\| \sum_{|n|=N+1}^{2N} c_n e^{i\lambda_n t}\|^2 &\geq \| \sum_{|n|=N+1}^{2N} c_n e^{i\lambda_n t}\|^2_{L^2(-\pi+\frac{\pi}{16(2N+1)}, -\pi+\frac{\pi}{8(2N+1)})} \\
&= \| \sum_{n=N+1}^{2N} 2(2\pi n)^{-\frac{1}{2}-\frac{t}{2\pi}} \sin((n+\frac{1}{4})(t+\pi) \\
&\quad +o(1))(1+o(1))\|^2_{L^2(-\pi+\frac{\pi}{16(2N+1)}, -\pi+\frac{\pi}{8(2N+1)})} \\
&\geq \| \sum_{n=N+1}^{2N} 2(2\pi n)^{-\frac{1}{2}-\frac{t}{2\pi}} B_1\|^2_{L^2(-\pi+\frac{\pi}{16(2N+1)}, -\pi+\frac{\pi}{8(2N+1)})} \\
&\geq (\frac{2N}{\sqrt{4\pi N}} B_1)^2 \frac{\pi}{16(2N+1)} \\
&\geq B_2
\end{aligned}$$

for all large N. This shows that the expansion of $\cos t$ does not converge in $L^2(-\pi,\pi)$. So the system $\{e^{i(V_n-\frac{t}{2\pi})t}\}_{n\in Z}$ fails to be a basis.

Theorem 5.5 *The system* $\{e^{\frac{1}{2\pi}W_n(-\frac{1}{\epsilon})t}\}_{n\in Z\setminus\{0\}}$ *is not a basis in* $L^2(-\pi,\pi)$.

Proof: Note that $\{e^{iV_n t}\}_{n\in Z} = \{e^{\frac{1}{2\pi}W_n(-\frac{1}{\epsilon})t}\}_{n\in Z\setminus\{0\}}$, so it suffices to prove that $\{e^{iV_n t}\}_{n\in Z}$ is not a basis.

We prove it by contradiction. Take $f \in L^2(-\pi,\pi)$, then $f(t)e^{-\frac{t}{2\pi}} \in L^2(-\pi,\pi)$. Thus there exist $\{c_k\}$ such that

$$\int_{-\pi}^{\pi} |f(t)e^{-\frac{t}{2\pi}} - \sum_{k=-n}^{n} c_k e^{iV_k t}|^2 dt \to 0$$

as $n \to \infty$. So

$$\int_{-\pi}^{\pi} |f(t) - \sum_{k=-n}^{n} c_k e^{i(V_k-\frac{t}{2\pi})t}|^2 dt = \int_{-\pi}^{\pi} |f(t)e^{-\frac{t}{2\pi}} - \sum_{k=-n}^{n} c_n e^{iV_n t}|^2 e^{\frac{t}{\pi}} dt \to 0.$$

Thus $\{e^{i(V_n - \frac{i}{2\pi})t}\}_{n \in Z}$ is a basis which is a contradiction.

Further discussion: when $a \neq -\frac{1}{e}$, the numerical results in File 5 suggest that $\{e^{W(k,a)t}\}$ is not a basis in $L^2(0,1)$. We are not going to give a proof here.

Proof of Proposition 2.3:

We give a sketch of the proof of Proposition 2.3 with $\rho_n = \rho_n(a)$ $a < 0$. But this method also can be applied to the case of $a > 0$,

Set

$$F(z) = \prod{}'(1 - \frac{z}{\rho_n}).$$

Then from Theorem 3.4, $F(z)$ is in the Paley-Wiener space and is expressible in the form

$$F(z) = \int_{-\pi}^{\pi} \phi(t) e^{izt} dt$$

with $\phi(t) = \phi(-t)$ $a.e.$ in $[-\pi, \pi]$. So it follows that $F(1) = F(-1)$.

