PITCH AND INTERVAL STRUCTURES IN GEORGE PERLE'S THEORY OF TWELVE-TONE TONALITY

by

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Abstract

George Perle, American composer and theorist, has authored an innovative theory called *Twelve-Tone Tonality* (1977; 2nd ed. 1996), a compositional system based on the conjunction of interval cycles and inversional symmetry. Perle's theory is the result of his search for an atonal analogue to the precompositional structures and hierarchical organization of tonal music. The theory is not widely known in the theoretical community, partly due to its compositional purview and idiosyncratic terminology.

This dissertation explicates Perle's theory in a reorganized, accessible format, and includes the most recent developments to the theory that appear in the second edition of *Twelve-Tone Tonality* (1996). The study also explores the properties of the *cyclic sets*, the fundamental entities of the theory, outside the context of twelve-tone tonality. Through the process of imbrication the cyclic sets generate close associations of pitch class set classes identified as *ICS families* (for "imbricated cyclic set"). The set classes in these families share a number of structural properties, including inversional symmetry, transpositional combination (as developed by Richard Cohn 1987), and equivalence in other modular universes. The study also introduces an original similarity relation, the R_{SYM} *relation*, to reflect the symmetrical nature of the intervallic similarity between pairs of set classes in the ICS families.

The dissertation provides detailed analyses of two etudes from Perle's *Six Etudes for Piano* (1973-76). The analyses differentiate between the abstract dimension of twelve-tone tonal constructs and their concrete realization at the musical surface, and show both local and long-range structure. Further, although the analyses are guided primarily by Perle's theory of twelve-tone tonality, they are supplemented by observations from the perspective of pitch class set theory, as developed by Allen Forte (1973) and extended by

Robert Morris (1995b). This multifaceted analytical approach highlights distinctive features of the etudes.

Through the presentation of the tenets of twelve-tone tonality, the theoretical exploration of the cyclic sets, and the analysis of selected works, the dissertation aims to show the depth and potential of the theory, both within and outside its own context.

Keywords: atonality, axis of symmetry, Richard Cohn, equivalence classes, Allen Forte, IcVSIM, interval cycles, inversional symmetry, Eric Isaacson, modular equivalence, Robert Morris, George Perle, pitch-class set theory, similarity relations, transpositional combination, twelve-tone tonality

To Brian, Michael, and Alison

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Chapter One

Introduction

George Perle (b.1915) is one of America's foremost contemporary composers. He is also a respected theorist, and has written extensively on twentieth-century music. Perle is a recognized scholar of Alban Berg's music, and is the author of one of the first analytical studies of the compositional method of the Second Viennese School, *Serial Composition and Atonality* (Perle 1962, 6th ed. 1991). In addition, Perle is the creator of a unique approach to composition, a system based on the conjunction of interval cycles and inversional symmetry, which he calls "twelve-tone tonality."

Perle's theory of twelve-tone tonality rests on a foundation of interval cycles. An interval cycle is an ordered series of pitch classes (pcs) based on a single recurrent interval, which is measured by the number of semitones it spans.[1] Perle alternates the elements of two inversionally related interval cycles to form the basic unit of his system, the *cyclic set*. The combination of two cyclic sets forms an *array*, from which Perle derives his compositional resources and which generates the hierarchy of structural relationships in the system. As Perle explains in his book *Twelve-Tone Tonality* (1977b, 2nd ed. 1996), his theory is the result of a search for an atonal analogue to the precompositional structures and hierarchical organization of tonal music.

In 1937, at the age of twenty-two, Perle first happened upon a score of Berg's *Lyric Suite* and recognized that the work uses the twelve pcs of the aggregate as an autonomous collection outside the domain of tonality. Perle observed that in the first movement both the linear and vertical pitch configurations are structured on the basis of perfect fifth cycles. Although he attempted to decipher Schoenberg's twelve-tone method from Berg's

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[1] The interval cycles are discussed in detail in chapter two of this dissertation, and are notated in example 2.1.

Lyric Suite, he was not successful. In 1939 Perle began composing a string quartet according to principles that he assumed constituted Schoenberg's method. But during a composition lesson with Ernst Krenek later that year, Krenek informed Perle that he had misunderstood Schoenberg's method, but called Perle's work a "discovery." Perle subsequently decided to continue with the development of his own method (1990a, 126-34).

Two years later Perle published his first article on his fledgling method, entitled "Evolution of the Tone-Row: The Twelve-Tone Modal System" (1941). Numerous analytical articles on other composers' works followed; those particularly relevant to the development of his own theoretical ideas will be discussed below. During a three-year span from 1969 to 1972 Perle collaborated with former pupil Paul Lansky; this collaboration resulted in an expansion of the theory's scope and compositional potential. In the preface to the second edition of *Twelve-Tone Tonality* (1996, xv-xvi) Perle relates how he gradually interpolated his theory in the appendices in the first three editions of his book *Serial Composition and Atonality*, until his editor suggested publishing the appendices as a separate book, which resulted in the first edition of *Twelve-Tone Tonality* in 1977. More general discussions of the theory appear in the articles "Berg's Master Array of the Interval Cycles" (1977a) and "The First Four Notes of Lulu" (1989). Perle gives a detailed exposition of the theory in the fifth chapter of his book *The Listening Composer* (1990a) and in the article "Symmetry, the Twelve-Tone Scale, and Tonality" (1992). A second edition of *Twelve-Tone Tonality* appeared in 1996, which includes the most recent extensions and refinements to the theory.

Perle's analytical studies of twentieth-century works

Through his analytical studies of the works of certain twentieth-century composers, especially those of Berg and Bartók, Perle discovered points of contact with his own theory, most notably in the use of interval cycles as an organizing force. Perle comments:

> My principal work as a theorist and, above all, as a composer since then [1941] has concerned itself with discovering and developing the implications of twelve-tone tonality, and not the least interesting result has been the realization of the extent to which these principles are anticipated in shared elements in the music of mainstream composers of the twentieth century—Scriabin, Stravinsky, Bartók, and Varèse, as well

> as Schoenberg, Berg, and Webern. The connections are evident not only in the extent to which the foundational concepts of the interval cycle are shared, but also in all sorts of surprisingly detailed ways (1990a, 163).

On the basis of his analytical studies, Perle has concluded that a normative referential language emerges from commonalities in the composers' works, one based on interval cycles and inversional symmetry.

In his article "Symmetrical Formations in the String Quartets of Béla Bartók" (1955) Perle discusses how Bartók uses symmetrical formations as a means of progression in the string quartets. Contrary to the impressionistic use of symmetrical formations to suppress a sense of key, motivic development, and harmonic motion, Perle contends that Bartók used them to promote these parameters actively. In his analysis of Bartók's *Fourth String Quartet* Perle identifies three symmetrical tetrachordal set classes (scs) derived from interval cycles. "Set x" (sc 4-1) is a segment of the interval-1 cycle, and "set y" (sc 4-21) is a segment of the interval-2 cycle. The third tetrachordal sc (sc 4-9, which Perle calls "fig.x" in this article) comprises two interlocking interval-6 dyads.[2] Perle describes how these symmetrical formations may be used to invoke modulatory procedures. All symmetrical sets imply a midpoint or axis of symmetry; transposing a symmetrical set establishes a new axial center. Further, repeatedly transposing such a set by its generating interval or intervals establishes a hierarchy of axial centers or "keys" of sets in relation the original set. Perle finds these three tetrachordal scs at the musical surface and in the background of Bartók's quartets, serving as generators and goals of motivic and harmonic motion.

In "Scriabin's Self-Analyses" (1984) Perle asserts that "the interval cycle is a means of symmetrically partitioning, and thus imposing an ordering upon, the functionally undifferentiated pcs of the twelve-tone scale" (1995b, 23). Perle contends that the octatonic collection (sc 8-28) forms the basis of Scriabin's musical language, and shows how Scriabin exploits the cyclic properties of this collection in both linear and vertical dimensions to define harmonic areas.

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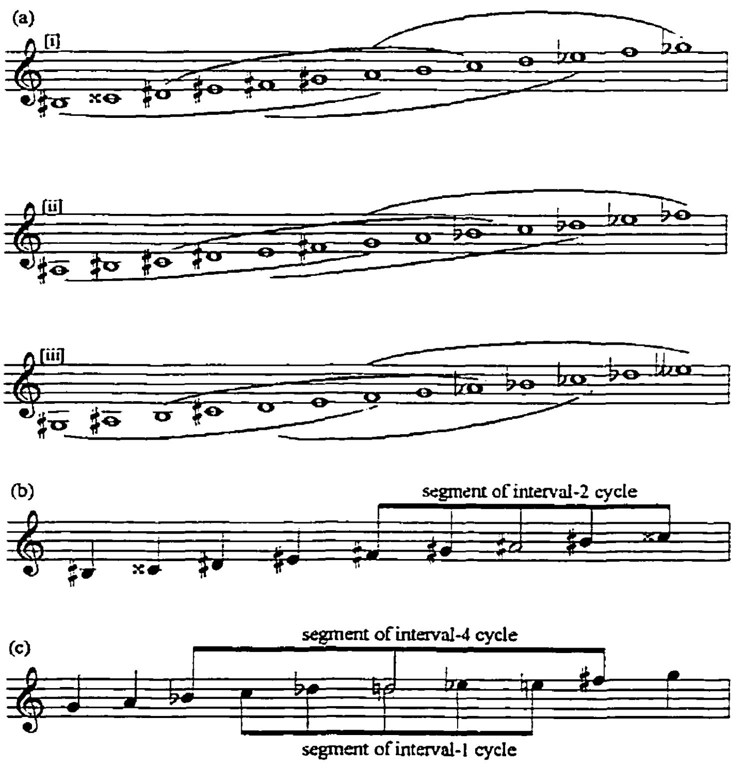
[2] Leo Treitler later labelled the set as cell Z, to correspond with the other two sets in the context of his discussion of Bartók's Fourth Quartet (Treitler 1959, 292-98).

The octatonic collection has only three distinct transpositions, as it comprises two interval-3 cycles. Perle argues that Scriabin often circumvents these restricted transpositional possibilities by employing a heptachordal subset of the octatonic collection, sc 7-31 (which Perle calls a "heptatonic scale"). Each octatonic collection contains eight distinct forms of this subset, thus yielding a total of twelve forms related by transposition and twelve by inversion. Perle introduces a configuration he calls a "master scale," which may be described as an octatonic collection that recognizes enharmonic spellings in order to present four transpositionally related heptachordal subsets. These master scales are illustrated in example 1.1a, with the heptachordal subsets indicated by slurs.[3]

Example 1.1. Scriabins's master scales (a), interval-2 cyclic segment created by altered pc within the 7-31 subset of the octatonic collection (b), and the interval-1 and interval-4 cyclic segments created by addition of non-octatonic pc to 7-31 subsets (c)



[3] Examples 1.1a and b are derived from Perle's examples 8 and 9, respectively (1995b, 7).

Perle shows how in Scriabin's Seventh Sonata (Op.64), the composer achieves harmonic contrast by chromatically altering a single pc in the heptachordal subset. This alteration creates a pentachordal segment of the interval-2 cycle which Scriabin exploits compositionally (example 1.1b). Perle also demonstrates how in the Five Preludes (Op.74) Scriabin often adds to an octatonic collection a pc from outside the collection, to create either an interval-1 or interval-4 cyclic segment (example 1.c). Perle states that in Scriabin's compositional process "symmetrical partitionings of the semitonal scale by means of interval cycles generate new, totally consistent, referential harmonic structures" (19).

Perle discusses the role of interval cycles in many of his writings on Alban Berg's music. As he describes in his article "Berg's Master Array of the Interval Cycles" (1977a), Perle detected early signs of Berg's interest in interval cycles in a letter Berg wrote to Schoenberg, dated July 27, 1920 (1995b, 207). In this letter Berg included a chart of an array of interval cycles, referring to it as an "oddity . . . a theoretical trifle." This array comprised twelve rows, each containing one of the twelve interval cycles. The cycles also lie in the columns of the chart. Figure 1.1 represents Berg's chart in pcs rather than pitches, and substitutes interval cycle integers instead of the diatonic interval names.[4]

Figure 1.1. Berg's array of interval cycles

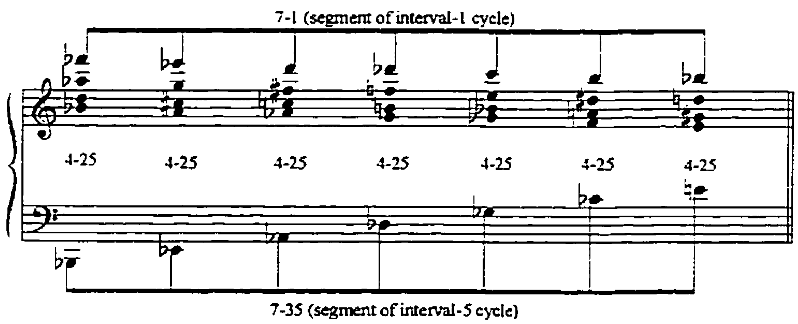
12	C	C	C	C	C	C	C	etc.					
11	C	B	B♭	A	A♭	G	G♭	F	etc.				
10	C	B♭	A♭	F♯	E	D	C	B♭	A♭	etc.			
9	C	A	F♯	E♭	C	A	G♭	E♭	C	A	etc.		
8	C	G♯	E	C	A♭	E	C	A♭	E	C	A♭	etc.	
7	C	G	D	A	E	B	G♭	D♭	A♭	E♭	B♭	F	C
6	C	G♭	C	F♯	C	F♯	C	G♭	C	F♯	C	F♯	C
5	C	F	B♭	E♭	A♭	C♯	G♭	C♭	E	A	D	G	C
4	C	E	A♭	C	E	G♯	C	E	A♭	C	E	A♭	C
3	C	D♯	F♯	A	C	D♯	G♭	A	C	E♭	F♯	A	C
2	C	D	E	F♯	A♭	A♯	C	D	E	F♯	A♭	B♭	C
1	C	C♯	D	E♭	E	E♯	G♭	G	A♭	A	B♭	B	C
↑ interval cycle →	0	1	2	3	4	5	6	7	8	9	10	11	12

[4] Dave Headlam also reproduces Berg's array in pitch-classes (1996, 54, figure 2.1).

Perle comments that "in fact, Berg's array of the interval cycles, far from being a mere 'trifle,' reflects a significant and persistent feature of his musical language, from the second song of Opus 2 . . . through to his last work, the twelve-tone opera *Lulu*" (209).

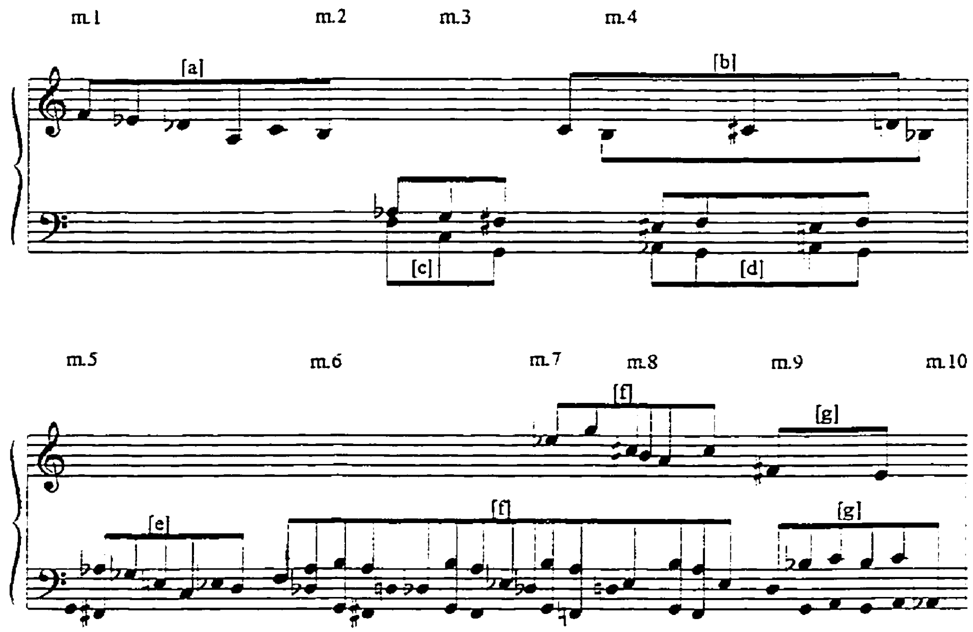
Perle argues that interval cycles in Berg's music function primarily to generate themes and to establish harmonic areas. Berg's interval cycles also provide means of progression, both locally, to fill in registral space, and at deeper levels of structure, to outline extended progressions. In his discussion of the second song of Berg's Op.2, Perle shows how the opening measures may be reduced to a series of descending vertical 4-25 sets ("French sixth" chords), as illustrated in example 1.2. These sets also unfold four linear (7-1) segments of interval-1 cycles,[5] while the bass line unfolds an ascending (7-35) segment of the interval-5 cycle. Perle comments that the simultaneous unfoldings of different cycles helps to mask the repetitiveness of the verticalities (210).

Example 1.2. Interval-cycle segments in Berg's Op.2, No.2



Perle also discusses the pervasive presence of interval cycles in the opening measures from the first movement of Berg's String Quartet, Op.3. He contends, however, that the first ten measures represent the only extended passage in the piece in which virtually every note can be viewed as belonging to an unfolding interval cycle. Example 1.3 is based on Perle's annotated excerpt of mm.1-10 (212, ex.8). Perle describes how at [a] a pentachordal subset of an interval-2 cycle unfolds in the second violin, with an additional

[5] The example beams the uppermost linear 7-1 set only.

Example 1.3. Unfolding interval-cycle segments in first ten measures of first movement, Berg's String Quartet, Op.3, No.3

non-cyclic note as the penultimate note of the gesture (indicated by the stemless closed notehead). The semitonal dyad this note forms with the final note then expands at [b], creating linear subsets of the interval-1 cycle. The viola and cello at [c] simultaneously unfold segments of different interval cycles (interval-1 and interval-5, respectively). The elements forming the final dyad between the two instruments are then chromatically altered at [d], thereby prolonging the gesture at [c], according to Perle. The opening gesture recurs at T_9 in m.5, at [e]. This transposition generates the pc collection from the other interval-2 partition of the aggregate, thereby defining a contrasting harmonic area to that of the opening gesture. Perle observes that the pc content of m.6 reverts to the first interval-2 collection (at [f]), with dissonant non-cyclic notes resolving to cyclic notes.[6] In m.9 the pc content derives from the second interval-2 cyclic collection (at [g]), again with non-cyclic notes resolving to cyclic notes in the next measure.[7] Perle concludes that

[6] Perle only discusses the dissonant F♯ resolving to F♮, but as example 1.3 [f] indicates, the D♮ is also a non-cyclic note that resolves immediately to the cyclic D♭ in m.6 and to the cyclic E♭ in m.8.

[7] Perle specifies that the cello (represented by pitches G and A in the bass staff at [g]) does not revert to the other interval-2 cycle until the A♭ at m.10.

"the harmonic language of [example 1.3] and its musical effect is well accounted for by an analysis that demonstrates the various kinds of compositional unfolding of interval cycles" (213).

In his examination of Berg's twelve-tone string quartet, the *Lyric Suite*, Perle notes that in this work Berg's "characteristic preoccupation with interval cycles merges with the serial concept" (226). The first movement's principal row, given in figure 1.2a, comprises all eleven intervals, and divides into two hexachords belonging to sc 6-32. Moreover, the row is symmetrical: the second hexachord is a T_6 retrograde of the first hexachord. In addition, the row possesses cyclic properties: each of its two hexachords alternates elements from inversionally related cycles. The movement actually contains three rows; the second and third rows derive from the principal row in that they each reorder the hexachordal pc content. The second row makes explicit the principal row's cyclic origin (figure 1.2b); the third row reorders the hexachordal pc content in a diatonic scalar order (figure 1.2c).

Figure 1.2. Three rows in the first movement of Berg's Lyric Suite

(a)	F	E	C	A	G	D	|	G♯	C♯	D♯	F♯	A♯	B
(b)	F	C	G	D	A	E	|	B	F♯	C♯	G♯	D♯	A♯
(c)	C	D	E	F	G	A	|	F♯	G♯	A♯	B	C♯	D♯

Perle compares the principal row to a *cyclic set*, which he defines as a row that alternates elements from inversionally related interval cycles (229).[8] Consequently, the row also contains alternating sums between adjacent elements. Perle views these properties as functioning referentially: "The cyclic set differs radically from the general series in that *any* partitioning generates totally systematic connections among pitch-class collections. . . . Any four-note chord comprising two dyads of interval 7 and two dyads of sum 9 or sum 4 will occur as a segment of the cyclic version of Berg's primary set-form [principal row]. Triadic partitionings produce segments in which each element of the set

[8] Cyclic sets form the foundation of Perle's theory of twelve-tone tonality. They are discussed in detail in this dissertation both within and outside the context of Perle's theory, in chapters two and three, respectively.

serves as the 'axis note' of a cyclic interval, the perfect fifth" (230). Through this statement Perle forms an unmistakable connection between Berg's cyclic practices (as evident in the principal row of the *Lyric Suite*, given in figure 1.3) and his own theory of twelve-tone tonality, which is based on the same cyclic set concept.

Figure 1.3. Cyclic intervals and adjacent sums in the principal row of the first movement of Berg's Lyric Suite

```
sums:          9     4     9     4     9              9     4     9     4     9
            F     E     C     A     G     D  |  G♯    C♯    D♯    F♯    A♯    B
intervals:  └───7───┘   └───7───┘              └───7───┘   └───7───┘
                  └───5───┘   └───5───┘              └───5───┘   └───5───┘
```

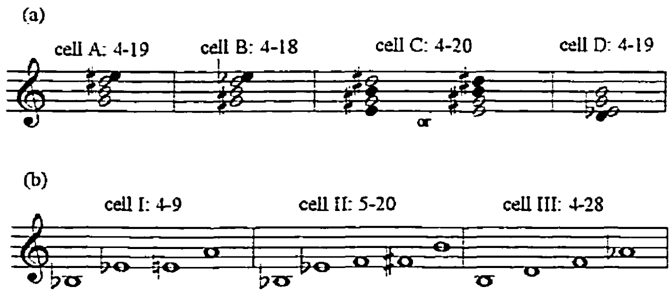
In his two-volume work *The Operas of Alban Berg* Perle discusses in detail the musical language of *Wozzeck* (vol.1, 1980) and *Lulu* (vol.2, 1985). In the latter volume, Perle comments on the difficulty, in atonal music, of establishing primary thematic and harmonic material and differentiating it from secondary or transitional material without the recourse to functional relationships available within tonality. Perle contends that Berg resolves the problem in his use of "basic cells" (which Perle also refers to as "intervallic cells"):

> The integrative element [of atonal music] is often a minute intervallic cell, which may be expanded through the permutation of its components, or through the free combination of its various transpositions, or through association with independent details. It may operate as a kind of microcosmic set of fixed intervallic content, statable either as a chord or as a melodic figure or as a combination of both. Its components may be fixed with regard to order, in which event it may be employed like the twelve-tone set, in its literal transformations. . . . Where it is stated as a simultaneity the order is not generally defined (1985, 87).