Set $G(z) = zF(z)$ and

$$G_n(z) = \frac{G(z)}{G'(\rho_n)(z - \rho_n)}.$$

Then $\overline{G(1)} = -\overline{G(-1)}$, and

$$\begin{aligned} G'(\rho_n) &= \int_{-\pi}^{\pi} \phi(t) e^{i\rho_n t} dt + \rho_n \int_{-\pi}^{\pi} it\phi(t) e^{i\rho_n t} dt \\ &= \int_{-\pi}^{\pi} \phi(-u) e^{i(-\rho_n)u} du + (-\rho_n) \int_{-\pi}^{\pi} iu\phi(-u) e^{i(-\rho_n)u} du \\ &= G'(-\rho_n) = G'(\rho_{-n}) \end{aligned}$$

The third equality follows from the fact that $\phi(t) = \phi(-t)$ *a.e.* in $(-\pi, \pi)$.

In the proof of Lemma 5.1, choose the test function $\sin t$ instead of $\cos t$, then we have

$$c_n = \frac{i\overline{G(1)}}{\rho_n \overline{G'(\rho_n)}} \frac{\rho_n^2}{\rho_n^2 - 1}.$$

So combining the symmetry of ρ_n and $G'_n(\rho_n)$, we have that

$$c_n = -\frac{i\overline{G(1)}}{\rho_{-n} \overline{G'(\rho_{-n})}} \frac{\rho_{-n}^2}{\rho_{-n}^2 - 1} = c_{-n}.$$

Note that $G'(\rho_n) = \lim_{x \to \rho_n} \frac{G(x)}{x - \rho_n}$. Then from Remark 3.2 we have that

$$|G'(\rho_n)| \sim c|n|^{-\frac{1}{2}}.$$

Since $G'(\rho_n)$ changes its sign everytime when n increases by one, we have that $\rho_n G'(\rho_n) \sim (-1)^n K|n|^{\frac{1}{2}}$. Thus

$$\sum_{n=1}^{N} (c_n e^{i\rho_n t} + c_{-n} e^{i\rho_{-n} t}) = \sum_{n=1}^{N} c_n \sin \rho_n t = K \sum_{n=1}^{N} \sin(n\pi + \rho_n t).$$

The rest of the proof will be straightfoward.

Appendix

File 1. The Residue Method

```
> read"File1.txt";
> #This program is to try solve the equation
> # y'(t) = a*y(t-1) with y(t)= sin(Pi*t) on 0<= t<=1
> #we use the residue method to compute the coefficients
> #and then  the numerical solution S_N(t).
>
> Rsum :=proc(n::posint, a::numeric)
> local c, k, j, m, s, t, Sn;
> with(linalg):
> alias(W=LambertW);
>
> c := array(-n+1..n);
> for k from -n+1 to n do
> c[k] := (Pi/(1+W(k,a)))*(W(k,a)*(W(k,a)+a))
>                 /(a*(Pi^2+W(k,a)^2));
> od;
> Sn := unapply(Sum(c[m]*exp(W(m,a)*t), m=-n+1..n),t);
> print('abs(Sn(0.5)-sin(Pi*0.5))'=evalf(abs(Sn(0.5)-sin(Pi*0.5))));
> print('abs(Sn(0.1)-sin(Pi*0.1))'=evalf(abs(Sn(0.1)-sin(Pi*0.1))));
>
> print('origin-aver'=evalf(int((abs(Sn(s)-sin(Pi*s))
>                                    )^2,s=0..1)));
```

```
>  end:
>  with(numapprox):
>  Rsum(1,0.5);
```

$$|Sn(.5) - \sin(.5\,\pi)| = .3633557275$$

$$|Sn(.1) - \sin(.1\,\pi)| = 1.365259975$$

$$origin - aver = .5749775966$$

```
>  Rsum(2,0.5);
```

$$|Sn(.5) - \sin(.5\,\pi)| = .06733704055$$

$$|Sn(.1) - \sin(.1\,\pi)| = .9201207000$$

$$origin - aver = .3314037499$$

```
>  Rsum(1,0.05);
```

$$|Sn(.5) - \sin(.5\,\pi)| = .6541496832$$

$$|Sn(.1) - \sin(.1\,\pi)| = 6.741099910$$

$$origin - aver = 12.04967847$$

```
>  Rsum(2,0.05);
```

$$|Sn(.5) - \sin(.5\,\pi)| = .1402610998$$

$$|Sn(.1) - \sin(.1\,\pi)| = 6.924495382$$

$$origin - aver = 21.17839096$$

File 2. The Projection Method

```
>  read"File2.txt";
>    interface(echo=2);
>   #This program is to try solve the equation
>   # y'(t) = a*y(t-1) with y(t)= sin(Pi*t) on 0<= t<=1
>   #To find the mininum difference in the mean of L^(0,1),
>   #we use the projection method to compute the optimal
>   #coefficients as well as the sum.
>   Psum :=proc(n::posint, a::numeric)
>   local A, B, C, D, i, j, k, k1, k2, m, s, t, Sn;
>   with(linalg):
>   alias(W=LambertW);
>
>   C := array(1-n..n);
>   D := array(1-n..n);
>
>   A  := evalm(matrix(2*n, 2*n, (i,j)->
>   evalf(int(exp(W(i-n,a)*t)*conjugate(exp(W(j-n,a)*t)),
>                                     t=0..1))));
>
>   B := evalm(inverse(A));
>
>     for k1 from 1-n to n do
```

```
>       D[k1] := unapply(add(B[k1+n,k2+n]*exp(W(k2, a)*t),
              k2=1-n..n), t);

>    od;

>

>   Sn := unapply(Sum(int(sin(Pi*t)*conjugate(D[m](t)),
        t=0..1)*exp(W(m,a)*s),

>                      m=1-n..n),s);

>

>   print('abs(Sn(0.5)-sin(Pi*0.5))'=evalf(abs(Sn(0.5)-sin(Pi*0.5))));

>   print('abs(Sn(0.1)-sin(Pi*0.1))'=evalf(abs(Sn(0.1)-sin(Pi*0.1))));

>

>   print('Psum-aver'=evalf(int((evalf(abs(Sn(s)-sin(Pi*s)))

>                                      )^2,s=0..1)));

>   end:
>  with(numapprox):
>  Psum(1,0.5);
```

$$|\mathrm{Sn}(.5) - \sin(.5\,\pi)| = .3006823191$$

$$|\mathrm{Sn}(.1) - \sin(.1\,\pi)| = .2136436567$$

$$Psum - aver = .08144540879$$

```
>  Psum(2,0.5);
```

$$|\mathrm{Sn}(.5) - \sin(.5\,\pi)| = .03512159932$$

$$|\mathrm{Sn}(.1) - \sin(.1\,\pi)| = .07176143571$$

$$Psum - aver = .01363608627$$

```
>  Psum(1,0.05);
```

$$|\mathrm{Sn}(.5) - \sin(.5\,\pi)| = .2749558332$$

$$|\mathrm{Sn}(.1) - \sin(.1\,\pi)| = .1114169245$$

$$Psum - aver = .06852310997$$

```
> Psum(2,0.05);
```

$$|\mathrm{Sn}(.5) - \sin(.5\,\pi)| = .1168893035$$

$$|\mathrm{Sn}(.1) - \sin(.1\,\pi)| = .1492003328$$

$$Psum - aver = .03216903371$$

File 3. The Derivative Method

```
> read"File3.txt";
>   interface(echo=2);
>  #This program is to try solve the equation
>  # y'(t) = a*y(t-1) with y(t)= sin(Pi*t) on 0<= t<=1
>  #To find the mininum difference in the mean of L^(0,1),
>  #we let the derivative of the mean to be zero to find
>  #the optimal coefficients.
>  Dsum :=proc(n::posint, a::numeric)
>  local A, B, C, i, j, u, k, m, s, t, Sn;
>
>  with(linalg):
>  alias(W=LambertW);
>  B := array(1..2*n);
>  C := array(1..2*n);
>
>  for k to 2*n do
>  C[k] := evalf(int(sin(Pi*t)*exp(conjugate(W(k-n,a))*t),t=0..1));
>  od;
>
>    A  := evalm(matrix(2*n, 2*n, (i,j)->
>    evalf(((abs(a))^2-conjugate(W(i-n,a))*W(j-n,a)
>      )/(conjugate(W(i-n,a))*W(j-n,a)
```

```
>       *(conjugate(W(i-n,a))+W(j-n,a)))))):
>
>     B := linsolve(A, C);
>
>   Sn := unapply(Sum(B[m]*exp(W(m-n,a)*t), m=1..2*n),t);
>
>
>   print('abs(Sn(0.5)-sin(Pi*0.5))'=evalf(abs(Sn(0.5)-sin(Pi*0.5))));
>   print('abs(Sn(0.1)-sin(Pi*0.1))'=evalf(abs(Sn(0.1)-sin(Pi*0.1))));
>
>
>   print('Dsum-aver'=evalf(int((abs(Sn(s)-sin(Pi*s))
>                                     )^2,s=0..1)));
>   end:
> with(numapprox):
> Dsum(1,0.5);
```