Perle discusses Berg's manipulation of these basic cells in both operas. In *Wozzeck* Perle demonstrates how the basic cells are employed specifically to define the sonata form of the music for the opening scene of Act II (1980, 145). The basic cells of *Lulu* have dramatic associations and functions throughout the opera (1985, 87). Examples 1.4a and b give the principal basic cells of *Wozzeck* and *Lulu*, respectively, as identified by

Perle.[9] All of these basic cells have cyclic structures. The basic cells in *Wozzeck* are formed from segments of either a single interval cycle with an added non-cyclic tone (cells A and D from sc 4-19, and cell B from sc 4-18), or combined interval cycles (cell C, from sc 4-20). Of the three basic cells identified in *Lulu*, the first (sc 4-9) may be viewed as a combination of two different interval-6 cycles (which Perle labels the Z-cell in Bartók's quartets), while the second (sc 5-20) may be viewed as a gapped segment of the interval-5 cycle; the third (sc 4-28) comprises a complete interval-3 cycle.

Example 1.4. Basic cells in Berg's Wozzeck *(a) and* Lulu *(b)*



In "The First Four Notes of Lulu" (1989) Perle recalls how his views have evolved over the course of the thirty years since his 1955 article on Bartók's use of symmetrical formations. At the end of the earlier article Perle remained uncertain as to whether symmetrical formations could form the basis of a common language for twentieth-century atonal music. But he now maintains that interval cycles and inversional symmetry are "a natural consequence of the replacement of a diatonic scale of unequal intervals between scale degrees by a semitonal scale of a single recurring interval" (1989, 284). Perle contends that these two concepts form a musical language that connects composers as diverse as Bartók and Berg, and demonstrates these connections in the opening of Berg's *Lulu* and Bartók's Fifth String Quartet.[10]

[9] Example 1.4a is adapted from Perle's Example 115 (1980, 146), while example 1.4b reproduces Perle's Example 28 (1985, 87).

[10] Perle makes a similar point at the end of *Twelve-Tone Tonality* (1977b, 171-72), although there he specifies that the sum and interval content of pc collections establish connections within and among twentieth-century works.

In his book *The Listening Composer* (1990a) Perle presents analyses of a number of twentieth-century excerpts to bolster his view of a common language based on principles of interval cycles and inversional symmetry. In the first chapter Perle provides a detailed analyis of Varèse's *Density 21.5*; he returns periodically to the analysis throughout the book to offer additional insights. Perle's analysis focuses primarily on the background structure of the piece, which he asserts comprises a succession of passages defined by octave boundary intervals. According to Perle, Varèse symmetrically partitions each of the octave boundaries into interval-6 cycles (as in C$^\sharp$-**G**-D$^\flat$). These in turn are partitioned into interval-3 cycles (as in C$^\sharp$-**E**-G-**B**$^\flat$-D$^\flat$), some of which are filled in at the surface by passing tones (as in C$^\sharp$-E-**F**$^\sharp$-G-**A**-B$^\flat$-**C**-D$^\flat$). Varèse consistently partitions the tritone divisions of the successive octave boundary intervals in this way until the last six measures of the piece, where he takes what Perle calls a "new harmonic direction" (1990a, 78). In the final passage (mm.56-61) the interval-6 partitions are not subdivided into interval-3 cycles, but into interval-2 cycles. The first three measures of this final passage contain pcs from one interval-2 cyclic collection, while the final three measures contain pcs from the other. Perle writes that these symmetrical partitionings "serve to unify a series of small-scale pitch relations and to comprehend them within an overall large-scale structure" (97).

In his tracing of common elements of a twentieth-century musical language, Perle maintains that

> To look for a series of direct 'influences' from one composer to the next as an explanation for these connections between Berg, Stravinsky, and Varèse is obviously fatuous and impossible to support on historical grounds. If such connections are to be explained by 'influences,' it is the common influence on all of them of the twelve-tone scale, the cyclic/symmetrical structure of which suggests corresponding cyclic/symmetrical structures derivable from the interval numbers that are factors of 12. Thus in the most natural way, the differentiating partitions of the universal pitch-class set emerge. . . . And though the qualitative transformation in the language of music that is implied in all this manifested itself rather suddenly, within a few years in the early part of this century, that transformation has a long prehistory in the tonal progressions that symmetrically partition the octave in the music of Schubert and Chopin and Liszt and Wagner (1990a, 92).

Perle sees a line of tradition issuing from nineteenth-century composers such as Schubert, Chopin, Liszt, Rimsky-Korsakov, and Wagner, who first symmetrically partitioned the octave within the context of tonality. Through his analytical work Perle traces the evolution of these practices in twentieth-century works by such composers as Scriabin, Bartók, Stravinsky, Varèse, and Berg, who to varying degrees exploited the concepts of interval cycles and inversional symmetry, both within and outside the context of tonality. Perle therefore views his own theoretical work as continuing the development of the musical language that he has traced in his studies of twentieth-century music.

Other analytical investigations of interval cycles in twentieth-century works

A number of theorists have sought to uncover structural applications of interval cycles in the pitch organization of twentieth-century music. In an analysis of Karol Szymanowski's Mazurka Op.50, No.3, Ann McNamee traces the unfolding of the circle of perfect fifths at a middleground structural level. In her analysis McNamee asserts: "The middleground motion over the span of the entire piece, which combines all of the foreground cycles, generates the complete circle of fifths" (1985, 71). McNamee generates the foreground cycles from two T_3-related dyads drawn from the boundary notes in the left and right hand in the first four measures of the piece, C♯-G♯ and E-B, which she calls "source dyads." The "foreground cycles" to which McNamee refers are actually four overlapping segments from the cycle of perfect fifths that emanate from the source dyads and divide into inversionally related segment pairs. McNamee's foreground cycles are given in figure 1.4.

Figure 1.4. Foreground cycles in McNamee's analysis of Szymanowski's Mazurka, Op.50, No.3

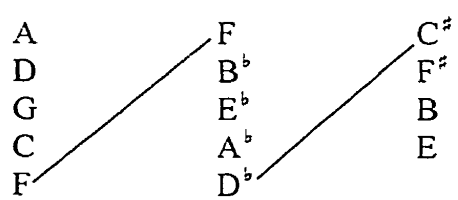
Cycle I:	E-B-F♯-C♯-G♯	Cycle III:	G♯-C♯-F♯-B-E- A-D-G-C-F
Cycle II:	B-E-A-D-G-C-F	Cycle IV:	C♯-G♯-D♯-[A♯]-F

McNamee traces the middleground progression of these four cycles, observing instances where they overlap, where they intersect through invariant cyclic elements, and also where they shift to an inversionally related cycle at the foreground through a "change of direction" of the cyclic progression. At the formal recapitulation, McNamee notes that all four foreground cycles combine to bring the piece to its conclusion.

Following J. Peter Burkholder, J. Philip Lambert divides the music of Charles Ives into two categories, music to be performed in public and "experimental" music (1990; 1997). The latter consists of musical sketches that explored new avenues of organization but were not intended for public consumption.[11] In this latter category, Lambert finds explicit usage of interval cycles in three basic formats: as single cycles, as combination cycles, and as "Omnibus" progressions (1997, 170).

Lambert observes a "wraparound technique" of interval cycles employed by Ives in several experimental works. In the sketch entitled *Song in 5's*, for example, a passage presents a succession of chords, each constructed of stacked interval 7s, as illustrated in figure 1.5 (1990, 49). A complete interval-7 cycle continues from one chord to the next by following a path from the top to the bottom of the first chord, wrapping around to the top of the second chord, and so on. The bottom note in each chord repeats as the top note of the next chord, thus the wraparound technique continues the cycle within and among the chords.

Figure 1.5. Interval-7 cycle in series of chords in Ives's Song in 5's



To expand his palette of available resources, Ives experimented by combining different interval cycles. The resulting "combination cycles" comprise a pair of transpositionally

[11] Lambert paraphrases this opinion expressed by J. Peter Burkholder in *Charles Ives: The Ideas Behind the Music* (New Haven: Yale University Press, 1985), 48.

related interval cycles, with elements of each cycle given in alternation, in a manner analogous to Perle's cyclic sets. Lambert labels a combination cycle according to the intervals formed by adjacent elements in the cycle, and places integers representing these intervals inside angle brackets. Hence some combinations of transpositionally related interval-5 cycles, for example, may include <32>, <98>, and <41>.[12] Figure 1.6 illustrates a <32> combination cycle derived from interval-5 cycles and a <95> combination cycle derived from interval-2 cycles (with the cycles differentiated by upper and lower case). The combination cycles, like single interval cycles, may be manifested in either the linear or vertical dimension.

Figure 1.6. <32> combination cycle from interval-5 cycles (a) and <95> combination cycle from interval-2 cycles (b)

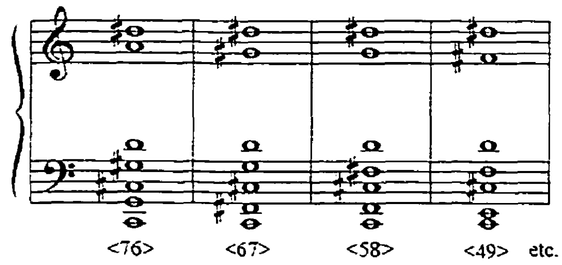
<32>	...	C	$e^{\flat}$	F	$a^{\flat}$	$B^{\flat}$	$d^{\flat}$	$E^{\flat}$	$g^{\flat}$	...
<95>	...	C	a	D	b	E	$c^{\sharp}$	$F^{\sharp}$	$d^{\sharp}$	...

Lambert introduces another cyclic model employed by Ives that he names the *Ives Omnibus*: "The Ives Omnibus is a series of musical entities, often simultaneities, ordered according to a gradual expansion or reduction in the sizes of the formative intervals" (1997, 170).[13] Such a series may entail a progression of chords, wherein the first chord comprises only adjacent interval-1 cycle pcs, the second chord comprises only interval-2 cycle pcs, and the third chord only interval-3 cycle pcs, continuing in this manner to some designated interval. Alternatively, the series may involve a combination cycle, in which the generating intervals of the chords change systematically, as from <76> to <67> to <58> to <49> and so on, as illustrated in example 1.5. Lambert concludes: "Thus does the Omnibus bring together the main features of systematic composition: it is

[12] Lambert changes his nomenclature for combination cycles in his book *The Music of Charles Ives* (1997). In his earlier article (1990), Lambert did not employ angle brackets; he simply separated the integers by a slash (as in 3/2 and 9/5).

[13] Lambert reports that his use of the term *omnibus* was "inspired" by Victor Fell Yellin's use of the word in conjunction with cyclic chromatic chord progressions in music by Mozart and Schubert. Lambert says that while there are some similarities his usage of the term, he prefers the word for the sense of pervasiveness it conveys (1997, 228, n.5).

Example 1.5. Series of chords generated by combination cycles as an Ives Omnibus



a carefully calculated structural model based on a self-generating, pitch-class-saturated transformational pattern" (170).

Elliott Antokoletz, a former pupil of Perle, expands Perle's investigations into cyclic and symmetrical structures in twentieth-century works. In his study "Interval Cycles in Stravinsky's Early Ballets" (1986), Antokoletz shows how interval cycles underlie the pitch materials in *The Firebird, Petrushka,* and *The Rite of Spring*. In numerous examples drawn from the ballets Antokoletz identifies pc segments as belonging to specific types of cyclic collections: octatonic, whole-tone, or diatonic. Throughout the study Antokoletz shows that these segments may either be juxtaposed, merged gradually into one another, or interlocked in a hybrid collection. He then reorders these collections as simple, gapped, or overlapping segments of interval cycles. Antokoletz frequently partitions the aggregate into literally complementary sets from scs 7-35 and 5-35, which he refers to as "white-key" and "black-key" segments of the interval-5 cycle, respectively.[14] He also examines modal collections as representing distinct segments of the interval-5 cycle.[15]

Antokoletz employs similar analytical techniques in his book *The Music of Béla Bartók* (1984). Antokoletz discusses how Bartók's folk-based melodies, which are of modal origin, may be expressed as interval-cycle segments: the diatonic modes (members of sc 7-35) and the pentatonic mode (sc 5-35) may be reordered as segments of the

[14] Antokoletz does not utilize Allen Forte's pc sc labels as set forth in Forte's *The Structure of Atonal Music* (1973).

[15] For example, a modal change from a G-Dorian to a C-Dorian diatonic collection implies a shift by one increment along the interval-5 cycle.

interval-5 cycle (1984, 51-66). In addition, Bartók employs non-diatonic modes; Antokoletz contends that Bartók extracts segments of these non-diatonic modes and then extends the segments symmetrically by adding pcs to create larger octatonic, whole-tone, or diatonic formations, which also may be reordered to show their cyclic properties (204ff).

Building on Perle's study of Bartók's string quartets (1955) Antokoletz expands considerably Perle's initial observations about the symmetrical tetrachordal sets (scs 4-1, 4-21, and 4-9), which Antokoletz formally identifies as the X, Y, and Z cells, respectively. In his discussion of the various properties of the cells, Antokoletz illustrates how different transpositions of a cell may combine to generate an interval-cycle segment (69-77). These include the juxtaposition of two 4-1 sets (X-cells) a semitone apart to form an eight-note segment of the interval-1 cycle, two 4-21 sets (Y-cells) a whole tone apart to form an eight-note segment of the interval-2 cycle, two 4-9 sets (Z-cells) in a T_3-relation to form an octatonic collection, and so forth. In addition, other combinations of these cells may generate different cells. For example, the boundary notes of two Y-cells a semitone apart will form a Z-cell (0-6 and 1-7). Antokoletz also describes a cell's role in cyclic expansion, whereby a cell segmented in the music is augmented by the symmetrical addition of pcs to form a segment of an interval cycle. Antokoletz exposes the surface and middleground presence and interaction of the three cells in a number of Bartók's works, primarily the Fourth String Quartet. He traces the cells' interaction and progression, establishing different axial centers which he claims are analogous to tonal centers in traditional tonal music.

In Antokoletz's view interval cycles and symmetry constitute the fundamental structural principles underlying Bartók's works. In the preface of *The Music of Béla Bartók* Antokoletz writes that he

> senses in Bartók's total output an all-encompassing system of pitch relations . . . [one] primarily based on the *equal subdivision* of the octave in the total complex of interval cycles. The fundamental concept underlying this equal-division system is that of symmetry. The functions and interactions of symmetrical pitch collections are significant both in the generation of the interval cycles in a given composition and in the establishment of central tonal or sonic areas. Although Bartók's music is permeated by nonsymmetrical pitch collections (including the traditional major and minor triads) as well as symmetrical ones, properties of the former in the organic

> growth of a work can generally be understood as having latent symmetrical possibilities; that is, nonsymmetrical collections often emerge in the course of a composition as segments of larger symmetrical formations (xii).

Gary Karpinski extends Antokoletz's exploration of interval cycles in his dissertation "The Interval Cycles in the Music of Bartók and Debussy though 1918" (1991). Karpinski draws parallels between the types of cyclic collections the two composers employed, and the contexts in which they employed them, at both local and longer-range structural levels.

In his dissertation "Interval Cycles and Symmetrical Formations as Generators of Melody, Harmony, and Form in Alban Berg's String Quartet, Opus 3" (1989) Charles Porter finds "themes" derived from segments of interval cycles at foreground, middleground, and background levels of structure. Under the rubric of "theme" Porter recognizes both motives and complete melodies, regardless of their length (15). These themes occur chiefly as interval cycles and cyclic segments embedded in other longer-range interval cycles and cyclic segments. Porter concludes: "Berg's use of interval cycles in the linear dimension of Opus 3 . . . demonstrates at the very least an intuitive grasp of the importance of interval cycles to the development of a new musical language as directors of motion to or from a particular pitch. The use of interval cycles on the largest scale to generate formal division is an important kind of solution to the problem of large-scale form in an atonal idiom" (102).

In his book *The Music of Alban Berg* (1996) Dave Headlam acknowledges the customary division of Berg's music into tonal, atonal, and twelve-tone creative periods, yet maintains that Berg's use of interval cycles transcends these distinctions. He contends that these are more chronological distinctions, representing differences in degree rather than in kind (11). Headlam shares Perle's view that Berg's works are generally cycle-based with incorporated dissonances, which are analogous to non-chord tones in tonal music. Headlam asserts that since purely cyclic passages are static, Berg limits them to infrequent, short passages. More typically, Berg inserts one non-cyclic tone in a cyclic collection. According to Headlam, Berg does this for two reasons: (1) to generate forward motion toward some resolution of the dissonant note, although it is never attained, and

(2) to create contrasting cyclic collections over surface and long-range spans (60-62). Headlam labels such collections with a plus sign, as in a "whole-tone+ collection," or a "5-cycle+ collection." The dissonant note also allows for the possibility of reinterpretation, whereby collections may serve as pivots into other cyclic systems. For example, sc 4-2 (0124) can be interpreted as a 1-cycle+ or a 2-cycle+ collection (73).

While acknowledging that Berg establishes cycles over long-range spans, Headlam does not identify this as a cyclic prolongation: "In my view, the cyclic collections in Berg's atonal music are referential and the basis of the pitch language, but they are not prolonged in a tonal sense. Cyclic collections are quickly superseded, are not 'in force' in their absence, and require constant reiteration for their continuing referential status. Thus, I do not posit large-scale cyclic collections comprised of largely non-adjacent notes spanning a piece or large sections" (63-64).

In discussing how Berg's twelve-tone techniques differ from those of Schoenberg and Webern, Headlam contends that even in Berg's twelve-tone works the interval cycle is still the underlying principle, although now functioning in conjunction with considerations of aggregate completion and order positions. Headlam believes Berg considered the row to be a means rather than an end:

> Since the rows are not central, their treatment of relationship to the surface need not be consistent. Thus Berg can reorder rows and even add or omit notes without disturbing the language. Although he often carefully related derived materials to the original row, the use of row-derived materials in non-row contexts, the reordering of row segments, and the free addition of non-row-derived notes suggests that the basis of the language is not the rows but the smaller derived and non-derived materials, which are mostly, as in his atonal music, cyclic-based collections (197-98).

Theoretical investigations of interval cycles

Interval cycles have interested composers and theorists from perspectives other than the analytical. In his book *Composition with Pitch-Classes* (1987) Robert Morris discusses the twelve-tone operators of transposition, inversion, and multiplication (by a factor of 5) as generators of cycles of pcs. Morris describes how a twelve-tone operator (TTO) acting on each of the 12 pcs generates a list of mappings (1987, 128). Cycles are formed by linking together mappings that ultimately return the original pc. For example, the TTO T_4 maps pc 0 into pc 4, notated as $0 \rightarrow 4$. Morris links this with the T_4 mapping of $4 \rightarrow 8$, and then with the T4 mapping of $8 \rightarrow 0$. The resulting cycle is (0-4-8). Morris uses the term *periodicity* to denote the number of repeated applications of a TTO required to generate a cycle (126). In the above example, the TTO T_4 has a periodicity of 3, since three applications of T_4 are required to complete the cycle. The periodicity is notated as $(T_4)^3$, and expressed as "T_4 to the third power" (126).

For each TTO there is an inverse TTO which negates or "undoes" the original operation. The inverse of TTO T_4 is T_8. The cycles generated by inverse TTOs have the same length and content, but their pcs occur in retrograde order. The T_4 cycles are (0-4-8), (1-5-9), (2-6-10), and (3-7-11). The T_8 retrogrades are (8-4-0), (9-5-1), (10-6-2), and (11-7-3). Morris labels as *involutions* those TTOs that are their own inverses. The TTOs T_0 and T_6 are involutions, for example, as are all of the T_nI operations. The cycles of an involution have a periodicity of 1 or 2.

Morris asserts that the powers of a TTO form a *cyclic group* (151). The number of TTOs within a cyclic group is determined by the periodicity of the TTO. Table 1.1a lists the cyclic group of TTOs formed by the powers of T_2; this TTO has a periodicity of 6. Some cyclic groups contain cyclic subgroups, as indicated in table 1.1b and c. For example, the cyclic group of T_4 is a subgroup of cyclic group T_2 since the powers of T_4 are also the second, fourth, and sixth powers of T_2. These two cyclic groups thus coincide in the TTOs T_4, T_8, and T_0. In the same way, the cyclic group of T_6 is a subgroup of cyclic group T_2 since the powers of T_6 are respectively the third and sixth powers of T_2. These two cyclic groups coincide in the TTOs T_6 and T_0.

Table 1.1. The cyclic group of T_2 (a) and cyclic subgroups of T_4 (b) and T_6 (c)

TTO:	**T_2**	**T_4**	**T_6**	**T_8**	**T_{10}**	**T_0**
a. powers of T_2:	$(T_2)^1$	$(T_2)^2$	$(T_2)^3$	$(T_2)^4$	$(T_2)^5$	$(T_2)^6$
b. powers of T_4:		$(T_4)^1$		$(T_4)^2$		$(T_4)^3$
c. powers of T_6:			$(T_6)^1$			$(T_6)^2$

In appendix 2 of his book, Morris lists the cycles generated by the 48 TTOs.[16] They are divided into four types: T_n, T_nI, T_nM, and T_nMI. The TTOs of each type may be divided further into classes according to the structure of their cycles: six classes of T_n, two classes of T_nI, three classes of T_nM, and four classes of T_nMI. These classes are listed below in table 1.2. Within each class, the cycles generated by the specific TTO reveal transpositionally related cyclic patterns. Figure 1.7 illustrates how one pair of TTOs, T_1M and T_3M, belong to the same class based on the transpositional relationship between their corresponding cycles.

Table 1.2. Classes of TTOs according to transpositionally related cyclic structures

TTO	class membership	
T_n	(1) n = 0,6	(4) n = 3,9
	(2) n = 1,11	(5) n = 4,8
	(3) n = 2, 10	(6) n = 5,7
T_nI	(1) n = odd integer	
	(2) n = even integer	
T_nM	(1) n = 0,4,8	
	(2) n = 2,6,10	
	(3) n = odd integer	
T_nMI	(1) n = 0,6	(3) n = 2,4,8,10
	(2) n = 1,5,7,11	(4) n = 3,9

Figure 1.7. Transpositional relationship between cycles of TTOs T_1M and T_3M

T_1M:	(0-1-6-7)	(2-5-8-11)	(3-4-9-10)
T_3M:	(10-11-4-5)	(0-3-6-9)	(1-2-7-8)

John Clough explores interval cycles in the context of a diatonic universe of seven pcs in his article "Diatonic Interval Sets and Transformational Structures" (1979-80). He measures intervals between pcs in ascending diatonic steps, and disregards the chromatic

[16] A similar version of this appendix and much of the related discussion appears in an earlier article written in collaboration with Daniel Starr (Starr and Morris, 1977) and in a separate article by Starr (1978).

notes between certain steps, asserting that his depiction of the diatonic system is not as a subset of the twelve-pc universe. Instead, Clough depicts a universe of seven pcs, where all diatonic intervals of the same numerical name are considered equivalent, regardless of quality.[17]

Clough lists the ascending intervals between adjacent pcs in an ordered string, which he calls an *interval series* (IS). The *equal-interval series* (=IS) is a specific type of IS consisting of a string of intervals of the same size. Clough recognizes six classes of =IS (excluding interval zero). He notes that all six classes produce all seven pcs of the diatonic system before any pcs are repeated (1979-80, 468).

Clough argues that the melodic sequence is an example of diatonic serialism, although "in a weaker sense than that of classical 12-note serialism" (470).[18] Clough recognizes a sequence as having "serial" properties if it unfolds a complete diatonic interval cycle (=IS), or can be decomposed into two or more interval cycles, each of which produces all seven pcs before any repeat. Clough proposes three procedures for detecting such a serial process underlying a given melodic sequence. He identifies the first procedure as "decomposition into parallel lines." If a sequence contains two or more different intervals in a repeating interval series (RIS), it can be divided into the same number of parallel lines, each exhibiting a different =IS. Example 1.6a decomposes the RIS (14) into two parallel =IS, which may be interpreted as diatonic interval-5 cyclic segments.[19] The second procedure, "straightforward enumeration," applies to a sequence generated by a single interval. Example 1.6b shows how a single =IS sequence may be interpreted as a segment of a diatonic interval-2 cycle. When verticalized, the parallel chords that result also reveal three linear segments of diatonic interval-6 cycles. The third procedure is identified as "grouping." A sequence may be segmented into discrete groups of pcs which, when reordered, produce a succession of parallel chords corresponding to an =IS,

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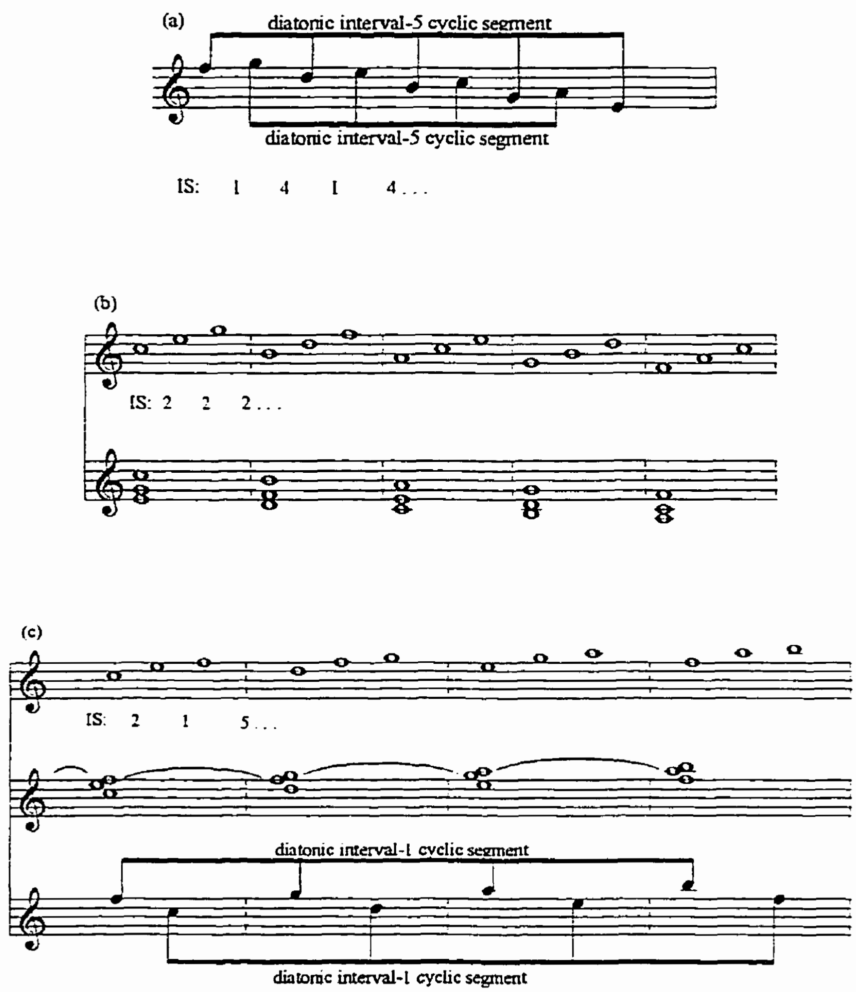
[17] Clough refers to this aggregate of seven pcs as the "total diatonic" or "TD" (1979-80, 469).

[18] Clough's use of the term "sequence" does not include considerations of register and rhythm.

[19] Examples 1.7a, b, and c are adapted from Clough's examples 6, 7, and 9 respectively (1979-80).

Example 1.6. Melodic sequences that illustrate "serial" properties in their component diatonic interval cycles



as illustrated in example 1.6c.[20] As with the second procedure, grouping results in parallel chords that reveal linear diatonic interval cycles.

Clough also discusses *regular extraction* and *regular interpolation* (476-80). The first term refers to a process of selecting pcs at regular intervals from an =IS in order to generate another =IS. The result of this process may be interpreted as a diatonic interval cycle embedded in another diatonic interval cycle. Conversely, the second term refers to a

[20] Clough further stipulates that repeated notes are treated as tied notes, and are not considered members of subsequent groups.

process of adding pcs to form fixed intervals between each of the intervals of a given =IS. Clough contends that both processes may generate hierarchical structures in the diatonic system.

Analytical studies of Perle's theory of twelve-tone tonality

Perle has won acclaim for his compositions, and his analytical contributions to the study of twentieth-century music are widely recognized, particularly in the area of Berg scholarship. But his theory of twelve-tone tonality is not well known in the theoretical community, perhaps in part due to its compositional purview, idiosyncratic terminology, and high level of abstraction. Only in the last decade or so have other expositions of Perle's theory appeared. These include dissertations by T. Patrick Carrabré (1993) and Steven Rosenhaus (1995), a chapter by Elliott Antokoletz in his textbook *Twentieth-Century Music* (1992), and articles by Dave Headlam (1995) and Charles Porter (1995). Porter's work focuses on the manifestation of the 5:4 proportion in the parameters of rhythm, tempo, phrase, form, and elapsed time in "Nocturne," the third movement of Perle's *Sonata a Quattro*, and does not extend to pitch.

In the chapter devoted to Perle's theory of twelve-tone tonality in *Twentieth-Century Music* (1992, 426-47), Antokoletz introduces the basic elements of the system and provides short analyses of excerpts from three works. These works are drawn from three different periods of Perle's compositional output, and are meant to be representative of the evolution of the system and its expanding possibilities. Antokoletz only briefly mentions the more abstract relations among the elements, focusing instead on the construction, interpretation, and relationships of chords in progressions at the compositional surface.

T. Partick Carrabré divides his dissertation "Twelve-Tone Tonality and the Music of George Perle" (1993) into three chapters. The first chapter presents the theoretical principles of Perle's theory. In the next two chapters Carrabré adopts a chronological approach in order to trace the theory's evolution. Beginning with the inception of Perle's theory in the 1940s, Carrabré divides Perle's compositional output into several periods: the twelve-tone modal system, the expanded modal system, the twelve-tone tonal system, as described in Perle's book *Twelve-Tone Tonality* of 1977, and the more mature system that includes Perle's formulation of synoptic arrays.[21]

Carrabré supplements each stage of his presentation with analyses, in which he employs voice-leading graphs that typically include multiple levels of reduction. Further, Carrabré employs a number of Schenkerian concepts, such as the *Urlinie*, prolongation, register transfer, cover tones, reaching-over, and initial ascent to the primary tone. The voice-leading graphs show diminutions of neighbour gestures, passing motions, arpeggiations, and segments of interval cycles. Carrabré modifies the Schenkerian terms to fit within the context of the twelve-tone tonality system. For example, three of his graphs show a descending fundamental line, but the lines are of chromatic semitonal steps to an axial note, rather than diatonic steps to a tonic.[22] Carrabré asserts that harmonic motion, like voice leading, is generally directed by long-range neighbour gestures, passing gestures, cyclic segments, T_6-related verticalities, or prolongation of dyads of the same interval or sum.

In his dissertation "Harmonic Motion in George Perle's *Wind Quintet No.4*" (1995) Steven L. Rosenhaus presents the main tenets of the system of twelve-tone tonality in the form of ten "general principles" based on information distilled from Perle's *Twelve-Tone Tonality* and Carrabré's dissertation. Rosenhaus then proceeds to discuss Perle's Wind Quintet No.4 (1987) in detail. In what he describes as an "eclectic approach," Rosenhaus represents the large-scale relationships of twelve-tone tonality in simple line graphs and relationships of synoptic arrays in tables. As did Carrabré, Rosenhaus illustrates prolongational relationships and voice leading in the form of reductive graphs. Rosenhaus opts not to perform an exhaustive analysis that accounts for every pitch of the quintet, stating that "with the inherent flexibility of the twelve-tone tonality system, arrays can change, and do change in the Quintet, frequently. The same may also be said of array alignments. . . . To attempt a chord-by-chord analysis under those circumstances would ultimately prove fruitless" (1995, 260). As a result, Rosenhaus limits his foreground

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[21] Synoptic arrays are fully explained in chapter two of this dissertation.

[22] Carrabré (1993) gives the graphs in his examples 2-11 (152), 3-12 (207), and 3-19 (230).

analysis mostly to cadence points, and focuses his attention instead on how the larger relationships of tonality and synoptic arrays direct the formal structure of the work.

Dave Headlam's article "Tonality and Twelve-Tone Tonality: The Recent Music of George Perle" (1995) combines a presentation of Perle's theory with a detailed analysis of the third movement from his *Piano Concerto No.1*. In a pedagogically oriented approach, Headlam unfolds the tenets of the theory gradually. He begins by making some general observations about the pervasiveness of certain intervals in the opening six measures of the movement. Headlam uses these observations to introduce some of the basic elements of the theory. He then reexamines the musical excerpt in the context of the theoretical elements he has just presented. Headlam repeats this three-stage presentation throughout the article, moving from the most basic elements to the more abstract relationships of the theory. He discusses briefly Perle's increased use of dissonance in the context of twelve-tone tonality, defined as pcs that do not belong to the prevailing array (307-8).[23] The only significant relationship not discussed by Headlam in this article is that of synoptic arrays. This is perhaps due to the fact that while Perle presents the topic in the first edition of *Twelve-Tone Tonality*, the concept of synoptic arrays is not fully formed until the second edition of 1996.

Headlam makes limited use of pc set theory, in order to relate segmented groups of pcs according to their transpositional or inversional equivalence. Headlam presents his analyses of the movement in reductive graphs. Although the graphs employ the Schenkerian notational conventions of beams, slurs, and stemless noteheads, the relationships they represent are not prolongational in the Schenkerian sense. Rather, Headlam's graphs focus on the patterns of sums and intervals and interval-cycle unfoldings within linear and vertical segments.

* * *

[23] Arrays in the context of twelve-tone tonality are combinations of cyclic sets, and are discussed in detail in chapter two of this dissertation.

Although there are some points of intersection, the approach taken in this dissertation departs from previous expositions of Perle's theory in several ways. Chapter two presents the tenets of twelve-tone tonality, taking into account the most recent developments of the theory as set forth in the second edition of *Twelve-Tone Tonality* (1996). The topics have been reordered here, however, for greater clarity. Some of the more abstruse topics are not included, due to my determination that they are not applicable, or not absolutely necessary to the discussion at hand, or because Perle himself has indicated that the topics are still not fully developed, or have not proved fruitful.

Following this exposition of Perle's theory of twelve-tone tonality, the next two chapters branch off in two very different directions; chapter three is of a speculative nature, while chapter four is analytical. Chapter three unfolds a theoretical investigation entirely outside the context of twelve-tone tonality, although it takes Perle's cyclic sets as its point of departure. The chapter explores the properties of the cyclic sets, utilizing selected tools of pc set theory to generate families of pc set classes from the cyclic sets. The chapter reveals close structural relationships within and among these families.

Readers interested in analytical applications of concepts from Perle's theory as presented in chapter two may wish to proceed directly to chapter four, which marks a return to the context of twelve-tone tonality. Chapter four presents analyses of the first and fourth etudes of Perle's *Six Etudes for Piano* (1973-76). The chapter aims to uncover compositional strategies at both local and deeper structural levels. The analyses challenge Rosenhaus's contention, in his study of Perle's *Wind Quintet No.4*, that "there are too many arrays to account for, and elements of prolongation . . . to rely solely on a chord-by-chord description. Such examinations would never describe the work as a whole, for most of its internal relationships operate at the highest, that is, synoptic, level" (1995, 245-46). In the detailed analyses of both etudes presented in chapter four, it is clear that the more local relationships are reflected in a variety of ways, not only at the compositional surface, but also at higher levels.

The analyses of the selected etudes are guided primarily by Perle's theory of twelve-tone tonality, but are supplemented by observations from the perspective of pc set theory, as developed by Allen Forte (1973) and extended by Robert Morris (1995b). The set-theoretical approach provides insights that complement those of the twelve-tone tonal approach. In addition, many compelling observations of Perle's music are evident only from the perspective of pc set theory. The combination of approaches provide a multi-dimensional view of the etudes.[24]

Finally, chapter five will review the insights and contributions of the previous chapters, discuss the critical reception of Perle's theory, and offer suggestions for areas of future research.

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[24] In addition to cyclic and symmetrical patterns, formations, and progressions, the analyses in chapter four uncover techniques more closely related to serial twelve-tone practices, such as the operations of transposition, inversion, retrograde, retrograde-inversion, and rotation. This finding speaks to the influence of the composers of the Second Viennese School, especially Berg, on Perle's own compositional methods.

Chapter Two

The Theory of Twelve-Tone Tonality

Perle's system of twelve-tone tonality evolved from some initial misconceptions he had formed about Schoenberg's twelve-tone system. Perle writes: "My understanding of the nature of the twelve-tone row and of how it could function as a source of pitch class relations differed in the most radical way from Schoenberg's" (1990a, 127).[1] First, Perle was unaware that Schoenberg, unlike Berg, did not employ the technique of rotation. Perle assumed that the last pitch class (pc) of the row had as its closest neighbours the first and penultimate pcs of the row. Second, Perle did not realize that the twelve pcs of a row followed a strict linear ordering. "Uninstructed as I was in Schoenberg's 'twelve-tone system,' I correctly assumed that the adjacencies comprised in the forty-eight different forms of a given twelve-tone row were a collective statement of the relations assigned to each element of the semitonal scale, but it did not occur to me that the tone row in itself was to be construed as a unitary linear structure" (1977b, ix).

Perle observed that any given pc has neighbours in the prime form of the row, and two other neighbours in the inverted form. For example, in figure 2.1a, C is flanked on either side by B and D in the prime form of the given row segment, and by F♯ and A in the inverted row. Perle reasoned that since these four neighbour notes constituted the closest relationships with the axial tone, then that tone could proceed in either direction to or from any of these pcs and could move freely between the prime and inverted forms of the

[1] Perle's first exposure to music composed in the twelve-tone method occurred in 1937, when he encountered Berg's *Lyric Suite*. Perle began an intensive study of the work, hoping to deduce Schoenberg's method from it. In 1937 very little was known in the United States about Schoenberg's twelve-tone method. Contemporary commentaries on the *Lyric Suite* gave fleeting reference to the work's twelve-tone aspect. Moreover, access to Schoenberg's own twelve-tone music was severely limited, due to the outbreak of World War II. Hence it was not until 1939, during a composition lesson with Ernst Krenek, that Perle learned of his misunderstanding of Schoenberg's method (1990a, 123-134).

row. Hence Perle interpreted the four neighbours as forming a cluster around a central tone, as in figure 2.1b. Each neighbour tone may in turn become an axial tone, likewise moving freely within its own neighbour-note cluster. We see this in figure 2.1c, with pc A now the axial tone. Perle believed that any given pc could move back and forth between neighbouring pcs in both the prime and inverted forms of the twelve-tone row.

Figure 2.1. Neighbour-note relationships around axial tones

(a)	row segment in prime form:	*B*	*C*	*D*	*F*	*A*	$A^{\flat}$	*G*	$F^{\sharp}$	$D^{\sharp}$	*E*	$B^{\flat}$	$C^{\sharp}$	*(B*
	row segment in inverted form:	B	$B^{\flat}$	$A^{\flat}$	F	$C^{\sharp}$	D	$D^{\sharp}$	E	G	$F^{\sharp}$	C	A	(B

(b) neighbour-note cluster around axial-tone C:

	$F^{\sharp}$	
B	*C*	*D*
	A	

(c) neighbour-note cluster around axial-tone A:

	C	
F	*A*	$A^{\flat}$
	B	

Perle was soon to learn of his error, but the notion of inversionally related row forms and neighbour-note relationships intrigued him, and sparked the genesis of his own theory of twelve-tone tonality.

This chapter will first introduce the more concrete elements of Perle's theory, and then proceed to a description of the basic relationships and means of progression among these elements followed by an exploration of the most abstract structural relationships. The chapter will conclude with a discussion of the theory's potential for further development.

Cyclic sets

Perle builds his system of twelve-tone tonality on a foundation of interval cycles. An interval cycle is an ordered series of pcs based upon a single recurrent interval, which is measured by the number of semitones it spans. The interval cycle is completed by the return of the initial pc. There are twelve different cycles, generated by the intervals of each of the six different interval classes (ics). These are notated in example 2.1, along with corresponding pc integer notation.[2] Cycles generated from complementary intervals form successions of pcs in retrograde order. Hence an ic-cycle represents both complementary intervals within the same ic. Members of an ic therefore include a given interval, its complement, and all octave equivalents. The octave divides equally into one semitonal or ic1-cycle, two whole-tone or ic2-cycles, three minor-third or ic3-cycles, four major-third or ic4-cycles, and six tritone or ic6-cycles. There is only one perfect-fourth or ic5-cycle.

Perle generates the basic unit of his system by alternating members of inversionally related interval cycles (example 2.2). He gives this construct the name *cyclic set*.[3] Perle differentiates between the complementary cycles by designating the ascending form the P-cycle (prime), and the descending form the I-cycle (inverted), and by listing the P-cycle members in italic font. In this formation any given pc is referred to as an *axis note* and is flanked by a *neighbour note* on either side. Each pair of neighbour notes or axis notes derives from the same interval cycle, and thus represents the set's *cyclic interval*. In addition, each axis note maintains the same combination of sums with its neighbours. In the cyclic set of example 2.2a the cyclic interval is 1, and the alternating adjacent sums are 0 and 1. These sums are called the "tonic sums."[4] The tonic sums pinpoint the specific pc alignment of the inversionally related cycles. Each pair of neighbour notes in example 2.2b preserves the cyclic interval 7, while each pair of adjacent members (the axis pc paired with each of its neighbours) forms a repeating tonic sum couple of 0,7. Hence the cyclic set needs only three consecutive elements for its identification. The segment *4* 8 *e*, for example, enclosed in a rectangle in example 2.2b, identifies the cyclic interval 7 (as e-4=7) and the two tonic sums 0 (as 4+8) and 7 (as 8+e). Moreover, the cyclic set is symmetrical: its retrograde ordering of pcs is identical to its prime ordering, and the

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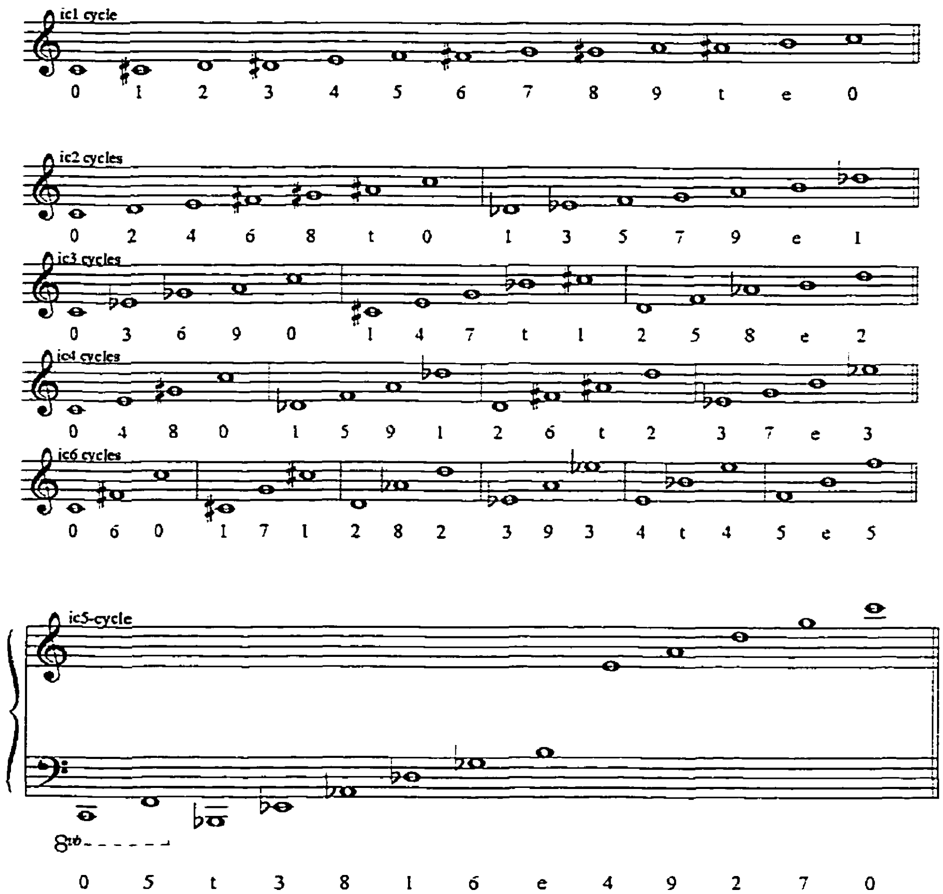
[2] Integers 0 to 11 represent the 12 pcs, with C=0, C♯/ D♭=1, and so on. Integers 10 and 11, when representing pcs, sums, or differences, will be replaced by "t" and "e" respectively.