$$|\mathrm{Sn}(.5) - \sin(.5\,\pi)| = .3006823194$$

$$|\mathrm{Sn}(.1) - \sin(.1\,\pi)| = .2136436568$$

$$Dsum - aver = .08144540873$$

```
> Dsum(2,0.5);
```

$$|\mathrm{Sn}(.5) - \sin(.5\,\pi)| = .03512159939$$

$$|\mathrm{Sn}(.1) - \sin(.1\,\pi)| = .07176143456$$

$$Dsum - aver = .01363608626$$

```
> Dsum(1,0.1);
```

$$|\mathrm{Sn}(.5) - \sin(.5\,\pi)| = .2793482641$$

$$|\mathrm{Sn}(.1) - \sin(.1\,\pi)| = .1309307322$$

$$Dsum - aver = .07135698326$$

```
> Dsum(2,0.1);
```

$$|\mathrm{Sn}(.5) - \sin(.5\,\pi)| = .07438300802$$

$$|\mathrm{Sn}(.1) - \sin(.1\,\pi)| = .1360686206$$

$$Dsum - aver = .02663195952$$

File 4. The Gram-Schmidt Method

```
> read"File4.txt";
>
>    interface(echo=2);
>
>   #This program is to try solve the equation
>   # y'(t) = a*y(t-1) with y(t)= sin(Pi*t) on 0<= t<=1.
>   #we use Gram-Schmedt process to orthogonalize the system,
>   #and then to get its Fourier coefficients,
>   #finally recombine the coefficients to get the optimal solution.
>
>   Nsum :=proc(n::posint, a::numeric)
>   local D, e, h, h0, j, k, k1, k2,m, s, t, Sn;
>
>   with(linalg):
>   alias(W=LambertW);
>
>   D := array(-n+1..n);
>   h0 := array(-n+1..n);
>   h := array(-n+1..n);
>   e := array(-n+1..n);
>
>   for k from -n+1 to n do
```

```
>    h0[k] := exp(W(k,a)*t);
>    od;
>
>    h[-n+1] := exp(W(-n+1,a)*t);
>    D[-n+1] := evalf(sqrt(int(evalf((abs(h[-n+1]))^2),t=0..1)));
>    e[-n+1] :=  exp(W(-n+1,a)*t)/D[-n+1];
>    for k1 from -n+2 to n do
>    h[k1] := h0[k1]-(add(evalf(int(evalf(h0[k1])*evalf(conjugate(e[j])),
>                                                 t=0..1)
>                                      )*e[j], j=-n+1..k1-1));
>    D[k1] := evalf(sqrt(int(evalf((abs(h[k1]))^2),t=0..1)));
>    e[k1] := evalf(h[k1])/D[k1];
>
>    od;
>    Sn:=unapply(add(evalf(int(evalf(sin(Pi*t))*evalf(conjugate(e[k2])),
>                                 t=0..1)
>                             )*evalf(e[k2]),
>                      k2=-n+1..n),
>               t);
>
>
>   print('abs(Sn(0.5)-sin(Pi*0.5))'=evalf(abs(Sn(0.5)-sin(Pi*0.5))));
>   print('abs(Sn(0.1)-sin(Pi*0.1))'=evalf(abs(Sn(0.1)-sin(Pi*0.1))));
```

```
>
>   print('norm-aver'=evalf(int(evalf(abs(evalf(Sn(t)-sin(Pi*t))
>                                             )
>                                          )^2,t=0..1)));
>   end:
> with(numapprox):
> Nsum(1,0.5);
```