[3] In the first edition of *Twelve-Tone Tonality* (1977b) Perle used the terms "cyclic sets" and "set forms" interchangeably. In the second edition (1996) he uses the term "cyclic set" exclusively throughout the new chapters of Part II.

[4] Perle refers to these sums as "adjacency sums" and "primary sums" in the first edition of *Twelve-Tone Tonality;* he identifies them consistently as "tonic sums" in Part II of the second edition of *Twelve-Tone Tonality* (1996).

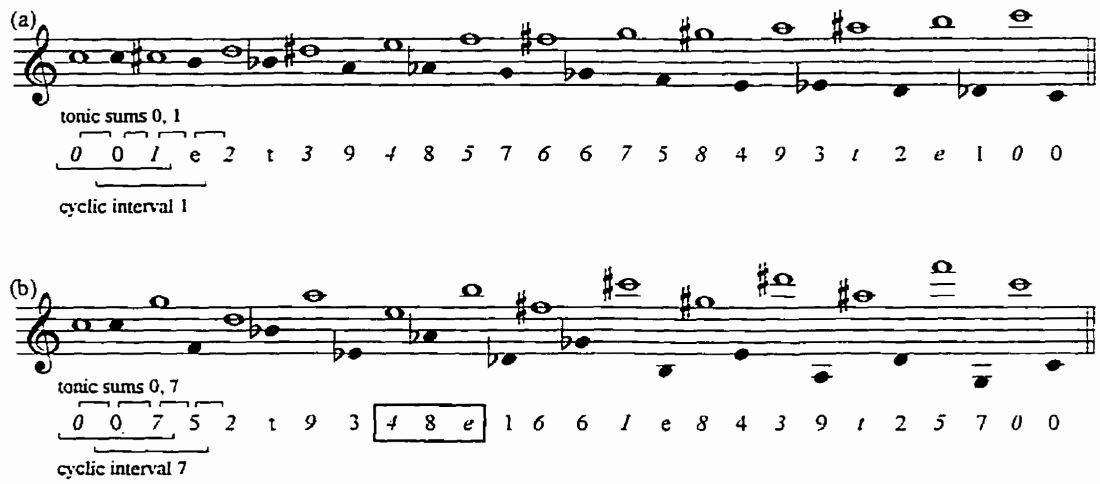
Example 2.1. Interval cycles



second half of the cyclic set is both the tritone transposition and the retrograde of the first half.[5]

Perle uses the tonic sums to name the cyclic sets, with each sum preceded by the lower-case letter **p** or **i**. Perle applies a formula which is based on whether the pcs involved belong to P or I cycles, and whether the tonic sums they form are even or odd. If the tonic sum is even and the left element of the dyad belongs to the P cycle, then the sum will be preceded by a **p**. But if the left element belongs to the I cycle, then an **i** precedes the sum. If the tonic sum is odd, the opposite situation obtains. Once the p- and i-designations have been determined, the tonic sums should then be ordered so that the

[5] The cyclic sets for all cyclic intervals (mod 12) are listed in appendix one.

Example 2.2. Cyclic sets of interval 1(a) and interval 7 (b)

cyclic interval may be determined consistently by subtracting the left sum from the right. Thus, in example 2.2a the cyclic set is labelled as p0p1; the cyclic set is constructed from inversionally related interval 1 cycles, and its tonic sums are 0 and 1. In the same way, the cyclic set in example 2.2b is p0p7; this cyclic set derives from inversionally related interval 7 cycles, and its tonic sums are 0 and 7.[6]

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[6] In the first edition of *Twelve-Tone Tonality* Perle contends that the ordering of the tonic sums in a cyclic set is irrelevant. Whether a cyclic set is identified as p0p7 or p7p0 the ordering of the pcs and the cyclic interval within the set are the same. To determine the cyclic interval Perle instructs the analyst to "subtract the even sum from the odd for p and vice versa for i. Thus p0p7 and i5i0 are both interval-7 cyclic sets" (1977b, 21). Yet Perle does not explain here how to determine the cyclic interval for a cyclic set that includes both p- and i-designations and whose tonic sums are either both even or both odd, as in p0i2. In a later article, however, Perle advises the analyst to "count up from an even 'p' or odd 'i' sum and down from an odd 'p' or even 'i' sum" (1993, 300, n.2). This rule obtains for both even and odd cyclic intervals. Hence the cyclic interval of the p0i2 set is 2, whereas p2i0's cyclic interval is t. Likewise, p0p5's cyclic interval is 5, while i5i0's cyclic interval is 7.

T. Patrick Carrabré suggests a simple method for determining the cyclic interval: subtract the tonic sum on the left from the tonic sum on the right (1993, 33). But this method may lead to the assumption that interval-7 generates p0p7, while interval-5 generates p7p0. The assumption that p0p7 and p7p0 are generated by different cyclic intervals contradicts Perle's intention. Clearly, the ordering of the tonic sums themselves is also significant. Perle acknowledges this fact in the newly added chapter 33 in the second edition of *Twelve-Tone Tonality*, wherein he stipulates that "even p-sums and odd i-sums [be placed] on the left, and odd p-sums and even i-sums on the right, so that the left tonic sum of the set is subtracted from its right tonic sum to determine the cyclic interval" (1996, 183).

Arrays and array segments

Perle vertically aligns two cyclic sets to form what he calls an *array*. The array takes its name from its component cyclic sets; hence both pairings in figure 2.2a and b are identified as the array p0p7/p4pe. The cyclic set p4pe in figure 2.2b has been shifted one position to the right to create a different alignment between the two cyclic sets.

Figure 2.2. Two alignments of array p0p7/p4pe

(a)	p0p7:	*0*	0	*7*	5	*2*	t	*9*	3	*4*	8	*e*	1	*6*	6	*1*	e	*8*	4
	p4pe:	*7*	9	*2*	2	*9*	7	*4*	0	*e*	5	*6*	t	*1*	3	*8*	8	*3*	1
(b)	p0p7:	*0*	0	*7*	5	*2*	t	*9*	3	*4*	8	*e*	1	*6*	6	*1*	e	*8*	4
	p4pe:	4	*7*	9	*2*	2	*9*	7	*4*	0	*e*	5	*6*	t	*1*	3	*8*	8	*3*

Perle segments arrays into units of varying sizes, primarily dyads, trichords, tetrachords, and hexachords.[7] He identifies the main unit as the *axis-dyad chord*, a collection of six pcs formed by pairing trichordal segments from each of the cyclic sets. In figure 2.2a, a rectangle encloses one such axis-dyad chord, formed by aligned trichords *9* 3 *4* and *4* 0 *e*. An axis-dyad chord comprises three vertical dyads. The middle dyad is the *axis dyad* (identified as 3/0 in the segmented axis-dyad chord of figure 2.2a), which is surrounded by the outer *neighbour dyads* (9/4 and 4/e in the same figure). Two consecutive neighbour dyads omitting the axis dyad form a neighbour note chord (figure 2.3a). Perle usually refers to it as a *cyclic chord* because its horizontal dyads contain the cyclic intervals of each cyclic set (4-9=7 and e-4=7). This pair of cyclic intervals forms the array's *interval system*, which is represented by a pair of integers denoting the cyclic intervals. Perle constructs arrays with cyclic sets of either the same or different cyclic intervals. Thus, the interval system of p0p7/p4pe is identified as 7,7, whereas the interval system of i7i8/i3p5 is 1,2.[8]

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[7] Perle recognizes repetitions of a pc within a segment as *independent* entities rather than multiple instances of a single member of a collection. Since any collection may include pc duplication, a hexachordal collection in Perle's terms may contain less than six distinct elements. The same obtains for dyads, trichords, and tetrachords. Further, these collections must comprise contiguous pcs, but they do not necessarily occur as verticalities or in single lines.

[8] Arrays are formed by a combination of any two cyclic sets (as listed in appendix one), including those

Another typical segmentation of the array yields the *sum tetrachord*, which consists of an axis dyad and only one of its neighbour dyads (figure 2.3b). Given any sum tetrachord, it is possible to ascertain two of the four tonic sums (one from each cyclic set), thus indicating the particular pc alignments of the two cyclic sets. The cyclic intervals comprising the interval system cannot be determined, however, and thus there is no way to identify the array itself from an isolated sum tetrachord.

Figure 2.3. Cyclic chord and sum tetrachord of p0p7/p4pe axis-dyad chord

(a) cyclic chord:	*9*	–	*4*		(b) sum tetrachord:	*9*	3	–
	4	–	*e*			*4*	0	–

Each array contains twelve different axis-dyad chords; that is, for each of the twelve trichords of one cyclic set there are twelve possible alignments of triadic segments in the other cyclic set.[9]

Certain alignments of the cyclic sets produce a special type of axis-dyad chord Perle calls the *tonic axis-dyad chord*. It is the only array segment granted hierarchical status in Perle's system, due to its high degree of pitch-class duplication. To achieve tonic status an axis-dyad chord must meet several conditions. First, the sum of its axis dyad must duplicate a tonic sum (as in figure 2.4a, wherein the axis-dyad of sum 7 duplicates the tonic sum of p7). Second, the pcs of the axis dyad must also be present simultaneously in a trichordal segment of one of the cyclic sets (figure 2.4b). A third condition arises when the cyclic interval is the same for each of the cyclic sets of the array (figure 2.4c). When this is the case, the pcs of the neighbour dyads must be found in a tetradic segment of the other cyclic set. But if an array's interval system comprises different cyclic intervals, the cyclic chord cannot be found as a tetradic segment of either cyclic set; each cyclic set comprises only one of the two cyclic intervals that form the cyclic chord. Hence the third

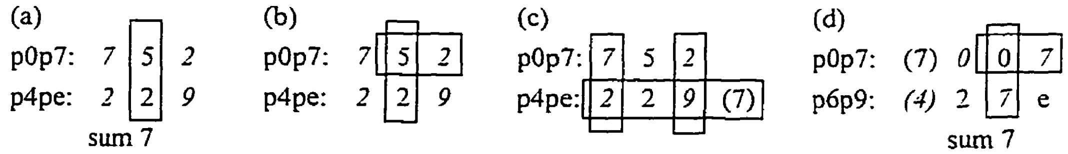
arising from either the same or different partitions of the same cyclic interval, or from different cyclic intervals.

[9] Although there are actually twenty-four possible alignments, twelve of these are duplicates, since the second half of the array is a retrograde of the first half, and will thus render the same twelve axis-dyad chords.

condition of tonic chords does not apply to chords from arrays of differing cyclic intervals (figure 2.4d).[10]

Thus, duplicated tonic sums and pcs are the distinguishing features of a tonic chord in twelve-tone tonality. Yet it is not merely the phenomenon of horizontal and vertical duplication that gives the tonic chord its significance; rather, it is the ease with which such a chord may be used as a pivot chord to other arrays, as will be discussed below in the section on modulation.

Figure 2.4. Attributes of tonic axis-dyad chords



Transposition and inversion

For each cyclic interval there are twelve cyclic sets, which Perle subdivides into two groups: six related by transposition and six by semi-transposition.[11] The labels of the cyclic sets related by transposition have matching p and/or i designations. Their corresponding tonic sums differ by a constant even integer. This integer is actually an even *sum*, since it represents the addition of either the same odd or even integer to each of the two component pcs of the tonic sum. Thus, the labels p0p7 and p2p9 indicate that the two cyclic sets are transpositionally equivalent. Figure 2.5a illustrates the transpositional relationship between p0p7 and p2p9. The T_2 applied to the tonic sums of the cyclic sets indicates the application of T_1 to each of the component pcs. Hence the relationship between the tonic sums may be expressed as T_{TS2}.[12]

[10] Instead, the cyclic chord is a tetrachordal segment from a *derived set*. See chapter 18 of *Twelve-Tone Tonality* (1977b, 69-72).

[11] Perle's concepts of semi-transposition and semi-inversion (to be discussed below) are not labelled as such until the second edition of *Twelve-Tone Tonality* (1996, 122, n.34).

[12] This dissertation introduces an unconventional notational practice in the discussion of the various parameters of twelve-tone tonality. The operation of transposition is typically associated with pitches and

Conversely, the cyclic sets p0p7 and i1i8, whose p/i designations do not match, are not transpositionally equivalent. Their corresponding tonic sums both differ by an *odd* integer, indicating that each pc within the cyclic set could not be transposed by the same value. Instead, the two cyclic sets are said to be related by semi-transposition; that is, the axis pcs are not transposed by the same interval as neighbour pcs (figure 2.5b).

Figure 2.5. Cyclic sets related by transposition (a) and semi-transposition (b)

(a)									
p0p7:	*0*	0	*7*	5	*2*	t	*9*	3
	+ 1	1	1	1	1	1	1	1
p2p9:	*1*	1	*8*	6	*3*	e	*t*	4

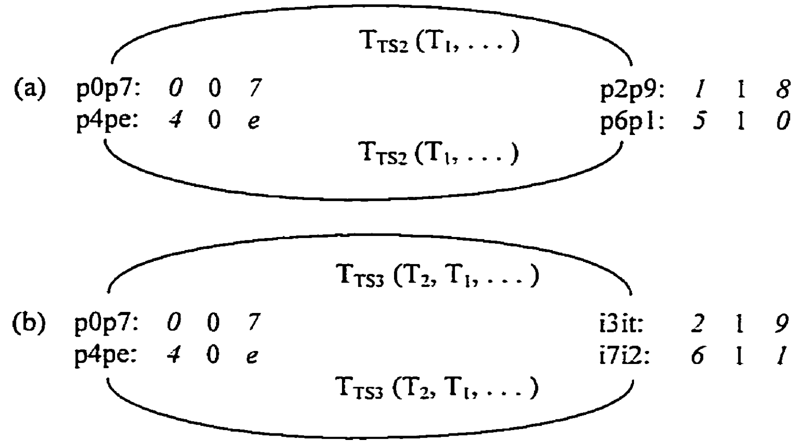
(b)								
p0p7:	*0*	0	*7*	5	*2*	t	*9*	3
	+ 0	1	0	1	0	1	0	1
i1i8:	*0*	1	*7*	6	*2*	e	*9*	4

Analogous to the transposition of cyclic sets, the transposition of arrays entails adding a consistent even integer to each of the four tonic sums, thereby preserving their p/i aspects. For example, the addition of 2 to each tonic sum in transposing the array p0p7/p4pe to p2p9/p6p1 translates into the addition of 1 to each pc, as illustrated in the two axis-dyad chords of figure 2.6a.[13] On the other hand, adding the same odd number to each tonic sum in an array changes each of the corresponding tonic sums' p/i designations. Since an odd sum is not divisible into two identical even or odd values, the addition of an odd integer to a tonic sum involves two different values being added to the individual pcs. Thus, a uniform transposition is not possible; that is, the axis-dyad and the cyclic chord will not be transposed by the same interval, as may be seen in the addition of

pcs. If, however, transposition is acknowledged as an arithmetical operation (addition) between two objects, then the term may be applied to different types of objects, in a manner analogous to the transposition of pitches and pcs. In this dissertation the term transposition may be used to describe the relation between tonic sums, interval systems, modes, or keys, as discussed below. To make the context clear, the particular parameter under transposition will appear in abbreviated form in the subscript after the T-label. For example, T with no subscript will refer to transposition of pcs; T_P will refer to transposition of pitches; T_{TS} will refer to tonic sums in cyclic sets; T_{IS} will refer to interval systems; T_M, to modes; and T_K, to keys. Furthermore, different elements within a parameter may be transposed by complementary values, which will be indicated by a ± symbol preceding the value.

[13] Perle's operations of transposition and semi-transposition are thus two-stage processes involving multiple operators. The transposition of pcs triggers a transposition in the corresponding tonic sums (and vice versa), although not by the same values.

Figure 2.6. Axis-dyad chords related by transposition (a) and semi-transposition (b)



3 to each tonic sum of p0p7/p4pe in figure 2.6b. Both transposition and semi-transposition preserve the cyclic intervals of cyclic sets and arrays so related.[14]

To invert cyclic sets and arrays Perle subtracts the tonic sums from a consistent even integer (figure 2.7a).[15] The tonic sums of the inverted cyclic sets and arrays will retain the same p/i designations, but will comprise the complementary cyclic interval. Since an *odd* integer is not divisible into two identical even or odd values, it is not possible to subtract the two component pcs of a tonic sum from the same value. The individual pcs must be

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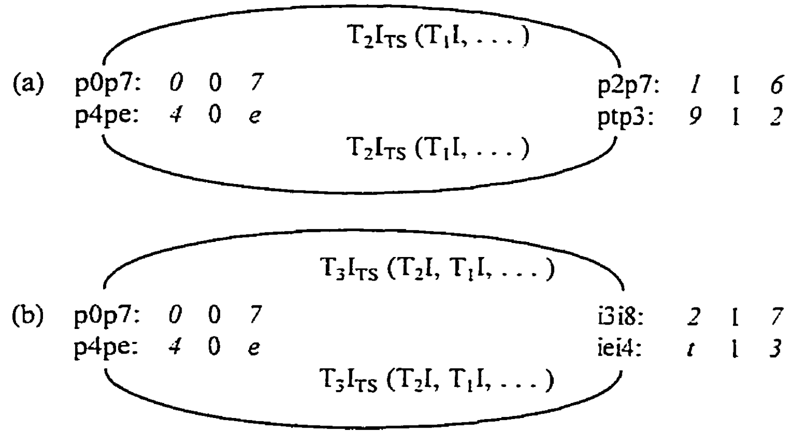
[14] In the second edition of *Twelve-Tone Tonality* Perle states that he will not differentiate between transposition and semi-transposition beyond this point, leaving it to the reader to "recognize the distinction" (1996, 185).

[15] Although the particular inversion notation of T_nI suggests a hierarchical priority of inversion about pc 0 (followed by transposition by a value of n), Perle views inversion as an sum-preserving operation that may occur about any axial pcs. From a similar viewpoint Lewin (1977) proposes an inversion symbol that is not dependent on pc labels, and does not prioritize inversion about pc 0. His expression $I^{uv}(s)$ denotes the inversion of a pc s about an axis u/v. (In more recent writings Lewing verticalizes the u/v symbols in his notation.) For instance, the axial pcs 5/6 lie between inversionally related pc pairs 0/e, 1/t, 2/9, 3/8, and 4/7. Nonetheless, if n represents the sum of the axial pcs u and v, the same inversionally related pc pairs are produced by the operation T_nI (s).

This study introduces a new notational system to denote transpositional operations on other objects in addition to pcs (see n.12 above). As illustrated in figure 2.7, this notation is also used for inversional operations. Using Lewin's symbols in conjunction with this study's notational system, the inversional operation in figure 2.7a would be notated as $I_{TS}^{1/1}$. But since both the expressions $I^{uv}(s)$ and T_nI (s) yield the same result, I have elected to represent transposition and inversion operations of the parameters of twelve-tone tonality by the symbols T_n and T_nI, in an effort to maintain a degree of uniformity of operational symbols.

Figure 2.7. Axis-dyad chords related by inversion (a) and semi-inversion (b)



subtracted from different values (figure 2.7b). Hence the resulting cyclic sets or arrays will not be inversionally equivalent; instead, they are related by semi-inversion.[16]

All pairs of cyclic sets with the same tonic sums but with opposite p- and i-designations are semi-equivalent. Sets so related have the same ordering of pcs, but these originate in opposite P and I cycles. Thus, semi-equivalent cyclic sets comprise complementary cyclic intervals.[17]

The cognate relation

Perle introduces the *cognate relation* as another form of association between cyclic sets and arrays, in which adjacent pairs of pcs are held invariant between cyclic sets. More specifically, two cyclic sets are in the cognate relation if they are related inversionally and if they share a "single series of dyads" (1977b, 20), which implies that they share an invariant tonic sum. Two such cyclic sets are p0p7 and p2p7, of cyclic

[16] As with transposition and semi-transposition, inversion and semi-inversion are two-stage operations with multiple operators. Please see n.13 above.

[17] For example, in the segment (*9 6 1*) from the cyclic set i3p7 (cyclic interval 4) the neighbour notes derive from the P cycle and the axis note derives from the I cycle. The same segment in i7p3 of the complementary cyclic interval 8 (9 *6* 1) derives its neighbour notes from the I-cycle and its axis note from the P-cycle.

intervals 7 and 5 respectively. These cyclic sets share complementary cyclic intervals and the tonic sum 7 (figure 2.8a). Sets that share one of their tonic sums will also share that sum's component pcs. The cognate relation is apparent when the two cyclic sets are arranged so as to align the invariant pcs. The latter will appear in pairs of adjacent linear dyads of retrograde order. Each vertical dyad will add up to the shared tonic sum, referred to here as the *cognate sum.*

Figure 2.8. Cognate relation between cyclic sets

(a)

p0p7:	7	*0*	0	*7*	5	*2*	t	*9*
p2p7:	0	*7*	7	*0*	2	*5*	9	*t*
sum:	7	7	7	7	7	7	7	7

(b)

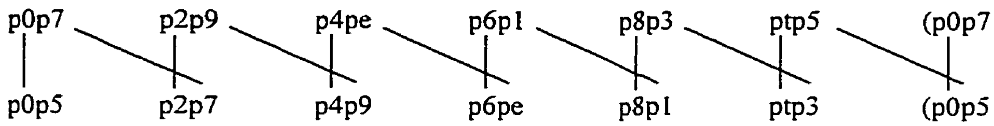
p0p7:	*0*	0	*7*	5	*2*	t	*9*	3
i5i0:	0	*0*	5	*7*	t	*2*	3	*9*
sum:	0	0	0	0	0	0	0	0

The cognate relation also obtains between cyclic sets of the *same* cyclic interval if the sets share a tonic sum.[18] Figure 2.8b illustrates the cognate relation between cyclic sets p0p7 and i5i0. These sets share the same cyclic interval 7 and tonic sum 0, and are related by semi-transposition. In his chapter on Perle's twelve-tone tonal system in *Twentieth–Century Music* Elliott Antokoletz observes that "all 12 forms of the cyclic set—the 6 primes and the 6 inversions—form a closed system through the cognate relation, just as the circle of fifths forms a closed system for the scales of the major-minor system" (1992, 433). Figure 2.9 illustrates this phenomenon in its arrangement of the twelve cyclic sets of interval 7, with sets related by transposition organized into rows, and those related by inversion in columns. The vertical and diagonal lines indicate the cognate relation.