$$|\mathrm{Sn}(.5) - \sin(.5\,\pi)| = .3006823188$$

$$|\mathrm{Sn}(.1) - \sin(.1\,\pi)| = .2136436567$$

$$norm - aver = .08144540879$$

```
> Nsum(2,0.5);
```

$$|\mathrm{Sn}(.5) - \sin(.5\,\pi)| = .03512159941$$

$$|\mathrm{Sn}(.1) - \sin(.1\,\pi)| = .07176143453$$

$$norm - aver = .01363608627$$

```
> Nsum(1,0.05);
```

$$|\mathrm{Sn}(.5) - \sin(.5\,\pi)| = .2749558339$$

$$|\mathrm{Sn}(.1) - \sin(.1\,\pi)| = .1114169249$$

$$norm - aver = .06852310997$$

```
> Nsum(2,0.05);
```

$$|\mathrm{Sn}(.5) - \sin(.5\,\pi)| = .1168893007$$

$$|\mathrm{Sn}(.1) - \sin(.1\,\pi)| = .1492003252$$

$$norm - aver = .03216903370$$

File 5. The Convergence of Optimal Coefficients

```
>  read"File5.txt";
>    interface(echo=2);
>   #This program is to test basis property of the solution
>   #system of the equation y'(t) = a*y(t-1) with some initial
>   #conditions on 0<= t<=1.
>   #We use the projection method to compute the optimal coefficients
>   #If some coefficients does not converge, then the system is not
>   # a basis.
>
>   Coeconv :=proc(n::posint, a::numeric)
>   local A, B, C, D, i, j, k1, k2, m, t;
>   with(linalg):
>   alias(W=LambertW);
>
>   C := array(1-n..n);
>   D := array(1-n..n);
>   A  := evalm(matrix(2*n, 2*n, (i,j)->
>   evalf(int(exp(W(i-n,a)*t)*conjugate(exp(W(j-n,a)*t)),
>                                   t=0..1))));
>   B := evalm(inverse(A));
>
>   D[1] := (evalf(add(B[1+n,k2+n]*exp(W(k2, a)*t), k2=1-n..n)));
```

```
>   print('ct[1]'=evalf(int(evalf(t*conjugate(D[1](t))), t=0..1)));

>   print('ccos[1]'=evalf(int(evalf(cos(t)*conjugate(D[1](t))),
t=0..1)));

>

>     if (n> 1) then

>     D[2] := (evalf(add(B[2+n,k2+n]*exp(W(k2, a)*t), k2=1-n..n)));


>    print('ct[2]'=evalf(int(t*conjugate(D[2](t)), t=0..1)));

>    print('ccos[2]'=evalf(int(cos(t)*conjugate(D[2](t)), t=0..1)));

>     fi;

>   end:
> with(numapprox):
> Coeconv(2,-0.5);
```

$$ct_1 = .2077749194 - .1242557283\, I$$

$$ccos_1 = -.04798355578 + .03584265369\, I$$

$$ct_2 = .07623320530 + .09179614370\, I$$

$$ccos_2 = -.01890769139 - .02353101851\, I$$

```
> Coeconv(3,-0.5);
```

$$ct_1 = .1208783593 - .1969867657\, I$$

$$ccos_1 = -.02577648368 + .06089360636\, I$$

$$ct_2 = .1350723909 - .009862843495\, I$$

$$ccos_2 = -.03887646064 + .004948514171\, I$$

```
> Coeconv(4,-0.5);
```

$$ct_1 = .07445876431 - .2175948340\, I$$

$$ccos_1 = -.01186459614 + .06809525901\,I$$

$$ct_2 = .1185687371 - .06913565114\,I$$

$$ccos_2 = -.03449102826 + .02340961069\,I$$

```
> Coeconv(5,-0.5);
```

$$ct_1 = .04649580956 - .2260129792\,I$$

$$ccos_1 = -.003143868204 + .07103882193\,I$$

$$ct_2 = .09624496157 - .09728085917\,I$$

$$ccos_2 = -.02765950451 + .03242743309\,I$$

```
> Coeconv(6,-0.5);
```

$$ct_1 = .02787899728 - .2302756807\,I$$

$$ccos_1 = .002773132955 + .07252626710\,I$$

$$ct_2 = .07801014687 - .1116472180\,I$$

$$ccos_2 = -.02190809413 + .03709796715\,I$$

```
> Coeconv(7,-0.5);
```

$$ct_1 = .01457952188 - .2327675230\,I$$

$$ccos_1 = .007050166945 + .07339657282\,I$$

$$ct_2 = .06397461413 - .1196452948\,I$$

$$ccos_2 = -.01741035580 + .03971741376\,I$$

```
> Coeconv(8,-0.5);
```

$$ct_1 = .004765866572 - .2346681575\,I$$

$$ccos_1 = .01233647105 + .07751101354\,I$$

$$ct_2 = .05313114586 - .1244204291\,I$$

$$ccos_2 = -.01391479396 + .04516530916\,I$$

```
> Coeconv(9,-0.5);
```

$$ct_1 = -.003252411189 - .2355237735\, I$$

$$ccos_1 = .01280937236 + .07427351576\, I$$

$$ct_2 = .04455981150 - .1275096996\, I$$

$$ccos_2 = -.01108128893 + .04227360229\, I$$

```
> Coeconv(10,-0.5);
```

$$ct_1 = -.009504083352 - .2363574870\, I$$

$$ccos_1 = .01488777162 + .07463296481\, I$$

$$ct_2 = .03767804280 - .1295616089\, I$$

$$ccos_2 = -.008870667353 + .04300767909\, I$$

```
> Coeconv(11,-0.5);
```

$$ct_1 = -.01703814826 - .2353185186\, I$$

$$ccos_1 = .01907796955 + .0615844708 4\, I$$

$$ct_2 = .02683255631 - .1287477137\, I$$

$$ccos_2 = -.007580366371 + .03261671657\, I$$

```
> Coeconv(12,-0.5);
```

$$ct_1 = -.02223619477 - .2041864513\, I$$

$$ccos_1 = .07483026588 + .1885996845\, I$$

$$ct_2 = .02223288944 - .1451974626\, I$$

$$ccos_2 = .03561811726 + .01953455109\, I$$

Bibliography

[1] E. Artin, *The gamma function*, New York, Holt, Rinehart and Winston, 1964.

[2] S. A. Avdonin and I. Joo, *Riesz basis of exponentials and sine-type functions*, Acta Math. Hung., Vol. 51 (1), 1988, 1-14.

[3] R. Balan, *Stability theorems for Fourier frames and wavelet Riesz bases*, J. Four. Anal. Appl., Vol. 3, 1997, 499-504.

[4] R. Bellman and K. L. Cooke, *Differential-Difference Equations*, Academic Press, 1963.

[5] H. Benzinge, *Nonharmonic Fourier series and spectral theory*, Trans. Amer. Math. Soc., Vol. 299 (1), 1987, 245-259.

[6] R. P. Boas, JR, *Entire Functions*, Academic Press, 1954.

[7] R. P. Boas, JR, *Integrability along a line for a class of entire functions*, Trans. Amer. Math. Soc., Vol. 73, 1952.

[8] M. Bownik, *Tight frames of multidimensional wavelets*, J. Four. Anal. Appl., Vol. 3, 1997, 525-542.

[9] M. L. Cartwright, *On certain integral functions of order one*, Proc. London Math. Soc., (2) Vol. 33, 1931.

[10] P. G. Cazassa and O. Christensen, *Perturbation of operators and applications to frame theory*, J. Four. Anal. Appl. Vol. 3, 1997, 543-557.

[11] P. G. Cazassa and O. Christensen, *Riesz frames and approximation of the frame coefficients*, Approx. Theory Appls., (2) Vol. 14, 1998, 1-11.

[12] P. G. Casazza and N. J. Kalton, *Generalizing the Paley-Wiener perturbation theory for Banach spaces*, Proc. Amer. Math. Soc., Vol. 127, 1999, 519-527.

[13] O. Christensen, *Frames and the Projection method*, Appl. Comp. Harm. Anal., Vol. 1, 1993, 50-53.