Although Perle stipulates that cognate sets must be related inversionally, a partial cognate relation might also be said to exist between sets whose cyclic intervals are neither complementary nor the same. In such sets the common pcs forming the invariant tonic

[18] Such cognate relations are possible through the relation of semi-equivalence. The semi-equivalent cyclic sets p2p7 and i7i2 share the same tonic sums, but comprise complementary cyclic intervals. As such, their pcs occur in the same order, although they derive from opposite P / I cycles. Hence just as p0p7 shares a cognate relation with the inversionally related p2p7, it also shares a cognate relation with the semi-transposed i7i2. In figure 2.8b above, p0p7 shares a cognate relation with the semi-transposed i5i0, just as it does with the inversionally related p0p5.

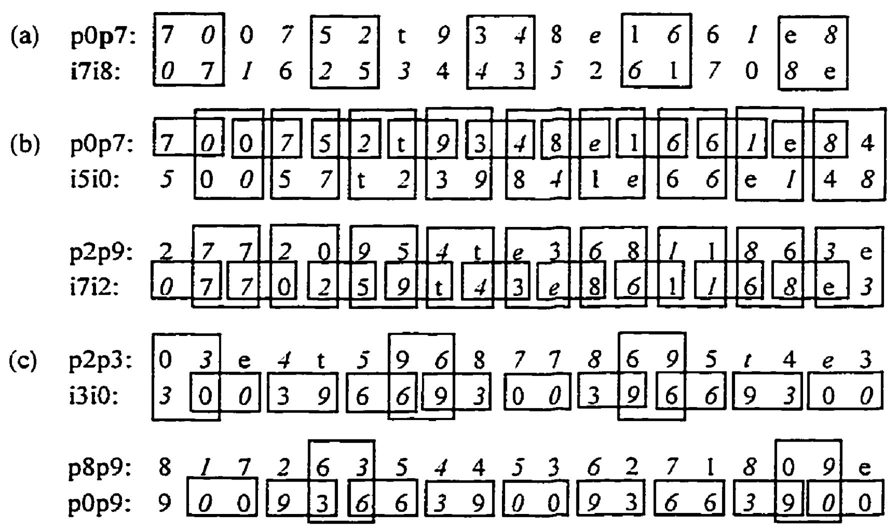
Figure 2.9. Closed system of cyclic sets through the cognate relation



sums do not occur in adjacent dyad pairs; rather, they coincide only when the dyadic pairs add up to the cognate sum. Figure 2.10a illustrates this partial relation in the array p0p7/i7i8, whose cyclic sets share tonic sum 7 and whose interval system is 7,1.

The cognate relation may be detected not only between cyclic sets within an array, but also between *arrays* that share a tonic sum (figure 2.10b). The degree to which the cognate relation is present depends on whether the arrays' interval systems contain identical or complementary cyclic intervals between the cyclic sets of each array and between the arrays themselves. For example, the cognate relation is maximized between arrays p0p7/i5i0 and p2p9/i7i2, which share tonic sum 7 (figure 2.10b). Both arrays have interval systems of 7,7 and are related by transposition; each array comprises cyclic sets related by semi-transposition.

Figure 2.10. Partial cognate relations



As illustrated in figure 2.10c, the cognate relation is weaker between the arrays p2p3/i3i0 and p8p9/p0p9, which share the cognate sum 0, since their individual interval systems (1,9) contain different component cyclic intervals. Nonetheless, these partial cognate relations demonstrate that arrays which share a tonic sum are related to some degree through invariant adjacent dyads.

Secondary differences and sums

Perle forms arrays by aligning the cyclic sets in one of two ways: either their respective P-cycles and I-cycles are aligned (as in figure 2.2a), or the P-cycle of one cyclic set is aligned with the I-cycle of the other, and vice versa (figure 2.2b). Perle characterizes an array as being in a *difference alignment* when the P-cycles and I-cycles of its component cyclic sets are aligned. As a result, the array's vertical dyads yield a consistent pattern of alternating intervals referred to as its *secondary differences*. The first alignment of array p0p7/p4pe in figure 2.11a produces a repeating pattern of secondary differences 1/9.[19] Rotating one of the cycles in a difference alignment relative to the other by an even number of places generates other patterns of secondary differences.[20] In comparison to its setting in figure 2.11a, the p4pe cyclic sets in figures 2.11b and c rotate their members by two and four places to the left, respectively. In figure 2.11d the cyclic set p4pe rotates its members two places to the right.

Whereas parallel cycles are aligned in figure 2.11, opposite cycles may also be aligned: that is, P-cycles with I-cycles and vice versa. Perle refers to this as a *sum alignment*, which preserves a consistent pattern of alternating *secondary sums.* Figure 2.12 demonstrates two of the possible sum alignments of p0p7/p4pe. Figure 2.12b rotates

[19] Axis-dyads and neighbour-dyads in a difference alignment belong to opposite P/I cycles. As a result, Perle calculates differences or intervals in opposite directions for axis dyads and neighbour dyads. Thus, either one may subtract the top integer from the bottom for those dyads from the P-cycle and subtract the bottom integer from the top for those dyads from the I-cycle, or vice versa. (This dissertation adopts the former order.) This stipulation plays a vital role in the calculation of an array's mode, to be discussed below.

[20] Rotating a cyclic set by an odd number would align the P-cycle with the I-cycle, thereby converting the difference alignment to a sum alignment.

Figure 2.11. Various difference alignments of array p0p7/p4pe resulting from rotation of cyclic set p4pe

(a)

p0p7:	0	0	7	5	2	t	9	3
p4pe:	1	3	8	8	3	1	t	6
diff:	1	9	1	9	1	9	1	9

(b)

p0p7:	0	0	7	5	2	t	9	3
p4pe:	8	8	3	1	t	6	5	e
diff:	8	4	8	4	8	4	8	4

(c)

p0p7:	0	0	7	5	2	t	9	3
p4pe:	3	1	t	6	5	e	0	4
diff:	3	e	3	e	3	e	3	e

(d)

p0p7:	0	0	7	5	2	t	9	3
p4pe:	6	t	1	3	8	8	3	1
diff:	6	2	6	2	6	2	6	2

Figure 2.12. Various sum alignments of array p0p7/p4pe resulting from rotation of cyclic set p4pe

(a)

p0p7:	0	0	7	5	2	t	9	3
p4pe:	1	t	6	5	e	0	4	7
sums:	1	t	1	t	1	t	1	t

(b)

p0p7:	0	0	7	5	2	t	9	3
p4pe:	6	5	e	0	4	7	9	2
sums:	6	5	6	5	6	5	6	5

the p4pe cyclic set of the array two places to the left.[21]

Whereas cyclic intervals and tonic sums define the structure of the individual cyclic sets, secondary differences and sums define the particular relationship between the two cyclic sets of the array.

Modes

Perle describes the relationships between an array's cyclic sets in terms of the array's *mode* and *key*. The mode reflects the basic intervallic relationship between the cyclic sets; Perle names the mode according to the difference between the corresponding tonic sums of the cyclic sets in a difference alignment. Thus, the array p0p7/p4pe has a modal designation of 8,8, since 0-4=8 and 7-e=8 (figure 2.13a).

The mode manifests itself in the collection of secondary difference patterns generated by all the difference alignments of the cyclic sets in an array. Figure 2.11 subjected the

[21] Rotating an odd number of places would align parallel cycles, thereby converting the sum alignment to a difference alignment.

array p0p7/p4pe to successive realignments of its cyclic sets, which generated four difference alignments and four pairs of secondary differences, 1/9, 8/4, 3/e, and 6/2. These pairs are related by transposition, with each pair maintaining a difference of eight between its component elements. The remaining difference alignments for this particular array will also show differences of eight: 2/t, 4/0, 5/1, 7/3, 9/5, t/6, e/7, and 0/8. This is because every array has a consistent intervallic pattern underlying the various difference alignments of its cyclic sets. This intervallic pattern is the mode. Thus, an alternative method of determining the array's mode is to find the difference between the axis-dyad's interval and each of its neighbour-dyad's intervals (figure 2.13b).[22]

Figure 2.13. Calculation of an array's mode

(a)

	p0	p7
	- p4	- pe
diff:	8	8

(b)

p0p7:	*0*	0	*7*
p4pe:	- *3*	1	*t*
diff:	3	e	3
	└ 8 ┘		
		└ 8 ┘	

(c)

i1i6:	*1*	0	*6*
iei2:	- *5*	6	*8*
diff:	4	6	2
	└ 2 ┘		
		└ 4 ┘	

Modes offer a means of relating arrays from either the same or different interval systems. Arrays of any combination of interval systems may belong to the same mode if they preserve the same differences between their corresponding tonic sums. The array iei2/p3p6 (interval system 3,3) also belongs to mode 8,8, as does i0i1/p4p5 (interval-system 1,1). Thus, mode 8,8 contains all arrays whose adjacent secondary differences differ by eight. In the same way, mode 0,0 contains all arrays whose adjacent secondary intervals differ by zero. A mode with two different elements in its name, such as mode 2,4, results from arrays whose interval systems comprise different cyclic intervals. Examples include array i1i6/iei2 of interval system 5,3 (figure 2.13c) and i1i7/iep3 of interval system 6,4. Both of these arrays belong to mode 2,4. Consequently, the alternating secondary differences emerging from any difference alignment of these arrays will display alternating differences of 2 and 4.

[22] Modal designation offers further evidence of the necessity for consistent ordering of tonic sums. Had the array p0p7/p4pe been ordered as p0p7/pep4 instead, the mode would have been identified as 1,3, implying an alignment of secondary difference pairs showing differences of 1 and 3, rather than of 8 and 8.

Each array belongs to only one mode. Each mode contains twelve interval systems, and thus 144 arrays (twelve from each of the twelve interval systems). Finally, there are 144 modes (twelve with identical elements in their names, twelve modes whose elements differ by one, twelve modes whose elements differ by two, and so on).

Keys

In Perle's theory the *key* indicates the relationship between an array's cyclic sets in a sum alignment, and is calculated as the sum of oppositely aligned tonic sums. Thus, the key of array p0p7/p4pe is identified as e,e, since 0+e=e and 7+4=e (figure 2.14a).

An array in a sum alignment generates a consistent pattern of alternating secondary sums between its axis dyads and neighbour dyads. For example, the array p0p7/p4pe may be aligned to show any of the following secondary sum patterns: 0/e, 1/t, 2/9, 3/8, 4/7, and 5/6. These pairs are all related by sum transposition, with each pair showing a sum of eleven. Every array contains a consistent symmetrical pattern underlying the various sum alignments of its cyclic sets, which defines the array's key. Thus, the key may also be calculated as the sum of the axis dyad's sum and each of its neighbour dyad's sums (figure 2.14b).[23]

Figure 2.14. Calculation of an array's key

(a)		p0	p7	(b)	p0p7:	*0* 0 *7*	(c)		i1	it	(d)	i1it:	*9* 4 *6*
		+ pe	+ p4		p4pe:	+ 6 *5* e			+ i6	+ i5		i5i6:	+ 3 *3* 2
	sum:	e	e		sum:	6 5 6		sum:	7	3		sum:	0 7 8
						└e┘							└7┘
						└e┘							└3┘

A key may relate arrays of the same or different interval systems. The same key relationships will be generated by any of the other sum alignments that show the same patterns of secondary sums. Therefore key e,e comprises all arrays whose adjacent secondary sums add up to eleven. Likewise, key 4,4 contains all arrays whose secondary

[23] Key designation also supports the argument for consistent ordering of tonic sums. Had the array p0p7/p4pe been ordered as p0p7/pep4 instead, the key would have been identified as 4,6. Please see n.21.

sums add up to four. Key 7,3 contains arrays whose adjacent pairs of secondary sums add up to seven and three in alternation (figures 2.14c and d). Just as there are 144 different arrays in each mode, there are also 144 different arrays in each key (twelve different arrays for each of twelve different interval systems), and 144 different keys in the twelve-tone tonality universe.

Thus, the tonic sums provide much information about the structure of an array. The specific combination of tonic sums in each cyclic set describes the particular pitch alignment of the cycles within each cyclic set. The differences between the tonic sums in each cyclic set indicate the cyclic intervals. The differences between parallel-aligned tonic sums disclose the underlying modal structure. The sum of oppositely-aligned sums yields the key.[24]

Finally, within an array the relationship between a mode's elements corresponds to the relationship between its key's elements and its cyclic intervals. For array p0p7/p4pe, the mode is 8,8, the key is e,e, and the interval system is 7,7. The difference between the elements in each aspect of the array is 0.[25] These relationships all underscore the internal consistency within an array.

Synoptic arrays

Relationships among arrays are established by transposition, inversion, and membership in the same interval system, mode, or key. Yet collections of arrays lacking these associations may still be related at a more fundamental level. Perle establishes such connections in his concepts of *synoptic arrays* and *tonality*. Arrays in the same synoptic relationship share structural similarities among their interval systems, whereas arrays in the same tonality share the same axis of symmetry.

Each synoptic array contains what Perle calls a *master array*, a group of arrays sharing the same difference or sum relationship between their component cyclic intervals, despite

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[24] The particular combination of tonic sums also determines the array's synoptic mode, synoptic key, and tonality, as will be discussed below.

[25] This issue is discussed in more detail below, in the section on synoptic arrays.

differences in interval systems, modes, or keys. The synoptic array divides into the *synoptic mode* and *synoptic key*.[26] Arrays whose cyclic intervals show the same difference belong to the same master array of the synoptic mode, whereas arrays whose cyclic intervals show the same sum belong to the same master array of the synoptic key.[27]

Perle identifies seven different master arrays in the synoptic mode, numbered from 0 to 6, determined by the ic difference between the two cyclic intervals in the interval system. The arrays listed in table 2.1 all belong to master array 0 of the synoptic mode.[28] The underlying relationship among these seemingly disparate arrays is found in their interval systems. Each pair of cyclic intervals displays a difference of 0, as indicated in the fourth column, and so belongs to master array 0 of the synoptic mode. Thus all arrays in the same master array of the synoptic mode have interval systems that are related by transposition. The same relationship is exhibited in the arrays' modes and keys as well.

Table 2.1. Representative arrays belonging to master array 0 of the synoptic mode

Array	SM	IS	IS diff.	Mode	Key
(a) i5i6/i3i4	0	1,1	1-1=0	2,2	9,9
(b) p0p7/p4pe	0	7,7	7-7=0	8,8	e,e
(c) i1i4/p6p9	0	3,3	3-3=0	7,7	t,t
(d) p2i6/pti2	0	4,4	4-4=0	4,4	4,4

[26] Perle first discusses synoptic key in the second edition of *Twelve-Tone Tonality* (1996, 195-97). In the first edition Perle's discussion dealt only with the synoptic mode, although the concept had not yet been assigned this particular label. Rather, Perle referred to the master arrays of the synoptic mode as "master modes and keys," or collectively as "master arrays" (1977b, 87, 95, 99). In the interim between the first and second editions both Carrabré (1993) and Rosenhaus (1995) introduced Perle's mature formulation of synoptic arrays in their Ph.D. dissertations, basing their information on independent correspondence with Perle.

[27] Perle further divides master arrays into *master modes* and *master keys*, to distinguish between arrays given in difference and sum alignments (1977b, 95). Thus, arrays in difference alignments that share the same difference between their component cyclic intervals belong to the same *master mode of the synoptic mode*, while the same arrays in sum alignments belong to the same *master key of the synoptic mode*. In the same way, arrays in difference or sum alignments that show the same sum between their component cyclic intervals belong to the same *master mode of the synoptic key*, or *master key of the synoptic key*, respectively.

[28] In Table 2.1 I have abbreviated the terms "interval system" and "synoptic mode" as "IS" and "SM," respectively. As well, the abbreviation "IS diff." represents the difference between cyclic intervals in the interval system.

Each of the seven master arrays of the synoptic mode contains a large grouping of arrays. For example, consider that there are twelve combinations of interval systems that differ by 1 (0,1; 1,2; 2,3; and so on), and that within each of these interval systems there are 144 different arrays. In interval system 0,1, for example, there are twelve cyclic sets of cyclic interval 0 to be aligned with twelve different cyclic sets of cyclic interval 1. Thus, master array-1 of the synoptic mode contains 1,728 (as 12 × 144) different arrays whose interval systems differ by 1. This master array also contains those arrays whose interval systems differ by e, but this group may be shown to correspond with the former group by reordering the cyclic sets within each array. For example, the arrays of i3p5/i5i6 and i5i6/i3p5 both belong to master-array 1 of the synoptic mode.

In contrast to the synoptic mode, in each master array of the synoptic key Perle includes all arrays whose pairs of cyclic intervals add up to the same *sum*. He identifies seven such master arrays, numbered from 0 to 6.[29] Whereas all arrays belonging to the same master array of the synoptic mode display the same difference relationships among their interval systems, modes, and keys, arrays in the same master array of the synoptic key *do not* necessarily exhibit analogous sum relationships. The reason for this lack of symmetry is that although complementary interval-system sums are subsumed under a single master array designation, they actually generate non-corresponding sets of relationships within the same master array of the synoptic key. Only those arrays sharing the same interval-system sums will exhibit symmetrical relationships.

Arrays with the same interval-system sum will necessarily display a symmetrical relationship in the corresponding elements of their interval systems, as seen in table 2.2, rows a-e. For example, the corresponding left elements in the interval systems of rows a and b are related by T_{IS-2}, while the corresponding right elements are related by T_{IS+2}. But an analogous relationship among the keys is further determined by each array's *aggregate sum*, defined as the sum of the four tonic sums in an array name. All arrays of the same aggregate sum will have keys with symmetrically related corresponding elements, as illustrated in rows a-d. Finally, if the sums of the component cyclic sets

[29] In determining the master array of the synoptic key, interval-system sums larger than 6 are replaced by their smaller complementary value, mod-12.

Table 2.2. Representative arrays belonging to master array 3 of the synoptic key

Array	SK	IS	IS sum	Agg. (cyclic set sums)	Key	Mode
(a) p0p7/p4pe	2	7,7	2	t (7+3)	e,e	8,8
(b) i1i6/i3i0	2	5,9	2	t (7+3)	1,9	t,6
(c) p2p5/p2p1	2	3,e	2	t (7+3)	3,7	0,4
(d) i1i7/i3pe	2	6,8	2	t (8+2)	0,t	t,8
(e) p6it/p2i0	2	4,t	2	6 (4+2)	6,0	4,t
(f) p4it/i1p5	2	6,4	t	8 (2+6)	9,e	3,5

within the arrays are the same, then both the modal elements and the corresponding tonic sums of the cyclic sets themselves will be symmetrically related. We see this in in rows a-c of table 2.2, wherein the fifth column lists the same cyclic set sums (7+3) for each array.

The aggregate sum in row d (sum t) is formed from different cyclic set sum pairs (8+2) than in rows a-c (7+3); consequently, the component elements of the mode in row d are not symmetrical with the corresponding modal elements in rows a-c. The aggregate sum in row e (sum 6) differs from the aggregate sum in rows a through d (sum t), and so the component elements of the key in row e are not symmetrically related to the corresponding key elements above them. The array given in row f, although a member of the same master array of synoptic key 2, shares no symmetrical relationships with any corresponding aspects of the arrays in rows a-e, since its interval system sum (sum t) is inversionally complementary.

As with the synoptic mode, there are 1,728 different arrays in each master array of the synoptic key. For example, within synoptic array 2 of the synoptic key there are twelve distinct pairs of interval systems that sum to 2 and twelve that sum to t. Within each interval system (such as 8,6) there are 144 different arrays, or 12 different cyclic sets of one interval each aligned with 12 different cyclic sets of the other interval.

Tonality

Tonality represents the other type of fundamental relation among arrays in Perle's theory, and is based on the concept of the *axis of symmetry*. Any symmetrical collection of pcs contains an invisible middle line, or axis, around which the various pcs are symmetrically positioned. The smallest such collection is a dyad. All dyads of the same sum comprise pcs symmetrically disposed across their axis. Each axis will be represented either by a pair of repeating numbers (for dyads of even sums) or by numbers that differ by 1 (for dyads of odd sums). For every even sum n the two axes are calculated as $n/2$ and $n/2 + 6$. For every odd sum n the two axes are calculcated as $(n\pm1)/2$ and $((n\pm1)/2) + 6$. In figure 2.15a, the axis for dyads of sum 8 is 4,4 (or t,t) while in figure 2.15b, the axis for dyads of sum 5 is 2,3 (or 8,9).

Figure 2.15. Dyadic axes of symmetry

(a) sum 8 dyads have axes of 4,4 and t,t:

4	*5*	*6*	*7*	*8*	*9*	*t*	*e*	*0*	*1*	*2*	*3*	*4*
4	3	2	1	0	e	t	9	8	7	6	5	4

(b) sum 5 dyads have axes of 2,3 and 8,9:

2	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*t*	*e*	*0*	*1*	*2*
3	2	1	0	e	t	9	8	7	6	5	4	3

Similarly, Perle asserts that axis-dyad chords whose sum tetrachords add up to the same sum are also symmetrically related and have the same axis of symmetry. This axis is formed from one of three combinations: sum tetrachords of two even sums, two odd sums, or one even and one odd sum. Arrays represent concatenations of axis-dyad chords; therefore all arrays with the same aggregate sum will be symmetrically related and have the same axis of symmetry.