[14] O. Christensen, *Frame perturbations*, Proc. Amer. Math. Soc., (4) Vol. 123, 1995, 1217-1220.

[15] O. Christensen, *A Paley-Wiener theorem for frames*, Proc. Amer. Math. Soc., (7) Vol. 123, 1995, 2199-2202.

[16] O. Christensen, *Frames and pseudo-inverses*, J. Math. Anal. Appl., Vol. 195, 1995, 401-414.

[17] O. Christensen, *Frames containing a Riesz basis and approximation of the frame coefficients using finite-dimentional methods*, J. Math. Anal. Appl., Vol. 199, 1996, 256-270.

[18] O. Christensen, *Operator with closed range, pseudo-inverses, and perturbation of frames for a subspace*. Canad. Math. Bull., (1) Vol. 42, 1999. 37-45.

[19] R. Corless, *Essential Maple*, Springer-Verlag, 1994, 167-168.

[20] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey and D. E. Knuth, *On the Lambert W function*, Advances in Comp. Math., Vol. 5, 1996, 329-359.

[21] Z. Cvetkovic and M. Vetterli, *Error analysis in oversampled A/D conversion and quantization of Weyl-Heisenberg frame expansions*, Mem. No. UCB/ERL M95/48, May.

[22] I. Daubechies, *Ten Lectures on Wavelets*, the Society for Industrial and Applied Mathematics, 1992.

[23] I. Daubechies, *From the original framer to present-day time-frequency and time-scale frames*, J. Four. Anal. Appl., Vol.3, 1997, 485-486.

[24] R. J. Duffin and J. J. Eachus, *Some notes on an expansion theorem of Paley-Wiener*, Bull. Amer. Math. Soc., Vol. 48, 1942, 850-855.

[25] R. J. Duffin and A. C. Schaeffer, *Power series with bounded coefficients.* Amer. J. Math., Vol. 67, 1945.

[26] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier Series*, Trans. Amer. Math. Soc., Vol. 72, 1952, 341-366.

[27] N. Fujii, A. Nakamura, and R. Redheffer, *On the excess of sets of complex exponentials*, Proc. Amer. Math. Soc., Vol. 127, 1999, 1815-1818.

[28] D. Gaier, *Lectures on complex approximation*, Birkhauser, 1985.

[29] G. R. Grozev and Q. I. Rahman, *Reconstruction of entire functions from irregularly spaced sample points*, Can. J. Math., Vol. 48, 1996.

[30] E. Hernandez and G. Weiss, *A First Course on Wavelets*, CRC Press, 1996.

[31] G. Hinsen, *Explicit irregular sampling formulars*, J. Comp. Appl. Math., Vol. 40, 1992, 177-198.

[32] G. Hinsen, *Irregular Sampling of Bandlimited L^p- functions*, J. Appro. Theory, Vol.72, 1993, 346-363.

[33] S. Jaffard and R. M. Young, *A representation theorem for Schauder bases in Hilbert space*, Proc. Amar. Math. Soc., (2) Vol. 126, 1998, 553-560.

[34] M. I. Kadec, *The exact value of the Paley-Wiener constant*, Sov. Math. Doklady, Vol. 5, 1964, 559-561.

[35] G. Kaiser, *A Friendly Guide to Wavelets*, Birkhauser, 1994.

[36] P. Koosis, *The logarithmic integral*, Vol.2, Cambridge Univ. Press, 1992.

[37] H. Landau, *Necessary density conditions for sampling and interpolation of certain entire functions*, Acta Mathematica, Vol.117, 1967, 37-52.

[38] B. JA. Levin, *Distribution of zeros of entire functions*, Amer. Math. Soc., 1980;

[39] N. Levinson, *On the closure of* $\{e^{i\lambda_n x}\}$ *and integral functions*, Proc. Cambridge Phil. Soc., Vol. 31, 1935, 335-346.

[40] N. Levinson, *Gap and Density Theorems*, AMS. Col. Public. 26, 1940.

[41] A. M. Minkin, *Reflection of exponents, and unconditional bases of exponentials*, St. Petersburg Math. J., Vol. 3, 1992.

[42] R. E. Paley and N. Wiener, *Fourier transforms in the complex domain*, AMS. Col. Public. 19, 1934.