Arrays so related fall into one of three categories of axial symmetry, which Perle identifies as the *three tonalities*. The first category is *Tonality 0*, which constitutes all arrays whose aggregate sums are 0, 4, or 8. Their axes consist of two repeated even integers, and so are simply transpositions of one another by an even value of T_n. Table 2.3 lists these in column a. The second group of arrays whose aggregate sums are represented by an odd integer constitute *Tonality 1* (table 2.3, column b). Their axes are also

Table 2.3. Three categories of tonality

(a) Tonality 0		*(b) Tonality 1*		*(c) Tonality 2*	
Agg. sum:	axes:	Agg. sum:	axes:	Agg. sum:	axes:
0	0/0 and 6/6	1	0/1 and 6/7	2	1/1 and 7/7
4	2/2 and 8/8	3	1/2 and 7/8	6	3/3 and 9/9
8	4/4 and t/t	5	2/3 and 8/9	t	5/5 and e/e
		7	3/4 and 9/t		
		9	4/5 and t/e		
		e	5/6 and e/0		

transpositionally related, and include one even and one odd integer that differ by 1.[30] Third, all arrays whose sum tetrachords add up to 2, 6, or t belong to *Tonality 2* (table 2.3, column c). Their axes consist of a pair of identical odd integers, also transpositionally related by an even value of T_n. Although Tonality 0 and Tonality 2 both contain arrays of even aggregate sums, the arrays of the former are not symmetrically equivalent to those of the latter, because their axes are formed from even/even and odd/odd integer pairs, respectively.[31]

Since arrays are extensions of axis-dyad chords, the axis of any axis-dyad chord represents the tonality of all the axis-dyad chords within the same array, and consequently, the tonality of the array itself. Further, Perle contends that all arrays of the same category of aggregate sum belong to the same tonality. Thus the symmetrical relations extend from pcs within a dyad to massive groupings of arrays. Table 2.4 compares three axis-dyad chords of arrays all in tonality 0. All three arrays have different interval systems, although the first two have the same aggregate sum. Although the array in column c has a different aggregate sum, the array still belongs to the same tonality, since its axis is a transposition of the other axes.

[30] In the second edition of *Twelve-Tone Tonality* Perle formally divides Tonality 1 into subcategories A and B, based on transpositional and inversional equivalence of the arrays: Tonality 1A includes those arrays of aggregate sums 1, 5, and 9, and Tonality 1B includes those of aggregate sums of 3, 7, and e (1996, 190).

[31] In the second edition of *Twelve-Tone Tonality* Perle provides formulae to determine the tonality of an array, with n being the value of the aggregate sum. Tonality 0 includes arrays of 4n; Tonality 1A includes arrays of 4n+1, Tonality 1B includes arrays of 4n-1, and Tonality 2 includes arrays of 4n+2 (1996, 190-91).

Table 2.4. Comparison of three axis-dyad chords of different arrays in the same tonality

array:	**(a) p6it/i7p9**			**(b) p0p7/i3it**			**(c) i1i2/i9i0**		
interval system:	4,2			7,7			1,3		
axis-dyad	*2*	4	*6*	*0*	0	*7*	*8*	5	*9*
chord:	4	*5*	2	3	*7*	8	1	*e*	t
dyadic sums:	6	9	8	3	7	3	9	4	7
dyadic axes:	3,3	4,5	4,4	1,2	3,4	1,2	4,5	2,2	3,4
	9,9	t,e	t,t	7,8	9,t	7,8	t,e	8,8	9,t
aggregate sum:	8			8			0		
array axes:	4,4			4,4			0,0		
	t, t			t, t			6,6		
tonality:	**0**			**0**			**0**		

Modulation

An array may modulate to another of the same or different interval system, tonality, master array of either the synoptic mode or synoptic key (or both). Modulation is often accomplished by replacing dyads of an axis-dyad chord with others of the same difference or sum. Perle identifies this as *modulation through substitution.* A second type of modulation occurs when various members of an axis-dyad chord are reordered, resulting in a reinterpretation of the axis-dyad chord as a segment of a different array. This type of modulation often involves rotation of segments in the cyclic sets. Perle refers to this as *modulation through reinterpretation.*[32]

Substitutional modulation involves either parallel or symmetrical transformations of both cyclic sets, with corresponding changes to the tonic sums. An examination of the tonic sums reveals the effects of substitutional modulation on array p0p7/p4pe (see table 2.5, rows a and b). In order to modulate to another array of the same interval system, the tonic sums that form each cyclic interval must be altered in a parallel manner, as represented by the formulas [+x,+x] or [-x,-x]. Applying the same formula to both cyclic

[32] These terms were not formally assigned until the second edition of *Twelve-Tone Tonality*, chapters 35 and 36 (1996, 198-223).

Table 2.5. Substitutional modulations from array p0p7/p4pe to arrays in the same interval system

Formula applied to cyclic sets of initial array p0p7/p4pe		Resulting array	Tonality	IS	SM	SK	Mode	Key
	---	p0p7/p4pe	2 (agg. t)	7,7	0	2	8,8	e,e
a.	+x,+x/+x,+x (x=2)	p2p9/p6p1	2 (agg.6)	7,7	0	2	8,8	3,3
b.	+x,+x/-x,-x (x=2)	p2p9/p2p9	2 (agg. t)	7,7	0	2	0,0	e,e
c.	+x,-x/+x,-x (x=2)	p2p5/p6p9	2 (agg. t)	3,3	0	6	8,8	e,e
d.	+x,-x/-x,+x (x=2)	p2p5/p2p1	2 (agg. t)	3,e	4	2	0,4	3,7
e.	+x,±0/+x,±0 (x=2)	p2p7/p6pe	2 (agg. 2)	5,5	0	2	8,8	1,1
f.	+x,±0/+x,±0 (x=1)	i1p7/i5pe	0 (agg. 0)	6,6	0	0	8,8	0,0

sets of an array will cause a modulation to another array of the same mode (table 2.5, row a). Applying both formulas to an array, one to each cyclic set, will bring about modulation to another array of the same key (table 2.5, row b).

In these types of modulation, either the axis dyads or cyclic chords of axis-dyad chords are replaced with others of the same differences or sums. For example, in figure 2.16a, first the axis dyad of a chord from p0p7/p4pe, and then the cyclic chord, are replaced by another of the same interval, effecting a modulation to a new array in the same mode. Conversely, modulations to arrays in the same key may be accomplished by substituting either an axis dyad or a cyclic chord of the same sum (figure 2.16b).[33]

Whereas modulation to an array of the same interval system may occur by replacing either the axis dyad or the cyclic chord in an axis-dyad chord, modulation to another array in a different interval system cannot occur by substituting another axis dyad, since it has no direct effect on the cyclic intervals. Rather, the cyclic chord may be replaced by

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[33] In his discussion of substitional modulation, Perle restricts the procedure to *tonic* axis-dyad chords. In such cases, the axis dyad of a tonic axis-dyad chord is replaced by another of the same interval but different sum. This new axis dyad must duplicate a tonic sum in the new array, which is transpositionally related to the original. Conversely, the axis dyad of the tonic axis-dyad chord may be replaced by another of the same sum but different interval. The axis dyad's sum still must duplicate a tonic sum in the new array, which is symmetrically related to the original array. Perle also discusses substituting the neighbour dyads of a tonic axis-dyad chord with others of either the same sums or intervals, to form a tonic axis-dyad chord in the new array. In its consideration of substitional modulation between non-tonic axis-dyad chords as well as axis-dyad chords, this dissertation goes beyond Perle's original conception.

Figure 2.16. Substitutional modulation with axis-dyad chords as pivots

(a)	p0p7:	0	0	7	p2p9:	0	2	7	p2p9:	2	0	9
	p4pe:	4	0	e	p6p1:	4	2	e	p6p1:	6	0	1
	diff:	4	0	4	diff:	4	0	4	diff:	4	0	4
(b)	p0p7:	0	0	7	i1i8:	0	1	7	i1i8:	1	0	8
	p4pe:	e	0	4	i3it:	e	e	4	i3it:	t	0	3
	sum:	e	0	e	sum:	e	0	e	sum:	e	0	e

another of the same differences or sums.

To modulate to an array of a different interval system, changes must be made to one or both cyclic intervals of the original array, necessitating a [+x,-x] or [-x,+x] formula applied to each cyclic set. Applying the *same* formula to both cyclic sets retains the same difference relationship between the two arrays, which thereupon belong to the same master array of the synoptic mode, and have the same modal and key designations (table 2.5, row c). Applying *both* formulas to an array, one to each cyclic set, results in a symmetrical modification to the array's interval system; consequently, this type of modulation produces arrays in the same master array of the synoptic key (table 2.5, row d). This type of modulation involves the contraction of one cyclic interval and the expansion of the other.[34]

A modulation may also result by changing only two of the tonic sums, involving some combination of the formulas [+x,±0], [-x,±0], [±0,+x], and [±0,-x] applied to the cyclic sets. Substitutional modulation in these cases occurs by holding a sum tetrachord invariant, and replacing the remaining neighbour-dyad with another of the same sum or difference. This type of modulation may change the tonality, depending on the value of the integer added to tonic sums. If the integer 2 is added to two different pcs the aggregate sum increases by a total of 4, which means that the new array is in the same tonality as the old array. This will obtain for all even values of x, since doubling any even numbers

[34] Perle discusses these types of alterations to the tonic sums of arrays, not in the context of modulation, but in the section on symmetrical equivalency relations in the newly added chapter 33 of the second edition of *Twelve-Tone Tonality* (1996, 185-89). Essentially, by altering the tonic sums of an array in a symmetrical operation (such that a value *n* is added to two of the tonic sums and subtracted from the other two), another array is generated, and the two arrays are said to be "symmetrically equivalent."

in mod 12 renders a value of 0, 4, or 8 (table 2.5, row e). If, however, an odd number is added to each pc in the neighbour dyad the tonality may shift, since doubling any odd number results in a value of 2, 6, or t. If the original array is in tonality 1, the modulation will occur within the same tonality, since tonality 1 includes all aggregate sums of odd numbers. But if the original array is in tonality 0, such a modulation will affect a change to tonality 2, and vice versa (table 2.5, row f).

It is also possible to modulate by retaining only a neighbour dyad while substituting another sum tetrachord with the same dyadic differences or sums. Other such combinations may be derived, but since the resulting axis-dyad chords retain fewer invariant pcs their effectiveness as pivot chords is reduced.

The other type of modulation, reinterpretative modulation, involves the reorganization of pcs within an axis-dyad chord to pivot into a new array. The simplest method is to invert one of the dyads. Inverting a neighbour dyad may change the cyclic intervals if the dyad does not contain duplicate pcs (figure 2.17a), while inverting the axis dyad does not affect the cyclic intervals (figure 2.17b). Although the dyadic inversion changes some of the array's tonic sums, there is no change to the aggregate sum, and consequently no change in tonality.

Figure 2.17. Reinterpretative modulation by inverting dyads

	(a)	(b)
i7i2: *7* 0 *2*	i7i6: *7* 0 *6*	i5i0: *7* t *2*
i9i4: *e* t *6*	i9i0: *e* t *2*	iei6: *e* 0 *6*
interval system 7,7	interval system e,3	interval system 7,7
Tonality 2 (agg. t)	Tonality 2 (agg. t)	Tonality 2 (agg. t)

Other types of reinterpretative modulation, however, may create a change in the aggregate sum, possibly with a corresponding shift in tonality. This results from a reordering of the component pcs, which alters the tonic sums and thus the cyclic interval. Either one or both cyclic sets may involve reordering of segments (figure 2.18a).[35]

[35] In the newly added chapter 36 of the second edition of *Twelve-Tone Tonality* Perle describes reinterpretative modulation as a "reinterpretation of the functional implications of a given collection of pitch classes" (1996, 206). That is, by rotating the elements in a trichordal segment of a pivotal axis-dyad chord, either the cyclic interval may be reinterpreted as a tonic sum, or a tonic sum may be reinterpreted as a cyclic interval.

Figure 2.18. Reinterpretative modulation by linear and vertical reordering

p0p7: *0* 0 *7*	(a) i7p7: *0* 7 *0*	(b) i7it: *7* 0 *t*
i9i4: *e* t *6*	i5i4: *e* 6 *t*	i5pe: *6* e *0*
interval system 7,7	interval system 0,e	interval system 3,6
Tonality 0 (agg. 8)	Tonality 1 (agg. e)	Tonality 1 (agg. 9)

As well, reinterpretative modulations may combine linear and vertical reordering with rotation for many modulatory possibilities (figure 2.18b).

Reinterpretations of tonic axis-dyad chords as pivots provide the smoothest modulations due to the greater degree of pc duplication. In discussing the characteristics of tonic chords Perle comments that "a single compositional representation of [a] pitch class is often interpreted as a point of intersection between the two [cyclic sets]" (1977b, 34). Conversely, he notes elsewhere that "a repeated pitch class in the compositional statement sometimes represents a single instance of that pitch class in the array, and vice versa" (1977b, 59-60). These principles create still more possibilities for reinterpretative modulation. Tonic chords may serve as pivot chords in either substitutional (figure 2.19a) or reinterpretative modulation (figure 2.19b). In both cases, the axis-dyad of the tonic chord must be replaced by one which duplicates a tonic sum in the new array. The axis dyad must also be present as a linear dyad in one of the triadic segments of the cyclic sets.

Figure 2.19. Modulation with tonic axis-dyad chords

(a)	**p0p7**: *0* 0 *7*	p4pe: *0* 4 *7*	i7**i2**: *0* 7 *7*	iei6: *0* e *7*
	p4pe: e *0* 4	**p8p3**: e *4* 4	iei6: e *7* 4	i7**i2**: e *3* 4
	sum 0	sum 8	sum 2	sum 2
(b)	i9**pe**: *1* 8 *3*	i7p5: *1* 6 *e*	p0i2: *1* e *3*	
	p2p9: 6 *3* e	i4i9: 8 *1* 3	p2**p5**: e *6* 8	
	sum e	sum 7	sum 5	

Although the pc duplication ensures a smooth modulation, the degree of relatedness between the arrays themselves varies with each reinterpretation. Perle comments on the advantage of this type of modulatory process: "Such reinterpretations of identical pc collections open a variety of choices to the composer at every moment in the progress of a composition" (1977b, 61). The diverse methods entailed by substitutive and reinterpretative modulation thus permit modulation to arrays within the same interval system and to arrays in other tonalities with equal facility.

* * *

Perle's theory of twelve-tone tonality continues to evolve. In the second edition of *Twelve-Tone Tonality* Perle solidifies terminology, refines many concepts (most notably those of mode, key, tonality, and modulation), and fully develops the concept of synoptic array structure. Perle also ventures into new territory in twelve-tone tonality, that of prolongation. Perle writes that "just as the universality of the triad presents diatonic tonal music with a normative principle that defines the meaning of dissonance—a controlled departure from the triad, which remains the referential norm even when it is momentarily absent—so can analogous departures from symmetrical relations provide a basis for prolongational procedures in twelve-tone music" (1996, xiv).

Perle incorporates passing tones, suspensions, and anticipations into the theory, with their definition and usage analogous to that of tonal practice, namely, as pitches that do not belong to the prevailing array segment. Suspensions and anticipations arise through rhythmic displacement of the pc elements. But Perle extends the notion of the passing tone. Citing Berg's practice as a model, Perle introduces a type of figuration he calls "cyclical passing notes," in which a span between two pitches is filled in by notes that belong to an interval cycle. They may unfold the interval cycle directly, as in figure 2.20a, or they may be reordered and then transposed to fit in the registral space to be filled, as in figure 2.20b. The interval cycle may or may not correspond with those that generate the cyclic sets of the array.

Figure 2.20. Cyclical passing notes prolonging motion from pc7 to pc6

(a)	7—6:	7-8-9-t-e-0-1-2-3-4-5-6	(direct ic1 cycle)
(b)	7—6:	7-9-e-1-2-4-6	(reordered ic5 cycle)

Figure 2.21 illustrates the prolongation of a symmetrical progression between two sum 9 dyads by the insertion of cyclical passing tones. Figure 2.21a gives the original symmetrical progression. In figure 2.21b, the resulting gaps are filled in by members of ic1 and ic3 cycles.

Figure 2.21. Prolongation of a symmetrical progression of sum 9 dyads

(a)	4—7	(b)	4-5-6-7
	5—2		5-8-e-2

Perle writes that "the concept of octave displacement as a means of opening space for the prolongation of a progression is as relevant to twelve-tone tonal composition as it is to diatonic tonal composition" (1996, 235).

Perle also introduces the concept of structural levels, specifically that of foreground and background cyclic sets (1996, 235-40). Any particular cyclic set possesses a "background cyclic set," formed from every *third* element of the given cyclic set, as demonstrated in figure 2.22. Cyclic sets formed from ic0 and ic4 cycles will have background cyclic sets of ic0; those formed from ic1, ic3, or ic5 cycles will have background cyclic sets of ic3, and those formed from ic2 or ic6 will have background cyclic sets of ic6.

Figure 2.22. Foreground cyclic set p0p7 (ic5 cycle) and its background cyclic set i5i2 (ic3 cycle)

p0p7:	*0*	0	*7*	5	*2*	t	*9*	3	*4*	8	*e*	1	*6*	6	*1*	e	*8*	4	*3*	9	*t*	2	*5*	7	*0*	0
i5i2:	*0*			5			*9*			8			*6*			e			*3*			2			*0*	

Cyclic sets that have been symmetrically transposed by increments of 2 will share the same background set. In figure 2.23, p4p3 is a +2/-2 sum transposition of the given cyclic set p0p7. Its background cyclic set, like that of p0p7, is also i5i2:

Figure 2.23. Foreground cyclic set p4p3 and its background cyclic set i5i2

p4p3:	2	2	1	3	0	4	e	5	t	6	9	7	8	8	7	9	6	t	5	e	4	0	3	1	2	2
i5i2:	2			3			e			6			8			9			5			0			2	

The background cyclic set i5i2 can thus serve as an invariant structure underlying a foreground progression from p0p7 to p4p3. Perle demonstrates how this principle extends to successions of arrays (1996, 238-40).

Perle's utilization of figuration as a means of prolongation and his recognition of background and foreground structures add a significant level of sophistication and elegance to the theory of twelve-tone tonality, and contribute significantly to its potential for further growth.

Chapter Three

Structural Properties of Cyclic Sets

Perle's system of twelve-tone tonality builds upon the foundation of the cyclic sets. This chapter undertakes a speculative investigation of the nature of cyclic sets in a direction *not* explored by Perle. The investigation moves decidedly outside the context of twelve-tone tonality, employing instead labels and tools of pc set theory.

The process of imbrication is applied to all of the cyclic sets generated from the six ic cycles, resulting in set classes (scs) of cardinalities three to nine. These scs form groups that I shall call *ICS families* (the acronym represents the expression *imbricated cyclic set*). Set classes within an ICS family are closely associated in terms of their cyclic origin and their possession of the properties of inversional symmetry and transpositional combination, the latter concept introduced by Richard Cohn (1987). These traits also obtain among scs of different ICS families. In addition, scs in the ICS families demonstrate a form of equivalence when mapped into other modular universes. Moreover, the scs also exhibit similarity in their ic vectors (icvs). The examination of the similarity relations between sc pairs of the ICS families with Forte's R_n relations inspires the creation of another similarity relation, which I shall call the *R_{SYM} relation*.

The partitions of the interval-class cycles

Of the six different ic cycles, only the ic1 and ic5 cycles exhaust the aggregate upon completion of their cycles, as illustrated in figure 3.1.[1] The ic1 and ic5 cycles are isomorphic under the multiplicative operation M5; that is, the M5 operation maps each pc in the ic1 cycle into a corresponding pc in the ic5 cycle, and vice versa. The other ic cycles do not exhaust all twelve pcs within a single cycle; rather, they partition the

[1] The six different ic cycles are notated in example 2.1 in chapter two of this dissertation.

Figure 3.1. Ic1 and ic5 cycles[2]

ic1:	0	1	2	3	4	5	6	7	8	9	t	e	(0
ic5:	0	5	t	3	8	1	6	e	4	9	2	7	(0

aggregate into a number of cyclic collections, as shown in figure 3.2. The ic3 cycle partitions the aggregate into three cyclic collections (figure 3.2a). The ic2 cycle partitions the aggregate into two different cyclic collections, one consisting of even integers, the other of odd (figure 3.2b). The ic4 cycle partitions the aggregate into four cyclic collections, two comprising even integers, and two comprising odd integers (figure 3.2c). Finally, the ic6 cycle partitions the aggregate into six cyclic collections (figure 3.2d).

Figure 3.2. Partitions of ic3, ic2, ic4, and ic6 cycles

(a)	ic3:	0 3 6 9 (0	1 4 7 t (1	2 5 8 e (2
(b)	ic2:	0 2 4 6 8 t (0	1 3 5 7 9 (1	
(c)	ic4:	0 4 8 (0 2 6 t (2	1 5 9 (1 3 7 e (3	
(d)	ic6:	0 6 (0 3 9 (3	1 7 (1 4 t (4	2 8 (2 5 e (5

Imbrication of cyclic sets

Allen Forte defines imbrication as "the systematic (sequential) extraction of subcomponents of some configuration. . . . [It] is essentially a pre-analytical technique. . . . Imbrication represents an elementary way of determining the subsegments of a primary segment" (1973, 83-84). Since the ic1 and ic5 cycles exhaust the aggregate, the imbrication of the cyclic sets formed from either of these cycles generates a single collection of scs. The imbrication of each of the cyclic sets of the remaining ic cycles,

[2] The single parenthesis in each cycle indicates the completion of the full cycle. Yet since it is a cycle, there is no primary initial pc; the cycle can be initiated on any one of its elements, with the full cycle completed upon the return of the initial pc. The initial pcs given in the figure were arbitrarily chosen.

however, generates different collections of pc sets, depending on the cyclic set's particular combination of partitions. The partitions of the ic3 cycle generate two categories of cyclic sets: (1) a combination of the same partition with itself in inversion, or (2) a combination of two different partitions. The two partitions of the ic2 cycle contain exclusively even or odd integers, again generating two different types of cyclic sets: (1) a combination of the same partition with itself in inversion, or (2) a combination of different partitions, which mixes even and odd integers together, a situation described below as being of *unlike parity*. The partitions of the ic4 and ic6 cycles also contain exclusively even or odd integers, and generate an additional type of cyclic set: (3) a combination of either two different even-integer or two different odd-integer partitions, described below as being of *like parity*.

For all the cyclic sets except that constructed from ic4, the second half of the cyclic set is the tritone transposition of the first half. Further, in cyclic sets of ic1 and ic5, and in those of the remaining ics that combine inversionally related forms of the same partition, the retrograde of the cyclic set is identical to the prograde, as demonstrated in figure 3.3.

Figure 3.3. Ic3 cyclic set formed from two inversionally related forms of the same partition

0 0 *3* 9 *6* 6 *9* 3 *(0* 0

These features influence the formation and order of scs produced by the imbrication of each of the cyclic sets. First, the scs may unfold in a prograde formation that repeats halfway through the cyclic set (at the tritone transposition), as demonstrated in the pentachordal imbrication of an ic3 cyclic set in figure 3.4.

Second, the scs may unfold in a palindromic formation that also repeats halfway through the cyclic set. Together, the two palindromes, either discrete or elided, form a configuration of nested palindromes in one complete cyclic set. Figure 3.5 illustrates the nested palindrome of scs generated by a trichordal imbrication of an ic1 cyclic set.

Figure 3.4. Prograde formation of pentachordal scs in an ic3 cyclic set

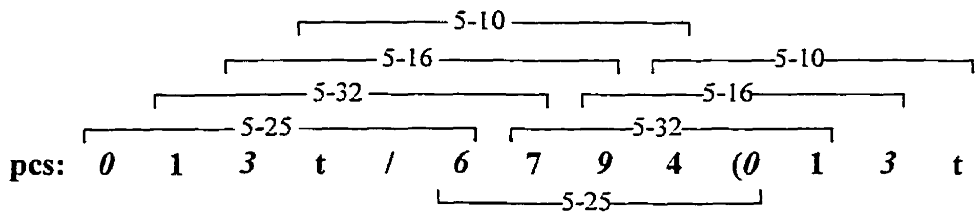
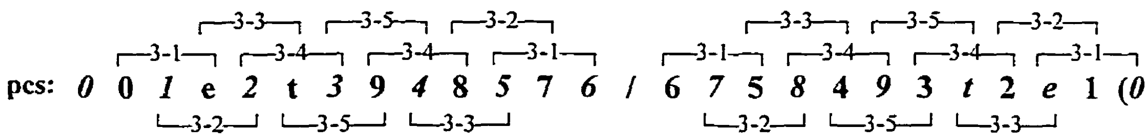


Figure 3.5. Nested palindrome of trichordal scs in an ic1 cyclic set



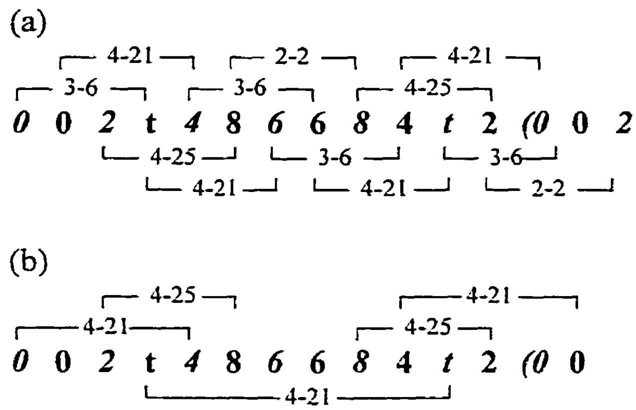
The ICS families

Since the structure of cyclic sets involves inversionally related interval cycles in alternation and produces an alternating pair of tonic sums, it is not surprising that the imbrication of each of these cyclic sets generates a consistent pattern of a small number of closely related scs.[3] I have organized the scs from each cyclic set imbrication into groups called *ICS families*. Imbrication yields a total of twelve trichordal, thirteen tetrachordal, fifteen pentachordal, fourteen hexachordal, eight heptachordal, six octachordal, and five nonachordal scs in all of the ICS-families combined.[4] Imbrication yields two different collections of scs, depending on whether pc duplications are included in or excluded from the segments. The former method produces scs of differing cardinalities, whereas the latter method produces scs of a single cardinality. Figure 3.6 illustrates these two methods applied to the cyclic set combining ic4 cycles of the *same*

[3] Cycles of a different kind are subjected to imbrication by Catherine Nolan (1989). For each trichordal sc, Nolan reordered the prime form in its six permutations, and then discarded the retrogrades. Each of the remaining three permutations comprises a distinct pair of adjacent intervals (except for the permutations of the symmetrical trichords). For each permutation, Nolan constructed cycles by extending each interval of the pair in alternation. Once the trichordal cycles were complete, Nolan imbricated each of the cycles, then observed the properties pertaining to each cycle's scs.

[4] Appendix two lists all the scs according to cardinality and ICS-family membership.

Figure 3.6. Tetrachordal imbrication including pc duplications (a) and excluding pc duplications (b)



partition. In figure 3.6a, tetrachordal imbrication that includes pc duplications produces scs of three different cardinalities, whereas in figure 3.6b, tetrachordal imbrication that excludes pc duplications produces only scs of cardinality four. This study uses the latter method, extending the imbricated segment to include the requisite number of distinct pcs for each cardinality.

Profiles of each of the ICS families are provided below, which emphasize their internal cohesion and kinships with other ICS families. The accompanying tables represent an exhaustive list of all the scs yielded by the imbrication of the cyclic sets.

Certain features are consistent in every ICS family.[5] Each family contains only those trichordal scs whose icvs show an entry in the ic that corresponds to the ic cycle forming the cyclic set. For example, the ICS-1 family's trichordal scs 3-1, 3-2, 3-3, 3-4, and 3-5 are the only trichordal scs with an ic1 entry in their icvs. In addition, the scs of cardinalities larger than three within each family show the highest value in the icv entry that corresponds to the ic cycle that forms the cyclic set. These values are highlighted in boldface in each table. Further, all the tetrachords, hexachords, and octachords in every ICS family are inversionally symmetrical; this is because they segment the consistently symmetrical structure of each cyclic set into scs of even cardinality. As well, all of the

[5] The term "family" is used in neither a group-theoretical nor a hierarchical sense; rather, it simply denotes the collection of scs that emerge from the imbrication of each cyclic set.

all-combinatorial hexachords appear in the imbricated cyclic sets, attesting to the symmetrical construction of the cyclic sets.[6]

The ICS-1 and ICS-5 families

There are twelve different alignments of two inversionally related ic1 cycles in the formation of cyclic sets. Nonetheless, regardless of alignment, imbrication consistently yields the same collection of scs unfolding in a nested palindromic formation in each ic1 cyclic set. Tables 3.1 and 3.2 list the scs that result from the imbrication of the ic1 and ic5 cyclic sets respectively, along with their icvs. These tables indicate that the isomorphic relation between the ic1 and ic5 cycles exists between the ICS-1 and ICS-5 families as well. The ICS-5 family comprises the same number of scs, which also unfold in a nested palindromic formation, and each sc in the ICS-1 family maps into a corresponding sc in the ICS-5 family under the operation M5. Moreover, in addition to

Table 3.1. The scs of the ICS-1 family

sc	icv	sc	icv	sc	icv	sc	icv
3-1	210000	4-1	321000	5-1	432100	6-1	543210
3-2	111000	4-3	212100	5-3	322210	6-z4	432321
3-3	101100	4-7	201210	5-6	311221	6-z6	421242
3-4	100110	4-8	200121	5-7	310132	6-7	420243
3-5	100011	4-9	200022				
7-1	654321	8-1	765442	9-1	876663		
7-5	543342	8-6	654463	9-5	766674		
7-7	532353	8-9	644464				

[6] Set class 6-1 is found in the ICS-1 and ICS-2 families, sc 6-7 in the ICS-1 and ICS-5 families, sc 6-32 in the ICS-2 and ICS-5 families, sc 6-35 in the ICS-2 and ICS-4 families, sc 6-8 in the ICS-2 family, and sc 6-20 in the ICS-4 family. For a discussion of the properties of all-combinatorial hexachords see Milton Babbitt (1955).

Table 3.2. The scs of the ICS-5 family

sc	icv	sc	icv	sc	icv	sc	icv
3-4	100110	4-8	200121	5-7	310132	6-7	420243
3-5	100011	4-9	200022	5-20	211231	6-z26	232341
3-7	011010	4-20	101220	5-27	122230	6-32	143250
3-9	010020	4-23	021030	5-35	032140	6-z38	421242
3-11	001110	4-26	012120				
7-7	532353	8-6	654463	9-5	766674		
7-14	443352	8-9	644464	9-9	676683		
7-35	254361	8-23	465472				

the ten invariant scs between the two families, the remaining scs form pairs in Forte's R_1 relations of icv similarity, to be discussed in detail below.

The ICS-3 family

As discussed above, the three partitions of the ic3 cycle generate two types of cyclic sets, which yield different pc sets under imbrication. Table 3.3a lists the sets resulting from imbricating cyclic sets of the *same partition* combination, while table 3.3b lists the sets resulting from imbricating cyclic sets of the *different partitions* combination. Only four different pcs emerge from the former combination; hence this combination's imbrication produces a single representative of sc 4-28 and two overlapping forms of sc 3-10, both of which contain entries in only ic3 and ic6 of their icvs. This result is due to the ic content of the ic3 cycle: adjacent pcs in an ic3 cycle form ic3, while non-adjacent pcs form either ic3 or ic6.

When the cyclic set combines two different partitions, eight different pcs emerge. Imbrication thus yields scs no larger than octachords, of which there is only one, the octatonic 8-28. Set class 7-31 occurs in eight overlapping forms related by transposition and inversion in alternation. Finally, the scs for each cardinalities three to six emerge in a prograde formation that repeats halfway through the cyclic set at the tritone transposition.

Table 3.3. The scs of the ICS-3 family

(a) same partition			
sc	icv	sc	icv
3-10	002001	4-28	004002

(b) different partitions							
sc	icv	sc	icv	sc	icv	sc	icv
3-2	111000	4-3	212100	5-10	223111	6-z13	324222
3-3	101100	4-10	122010	5-16	213211	6-z23	234222
3-7	011010	4-17	102210	5-25	123121	6-z49	224322
3-11	001110	4-26	012120	5-32	113221	6-z50	224232
7-31	336333	8-28	448444				

The ICS-2 family

The two partitions of the ic2 cycle generate two types of cyclic sets, each yielding different scs under imbrication. Table 3.4a lists the scs resulting from imbricating cyclic sets of the *same partition* combination, while table 3.4b lists the scs resulting from

Table 3.4. The scs of the ICS-2 family

(a) same partition							
sc	icv	sc	icv	sc	icv	sc	icv
3-6	020100	4-21	030201	5-33	040402	6-35	060603
3-8	010101	4-25	020202				

(b) different partitions							
sc	icv	sc	icv	sc	icv	sc	icv
3-1	210000	4-1	321000	5-1	432100	6-1	543210
3-2	111000	4-10	122010	5-2	332110	6-8	343230
3-7	011010	4-23	021030	5-23	132130	6-32	143250
3-9	010020			5-35	032140		
7-1	654321	8-1	765442	9-1	876663		
7-2	554331	8-10	566452	9-2	777663		
7-23	354351	8-23	465472	9-7	677673		
7-35	254361			9-9	676683		

imbricating cyclic sets of the *different partitions* combination. Despite the combination of *same* or *different* partitions, all imbricated scs except the hexachords occur in a nested palindromic formation. The hexachordal scs in the *different partitions* combination unfold in a prograde formation that repeats halfway through the cyclic set, at the tritone transposition.

Only six different pcs emerge from a combination of *same partition* ic2 cycles. Thus imbrication produces a single representative of cardinality six, the whole tone sc 6-35, and two overlapping forms of sc 5-33. All the scs in table 3.4a contain entries in only ic2, ic4, and ic6 of their icvs, reflecting the ic content of the ic2 cycle.

The ICS-4 family

The four partitions of the ic4 cycle generate three types of cyclic sets, again yielding different collections of scs under imbrication. Table 3.5a lists the sole sc that results from combining two inversionally related ic4 cycles of the *same partition* category. The combination of *different partitions* of the ic4 cycle divides into those of *like parity* and those of *unlike parity*. Only six different pcs emerge from either of the latter combinations. As listed in table 3.5b, imbrication of cyclic sets formed from the *different*

Table 3.5. The scs of the ICS-4 family

(a) same partition	
sc	icv
3-12	000300

(b) different partitions of like parity (even/even or odd/odd integers)							
sc	icv	sc	icv	sc	icv	sc	icv
3-6	020100	4-21	030201	5-33	040402	6-35	060603
3-8	010101	4-25	020202				

(c) different partitions of unlike parity (even/odd integers)							
sc	icv	sc	icv	sc	icv	sc	icv
3-3	101100	4-7	201210	5-21	202420	6-20	303630
3-4	100110	4-17	102210				
3-11	001110	4-20	101220				

partitions of like parity category produces the same scs as those produced by the ic2 cyclic set of the *same partition* (compare with table 3.4a), but with six overlapping forms of sc 5-33 here rather than two. Those scs of cardinalities three and four unfold in nested palindromic formations.

Imbrication of cyclic sets of the *different partitions of unlike parity* produces trichordal and tetrachordal scs unfolding in prograde formations that repeat. There are six overlapping forms of sc 5-21 and a single instance of sc 6-20. As shown in table 3.5c, none of these scs contain ic2 or ic6 entries in their icvs.

The ICS-6 family

The aggregate partitions into six different ic6 cycles, which combine to form four different types of ic6 cyclic sets. The ic6 cyclic set of the *same partition* category yields the dyadic sc 2-6, as given in table 3.6a. The ic6 cyclic sets from *different partitions of like parity* contain only four different pcs, which segment into whole-tone scs 3-8 and 4-25 (see table 3.6b). These two scs show entries in only the ic2, ic4, and ic6 of their icvs. These scs also belong to the ICS-2 and ICS-4 families.

The combination of different partitions of the ic6 cycle divides into those of like parity and those of unlike parity, with the latter further subdividing into those partitions whose corresponding elements differ by ic1 (as in {06}and {17}), and those that differ by ic3 (as in {06} and {39}). In each case, imbrication yields a single tetrachordal sc and one trichordal sc, the latter unfolding in a series of four overlapping instances related by transposition and inversion in alternation.

The ICS-3 family (*same partition*) intersects with the ICS-6 family (*different partitions, unlike parity, even/odd integers of ic3 difference*) in their common scs of 3-10 and 4-28. Both of these scs only contain icv entries in ic3 and ic6; the scs form ic6 cycles interlocked at ic3 (see table 3.6c and d).[7]

[7] This intersection is explored futher below, in the section on transpositional combination.

Table 3.6. The scs of the ICS-6 family

(a) same partition	
sc	icv
2-6	000001

(b) different partitions of like parity (even/even or odd/odd integers)			
sc	icv	sc	icv
3-8	010101	4-25	020202

(c) different partitions of unlike parity (even/odd integers of ic1 difference)			
sc	icv	sc	icv
3-5	100011	4-9	200022

(d) different partitions, unlike parity (even/odd integers of ic3 difference)			
sc	icv	sc	icv
3-10	002001	4-28	004002

Thus, the six ICS families contain scs that share a common cyclic origin and some commonalities within their icvs. Further, a number of scs belong to more than one family, due to the combination of partitions of the ic cycle to form the cyclic sets. In all six ICS families, all of the scs of cardinalities four, six, and eight are inversionally symmetrical, due to the even segmentation of the symmetrical cyclic set. But there is another property which all the scs except the trichords share, and that is the property of transpositional combination, a concept introduced by Richard Cohn (1987).

Transpositional combination in ICS families

Transpositional combination is an operation that adds each pc in one set to each pc in another set. These sets are identified as the *operands*. The set of larger cardinality that results from the operation is said to "bear the property" of transpositional combination (TC). Cohn frames the operation as a mathematical expression, with an asterisk denoting the TC operation, as 3-10 * 3-1 = 9-1; that is, (036) * (012) = (012345678). Cohn displays the TC operation in a matrix, with the pcs of one operand on the horizontal axis, and the pcs of the other operand on the vertical axis, as illustrated in figure 3.7. For operations involving a dyadic sc as an operand, Cohn omits the cardinality indicator of 2 and the hyphen. For example, the TC operation that adds the pcs of 4-2 to those of 2-5 is expressed as 4-2 * 5 = 8-14. This dissertation employs Cohn's notational practices.

*Figure 3.7. Matrix display of TC operation 3-10 * 3-1*

2	2	5	8
1	1	4	7
0	0	3	6
*	0	3	6

TC operations that comprise more than two operands are identified as TC chains. The order of the operands in a TC operation does not affect the resulting sc. TC is therefore both a commutative and an associative operation. A sc generated by the TC operation is described as being *factored* by its operands. Some scs may be factored in several different ways. For example, sc 6-9 may be factored by either 3-4 * 2 or 4-11 * 2. Cohn distinguishes between such scs and those that can be factored only by themselves and 1-1 with the terms *non-prime* and *prime*, respectively: "A set-class X is *prime* if its only factors are X and 1-1. Otherwise it is *non-prime*" (1987, 94).

In the ICS families all the scs of cardinalities four to nine, and five of the twelve trichords, bear the TC property.[8] It was observed above that all the scs of cardinalities four, six, and eight in the ICS families are inversionally symmetrical since they are segments of the consistently symmetrical structure of the cyclic set. Similarly, the TC operation may be understood as a symmetrical expansion of a sc segmented from the cyclic set. This particular TC operation employs a designated segment from the cyclic set as one operand and the cyclic ic as the other operand. In the TC operation of figure 3.8a, sc 3-4 from the ICS-5 family functions as one operand with cyclic ic5 as the other

[8] The trichords bearing the TC property are 3-1, 3-6, 3-9, 3-10, and 3-12. Cohn lists all scs in the mod-12 universe that possess the TC property, together with their factors, in Appendices 2 through 6 of his dissertation (1987, 498-619).

operand; together they produce a member of sc 5-20, also of the ICS-5 family. Figure 3.8b illustrates the symmetrical expansion of the 3-4 set within the ic5 cyclic set, wherein the pcs flanking the 3-4 segment are added to form a member of sc 5-20. In the same way, this process may be applied to the 5-20 set, resulting in a member of sc 7-14; in turn, the 7-14 set may expand into a member of the sc 9-9. Thus, TC is a generative operation.

Figure 3.8. TC operation as generator of scs in ICS-5 family

(a)	3-4 * 5 = 5-20	or	(015) * (05) = (0156t)
	5-20 * 5 = 7-14	or	(0156t) * (05) = (01356te)
	7-14 * 5 = 9-9	or	(01356te) * (05) = (0134568te)

(b) ic5 cyclic set segments: **0 5 7 t 2 3 9 8 4...**

9-9: 0 … 4; 3-4: t 2 3; 5-20: 7 t 2 3 9; 7-14: 5 7 t 2 3 9 8

Applying the TC operation recursively forms a TC chain of scs all belonging to the same ICS family. Although there are several ways to factor many TC scs of cardinality four or greater, the present study will limit TC equations to those that assign as one operand a sc belonging to a specific ICS family with its generating cyclic ic as the other operand. Generally, most trichords combine with the cyclic dyadic operand to form pentachords, which in turn combine with the dyadic operand to form heptachords, which then combine with the dyadic operand to form nonachords. In the same way, most tetrachords combine with the dyadic operand to form hexachords, which in turn combine with the dyadic operand to form octachords, and so on. The TC chains culminate in a final sc, beyond which a recursive application of the TC operation will not result in a different, larger set. In the ICS-3 family, for example, the culminating sc is 8-28; performing the TC operation with 8-28 as one operand and ic3 as the other results in another 8-28 set.

The TC chains in each ICS family are represented graphically in this chapter using tree diagrams. Although the ICS families described above include only cardinalities from three to nine, larger cardinalities appear in some TC trees to show the culmination of the recursive TC operations. Since the ic1 and ic5 cycles exhaust the aggregate in one complete cycle, the corresponding ICS-1 and ICS-5 families are each represented by a single TC tree, culminating in the aggregate 12-1. The other cyclic sets produce distinct groups of scs, according to the various combinations of cyclic partitions. Consequently, the scs generated by the different partition categories are represented in separate TC trees for each of the ICS families. In the ICS-2 family, for example, the scs derived from the cyclic set formed from the *same partition* category occupy one TC tree, while the scs derived from the cyclic set formed from *different partitions* occupy a second TC tree.

In the ICS family trees below, each line represents the TC operation *X* * *ic*, where *X* is the sc to the immediate left of the line functioning as one operand, and *ic* is the cyclic ic of the ICS family, functioning as the recursively applied operand. The TC operation results in the sc to the immediate right of the line. In figure 3.9, tracing the line from left to right beginning with sc 3-5 demonstrates the generative TC process, as 3-5 * 1 = 5-7, then 5-7 * 1 = 7-7, then 7-7 * 1 = 9-5, and so on to the culminating sc 12-1. Conversely, following the line from right to left indicates the factoring of the TC sc. Thus, for example, 9-5 = 7-7 * 1, and 7-7 = 5-7 * 1, and 5-7 = 3-5 * 1. The isomorphism between ic1 and ic5 cycles and their associated ICS families extends to the TC relations among their respective scs, as can be seen by comparing the two TC trees in figures 3.9 and 3.10.[9]

[9] The TC trees of figures 3.9 and 3.10 graph the scs given in tables 3.1 and 3.2, respectively.