[43] A. Pfluger, *On analytic functions bounded at the lattice points*, Proc. London Math. Soc., (2) Vol. 42, 1936.

[44] M. Plancherel and G. Polya, *Functions entieres et integrales de Fourier multiples*, Comment. Math. Helv. Vol. 9, 1937, 224-248; 10, 1938, 110-163.

[45] R. M. Redheffer, *Completeness of Sets of Complex Exponentials*. Advances in Mathematics, Vol. 24, 1977, 1-62.

[46] R. M. Redheffer and R. M. Young, *Completeness and basis properties of complex exponentials*, Trans. Amer. Math. Soc., Vol. 277, 1983, 93-111.

[47] A. M. Sedletskii, *Nonharmonic Fourier series*, Sib. Math. Zhur., Vol. 12, 1971, 1100-1114.

[48] A. M. Sedletskii, *On biorthogonal expansions in exponential functions*, Math. USSR. Izv., Vol. 6, 1972, 579-586.

[49] A. M. Sedletskii, *On the stability, completeness and minimality of a system of exponential functions in* L^2, Mat. Zame., Vol. 15, 1974, 213-219.

[50] A. M. Sedletskii, *Expansions in exponential functions*, Sib. Math. Zhur., Vol. 16, 1975, 820-829.

[51] A. M. Sedleckii, *Excesses of systems of exponential functions*, Math. Notes, Vol. 22, 1977, 941-947.

[52] A. M. Sedleckii, *On completeness of the system* $\{exp(ix(n+ih_n))\}$, Anal. Math., Vol. 4, 1978, 125-143.

[53] A. M. Sedletskii, *Excesses of close systems of exponentials in* L^p, Sibirskii Math. Zhurnal, Vol. 24, 1983, 164-175.

[54] A. M. Sedletskii, *Purely imaginary perturbations of the exponents* λ_n *in the system* $\{exp(i\lambda_n t)\}$. Sibirskii Math. Zhurnal, Vol. 26,1985, 151-158.

[55] A. M. Sedletskii, *Completeness and nonminimality of systems of exponentials in* $L^p(-\pi, \pi)$, Sibirskii Math. Zhurnal, Vol. 29, 1988, 159-170.

[56] S. Verblunsky, *On an expansion in exponential series*, Quart. J. Math., Vol. 7, 1956, 231-240.

[57] S. Verblunsky, *On a class of integral functions*, Quart. J. Math., Vol. 8, 1957, 312-320.

[58] S.Verblunsky, *On a class of Cauchy exponential series*, Rend. Circ. Mat. Palermo, Vol. 10, 1961, 5-26.

[59] S. Verblunsky, *On a class of infinite products*, Proc. Comb. Phil. Soc., Vol. 60, 1964, 847-854.

[60] S. Verblunsky, *On the stability of the set of exponents of a Cauchy exponential series*, Pacific J. Math., Vol. 16, 1966, 175-188.

[61] E. M. Wright, *The Linear Difference-differential Equation with Constant Coefficients*, Proc. Roy. Soc. Edinburgh, Sec. A, Vol. 62, 1949, 387-393.

[62] E. M. Wright, *Solution of the equation* $ze^z = a$, Bull. Amer. Math. Soc., Vol. 65, 1959, 89-93.

[63] R. M. Young, *A perturbation theorem for complete sets of complex exponentials*, Proc. Amer. Math. Soc., Vol. 55, 1976, 318-320.

[64] R. M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, 1980.

[65] R. M. Young, *On the pointwise convergence of a class of nonharmonic Fourier series*, Proc. Amer. Math. Soc., Vol. 89, 1983, 65-73.

[66] R. M. Young, *On a theorem of Ingham on nonharmonic Fourier series*, Proc. Amer. Math. Soc., Vol. 92, 1984, 549-553.

[67] R. M. Young, *On the stability of exponential bases in* $L^2(-\pi, \pi)$, Proc. Amer. Math. Soc., Vol. 100, 1987, 117-122.

[68] R. M. Young, *Interpolation and frames in certain Banach spaces of entire functions*, J. Four. Anal. Appl., Vol. 3, 1997, 639-645.