Figure 3.9. TC tree of the ICS-1 family

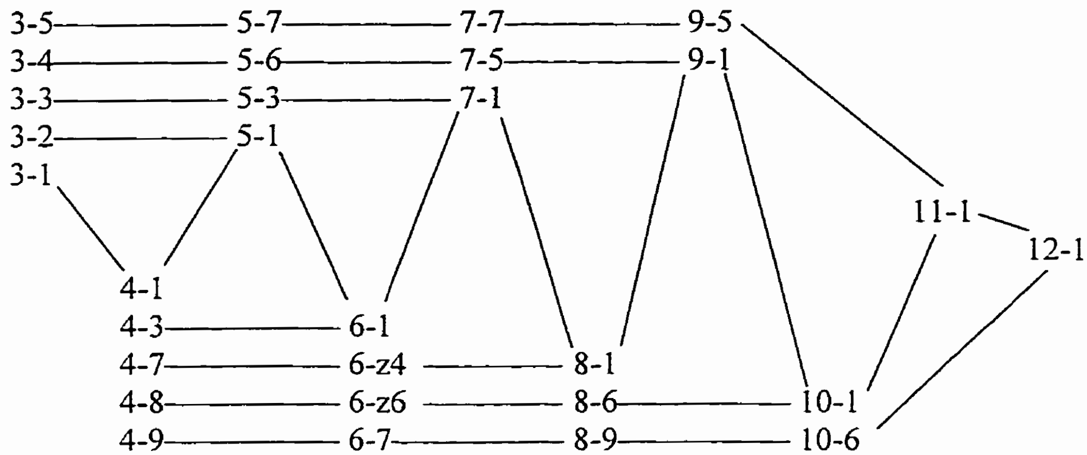
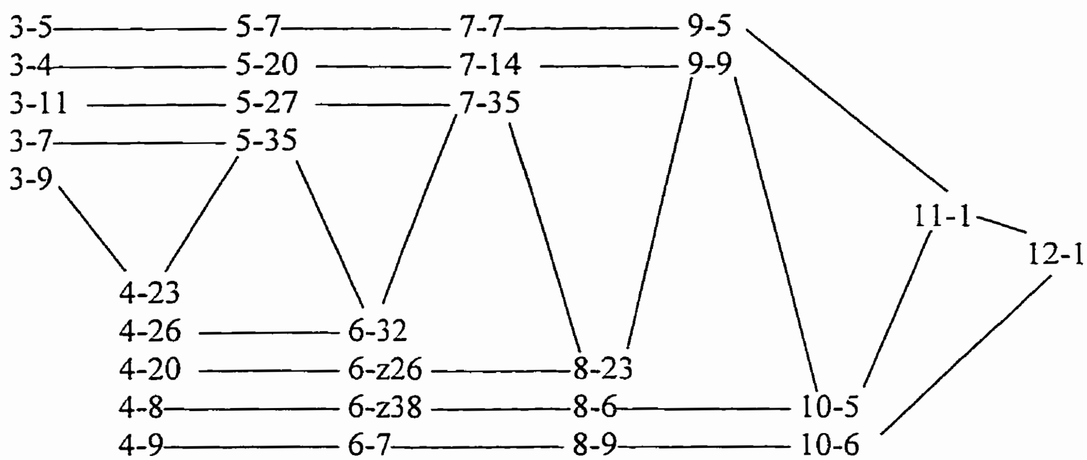


Figure 3.10. TC tree of the ICS-5 family

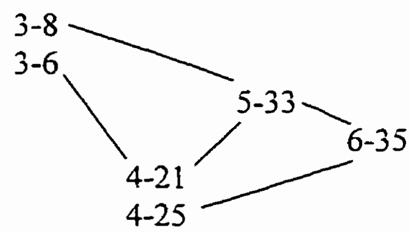


The two partition categories of the ic2 cyclic set produce two distinct groups of TC scs within the ICS-2 family, thereby generating two different TC trees, as illustrated in figure 3.11. Tree (a) graphs those sets from the *same partition* category. Tree (b) graphs those sets from the *different partitions* category.[10]

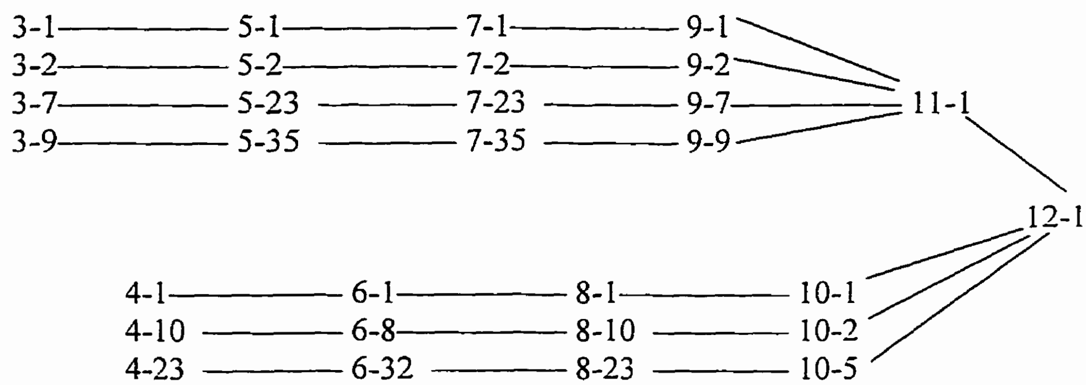
[10] The TC trees of figure 3.11 (a and b) graph the scs given in table 3.4 (a and b), respectively.

Figure 3.11. TC trees of the ICS-2 family

Tree (a): same partition



Tree (b): different partitions



Similarly, the two partition categories of the ic3 cyclic sets in the ICS-3 family generate two different TC trees (figure 3.12). Tree (a) contains only two nodes, representing the two scs derived from the *same partition* category. Tree (b) contains scs derived from the ic3 cyclic set formed from the *different partitions* category.[11]

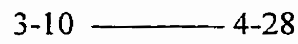
The three different partition categories of the ic4 cyclic set in the ICS-4 family generate three TC trees (figure 3.13). Tree (a) is a trivial example, since it contains just a single node.[12]

[11] The TC trees of figure 3.12 (a and b) graph the scs given in table 3.3 (a and b), respectively.

[12] The TC trees of figure 3.13 (a, b, and c) graph the scs given in table 3.5 (a, b, and c), respectively.

Figure 3.12. TC trees of the ICS-3 family

Tree (a):same partition



Tree (b): different partitions

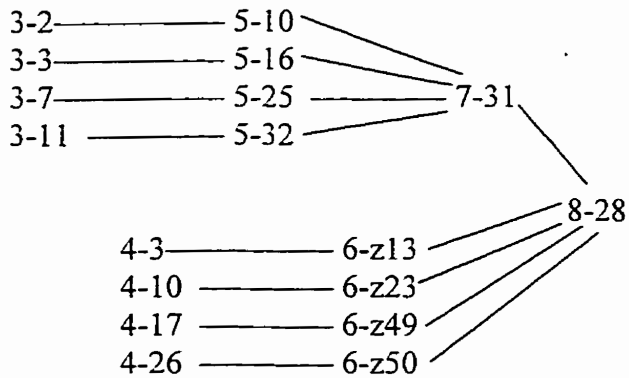
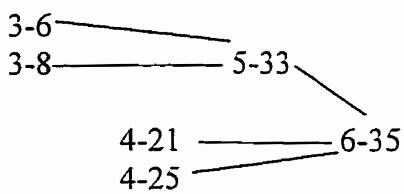


Figure 3.13. TC trees of the ICS-4 family

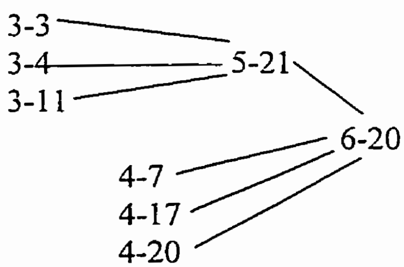
Tree (a): same partition

3-12

Tree (b): different partitions of like parity



Tree (c): different partitions of unlike parity



Finally, the ICS-6 family's scs derive from four different ic6 cyclic set partition categories, and so are represented by four different TC trees (figure 3.14). Tree (a) comprises a single node, while the other three trees each have only two nodes.[13]

Figure 3.14. TC trees of the ICS-6 family

Tree (a): same partition

2-6

Tree (b): different partitions of like parity

3-8———— 4-25

Tree (c): different partitions of unlike parity at ic1

3-5———— 4-9

Tree (d): different partitions of unlike parity at ic3

3-10 ———— 4-28

It was observed above that many of the scs belong to more than one ICS family due to some commonalities within their icvs and to intersecting combinations of partitions that form the cyclic sets. The TC property provides another explanation: these scs may be factored by more than one cyclic dyadic operand. For example, sc 5-7 belongs to both the ICS-1 and ICS-5 families, and may be factored as either 3-5 * 1 or as 3-5 * 5. Similarly, sc 4-26, which belongs to both the ICS-3 and ICS-5 families, is factored by dyadic operands as 3 * 5. Thus, the relations among the scs within and between ICS families are forged not only by cyclic origin, but also by the TC property.

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[13] The TC trees of figure 3.14 (a, b, c, and d) graph the scs given in table 3.6 (a, b, c, and d), respectively.

Similarity relations within ICS families

Forte introduces the R_1, R_2, and R_0 relations to measure the similarity between two scs of the same cardinality based on their ic content (1973, 48-49). Pairs of scs in either the R_1 or R_2 relation have four identical icv entries. Set classes in the R_1 relation feature an exchange of integers in the two variant entries; scs in the R_2 relation lack this feature in their variant entries. Set classes in the R_0 relation have no corresponding icv entries.

Forte's R_n relations by themselves have limited power in establishing similarity relations between scs in the ICS families. First, they do not account for sc pairs of the same cardinality with just one, two, or three identical icv entries. Second, Forte's R_n relations are absolute; they do not reflect relative degrees of similarity between scs. A cursory examination of the icvs of the scs in any of the ICS families reveals a systematic increase in the concentration of larger ics and corresponding decrease in smaller ics in a comparison of scs within and between cardinalities. This may be observed in table 3.7 wherein the changing icv entries of the ICS-1 tetrachords appear in boldface.[14]

Table 3.7. Systematic modifications of icvs in the tetrachordal sets of the ICS-1 family

sc	icv
4-1	**321**000
4-3	**2121**00
4-7	2**0121**0
4-8	20**0121**
4-9	200**022**

The assignment of Forte's R_n relations to the scs of the ICS-1 family produces few similarity relations, as shown in table 3.8. Out of all possible sc pairs of the same cardinality, only seven have relationships of maximum similarity (R_1 or R_2), while two have minimum similarity (R_0). The remaining scs receive no similarity measure at all, a seemingly counterintuitive situation in light of the systematic modification of the icvs in each ICS family, as illustrated in table 3.7. Forte's R_n relations thus appear insufficient

[14] Table 3.7 is a partial reproduction of table 3.1. The latter lists all the scs of the ICS-1 family.

Table 3.8. Sc pairs in the ICS-1 family in Forte's R_n relations

R relation	sc pairs in ICS-1 family			
R_1:	3-2/3-3	3-3/3-4	3-4/3-5	
R_2:	3-1/3-2	4-8/4-9	6-z6/6-7	8-6/8-9
R_0:	4-1/4-8	5-1/5-6		

for expressing the intervallic relationships of the scs in the ICS families.

The defining feature of Forte's R_1 relation is considered to be the ic exchange between the variant icv entries, as illustrated in the pentachordal sc pair in figure 3.15a. Examining Forte's R_1 relation from a different perspective reveals another significant feature: the variant entries are symmetrically related. Figure 3.15b illustrates the symmetrical correspondence between the variant entries in the same pair of scs.

Figure 3.15. R_1 relation between scs 5-10 and 5-16 with interchange feature emphasized (a), and with symmetrical relationship emphasized (b)

(a)	5-10:	2	2	3	1	1	1	(b)	5-10:	2	2	3	1	1	1
	5-16:	2	1	3	2	1	1		5-16:	2	1	3	2	1	1

Moreover, this feature consistently associates the icvs of all sc pairs in the R_1 relation, since the two pairs of variant entries will always display a symmetrical relationship. For every gain in one ic in the icv, there is a corresponding loss by the same value in the other ic. This is because scs in the same cardinality have the same total number of intervals. Since four icv entries are identical between R_1-related scs, then the two variant entries must form a gain/loss relationship.

This symmetrical relationship is captured graphically in the form of an *interval-difference vector*, a term introduced by Eric Isaacson (1990, 16). The interval-difference vector (idv) gives the difference between respective entries in the ic vectors. The idv removes the distinction between Forte's R_1 and R_2 relations. Whether the two variant entries use the same or different integers is immaterial, as seen in the idvs of figure 3.16.

Figure 3.16. Idvs for R_1-related sc pair 5-10 and 5-16 (a), and R_2-related sc pair 4-21 and 4-25 (b)

(a)

5-10:	2	2	3	1	1	1
5-16:	2	1	3	2	1	1
idv:	0	+1	0	-1	0	0

(b)

4-21:	0	3	0	2	0	1
4-25:	0	2	0	2	0	2
idv:	0	+1	0	0	0	-1

The R_{SYM} relation

Building on the symmetrical aspect of Forte's R_1 and R_2 categories, I propose the introduction of a new category of similarity relations between scs of the same cardinality, the *R_{SYM} relation*. The R_{SYM} relation permits a broader view of ic similarity between scs than that of Forte's relations in that it includes *pairs* of corresponding variant ic entries.

The R_{SYM} relation divides into two subcategories: R_{SYM-1} and R_{SYM-2}. Two scs in the R_{SYM-1} relation have two corresponding entries in their respective icvs that are identical, while the remaining four corresponding entries differ symmetrically. Figure 3.17a illustrates the R_{SYM-1} relation between scs 3-1 and 3-3; four of the idv entries show symmetrical differences of +1 or -1, while the remaining two entries show no difference. Although the symmetrical differences between their variant icv entries indicate that the scs are similar in ic structure, this sc pair does not share a similarity relation in any of Forte's terms.

Two scs in the R_{SYM-2} relation have no identical entries in their icvs, yet all the corresponding entries are symmetrically related.[15] Figure 3.17b illustrates the R_{SYM-2} relationship between scs 4-1 and 4-8, which lie in Forte's R_0 relation. All R_{SYM-2}-related scs are also R_0-related, but the associations are not automatically reciprocal: not all R_0-related sets share an R_{SYM-2} relation. For example, scs 4-11 and 4-28 have no corresponding entries in their icvs and so are R_0-related, but because not all the pairs of entries differ symmetrically, the sc pair is not R_{SYM-2}-related (as shown in figure 3.17c).

[15] The R_{SYM-1} relation does not necessarily imply a greater degree of similarity than the R_{SYM-2} relation, as discussed below.

Figure 3.17. Scs 3-1 and 3-3 in R_{SYM-1} (a), scs 4-1 and 4-8 in R_{SYM-2} and R_0 (b), and scs 4-11 and 4-28 in R_0 but not R_{SYM-2} (c)

(a)	3-1:	2	1	0	0	0	0
	3-3:	1	0	1	1	0	0
	idv:	+1	+1	-1	-1	0	0

(b)	4-1:	3	2	1	0	0	0
	4-8:	2	0	0	1	2	1
	idv:	+1	+2	+1	-1	-2	-1

(c)	4-11:	1	2	1	1	1	0
	4-28:	0	0	4	0	0	2
	idv:	+1	+2	-3	+1	+1	-2

The *IcVSIM* is a function derived from the idv introduced by Isaacson (1990) to quantify the similarity between two scs of any cardinality. It accomplishes this by finding the standard deviation from the mean in the idv of the two scs to be compared, assigning IcVSIM values ranging from 0.000 (indicating maximal intervallic similarity) to 3.578 (indicating maximal intervallic dissimilarity).[16]

The IcVSIM is useful in the present study because it enables the transformation of R relations into quantifiable measures of relative similarity. Relative similarity measures the *degree* of difference between corresponding icv entries, rather than the *number* of entries that differ. Hence scs in an R_1 relation are not necessarily more similar than sets in an R_2, R_{SYM-1}, or R_{SYM-2} relation. For example, the R_{SYM-1}-related scs 4-3 and 4-7 generate a fairly level idv of [0 +1 +1 -1 -1 0], and thus receive a lower IcVSIM value (0.816) than do the R_1-related scs 4-1 and 4-23. This latter pair generates a more jagged idv of [+3 0 0 0 -3 0], and thus receives a considerably higher IcVSIM value (1.732). Although only two entries differ between the latter sc pair's icvs, they differ to a greater degree (±3) than do the four variant entries of the former sc pair's icvs (±1). Hence, in this case the R_{SYM-1}-related pair of scs is judged to be more similar than the R_1-related pair.

Further, two different pairs of scs in an R_1 relation do not automatically receive the same IcVSIM value. Rather, IcVSIM values are consistent among any pair of R-related scs only if the variable entries of the idv differ by the same amount (although they need not occupy corresponding entry positions). Hence, R_1-related pairs 4-10 and 4-17 [idv: 0 +2 0 –2 0 0], and 4-5 and 4-27 [idv: +2 0 -2 0 0 0] both receive an IcVSIM value

[16] An IcVSIM value of 0.000 is assigned in three situations: (1) a sc compared with itself, (2) a sc compared with its z-correspondent, and (3) a sc compared with another of differing cardinality but which together generate a level idv, as in the paired scs 3-10 and 6-30, which generate idv [+2+2+2+2+2+2]. The sc pair of maximum dissimilarity is 6-35 and 8-28, the whole-tone and octatonic collections, respectively.

of 1.155. Although their idvs are not identical, the variable entries differ by ±2. The R_1-related pair of 4-1 and 4-23 discussed above receives an IcVSIM value of 1.732 because its variable entries differ by ±3.

The idvs of R-related scs therefore distinguish degrees of relative similarity, as indicated in table 3.9. The three columns under each category of R relation list in order the degree of similarity within that R relation, the amount of difference between the variable entries in the idvs, and the IcVSIM values for all sc pairs in that degree of R relation.

Table 3.9. Degrees of R relations between sc pairs of the same cardinality

sc pairs in R_1 or R_2			**sc pairs in R_{SYM-1}**			**sc pairs in R_{SYM-2}**		
degree	*idv variable*	*IcVSIM*	*degree*	*idv variable*	*IcVSIM*	*degree*	*idv variable*	*IcVSIM*
1^{st}	±1	0.577	1^{st}	±1,±1	0.816	1^{st}	±1,±1,±1	1.000
2^{nd}	±2	1.155	2^{nd}	±1,±2	1.291	2^{nd}	±1,±1,±2	1.414
3^{rd}	±3	1.732	3^{rd}	±2,±2	1.633	3^{rd}	±1,±2,±2	1.732
4^{th}	±4	2.309	4^{th}	±1,±3	1.826	4^{th}	±1,±1,±3	1.915
etc.			etc.			etc.		

This study employs Forte's R_n relations in a manner that differs from Forte's original intention. While Forte aims to show icv correspondence between scs of the same cardinality, he does not address quantitative issues or relative degrees of similarity beyond his distinction between maximum and minimum similarity. In the present study Forte's R relations are used as a point of departure for the development and exploration of R_{SYM} relations. These R_{SYM} relations establish similarity between scs based on symmetrical icv entries. As discussed above, the study employs Isaacson's IcVSIM to measure degrees of similarity. Isaacson contends that he makes no qualitative assertions through the assignment of an IcVSIM value per se: "It is worth observing that all of the similarity functions discussed here (IcVSIM included) measure only the quantitative similarity between sets; they make no attempt to account for the qualitative differences between IcVs. With the IcVSIM relation, even though two pairs of sets produce the same

IcVSIM value, that value may arise because of very different conditions in the two set pairs" (1990, 25).[17] The R_{SYM} relation combines qualitative assertions about symmetrical icv correspondence between sc pairs with quantitative assertions about the degrees of similarity.

R relations within ICS families

There are just ten distinct IcVSIM values for R-related scs in all the ICS families. Table 3.10 gives the R relations among the sc pairs in the ICS-1 family. A comparison of table 3.10 and table 3.8 demonstrates that the R_{SYM} relation establishes similarity relations among a broader range of sc pairs. Virtually all the scs in the ICS-1 family form an R relation with another sc of the same cardinality; the exceptions are 7-1 and 8-1.

Table 3.10. R relations between sc pairs in the ICS-1 family

R relation	sc pair			
1st-degree R_1:	3-2/3-3	3-3/3-4	3-4/3-5	
1st-degree R_2:	3-1/3-2	4-8/4-9	6-z6/6-7	8-6/8-9
1st-degree R_{SYM-1}:	3-1/3-3 3-2/3-5 4-7/4-8 7-5/7-7	3-1/3-4 3-3/3-5 5-1/5-3 9-1/9-5	3-1/3-5 4-1/4-3 5-3/5-6	3-2/3-4 4-3/4-7 5-6/5-7
1st-degree R_{SYM-2}:	6-1/6-z4			
2nd-degree R_{SYM-1}:	4-1/4-7	4-3/4-8	4-7/4-9	
2nd-degree R_{SYM-2}:	4-1/4-8	5-1/5-6		

[17] Isaacson reviews a number of measures of intervallic similarity (1990). The IcVSIM is utilized in the present study, however, because of the relevance of the idv to the R_{SYM} relations. Further, although Isaacson uses the IcVSIM to compare scs of different cardinalities, its usage here is restricted to scs of the same cardinality, since the present study is concerned with symmetrical icv entries.

Analogous relations obtain between sc pairs in the ICS-5 family, as seen in table 3.11. The only scs that form no R relations within the ICS-5 family are 7-35 and 8-23. They do form R_I relations with the 7-1 and 8-1 scs of the ICS-1 family, however. Moreover, scs in each category of R relation in one of these ICS families form R_I relations with corresponding scs in the same category in the other ICS family. Thus the isomorphic relationship between these two ICS families extends to their similarity relations as well.

Table 3.11. R relations between sc pairs in the ICS-5 family

R relation	sc pair			
1st-degree R_1:	3-4/3-5	3-4/3-11	3-7/3-11	
1st-degree R_2:	3-7/3-9	4-8/4-9	6-7/6-z38	8-6/8-9
1st-degree R_{SYM-1}:	3-4/3-7 3-5/3-11 4-23/4-26 7-7/7-14	3-4/3-9 3-9/3-11 5-7/5-20 9-5/9-9	3-5/3-7 4-8/4-20 5-20/5-27	3-5/3-9 4-20/4-26 5-27/5-35
1st-degree R_{SYM-2}:	6-z26/6-32			
2nd-degree R_{SYM-1}:	4-8/4-26	4-9/4-20	4-20/4-23	
2nd-degree R_{SYM-2}:	4-8/4-23	5-20/5-35		

For the remaining ICS families, the tables of R relations distinguish among the cyclic partition categories from which the scs are derived. The ICS-3 family comprises just one trichordal sc and one tetrachordal sc in its *same partition* category, so no scs of the same cardinality within this category may be examined for ic similarity. Each sc in the *different partitions* category forms an R relation with another sc, however, as listed in table 3.12a. In addition, R relations across the partition categories exist: the 3-10 sc from the *same partition* category forms 1st-degree R_{SYM-1} relations with the trichordal scs of the *different partition* category, as given in table 3.12b. Set class 4-28 alone forms no R relations within the ICS-3 family, since the idvs of the tetrachordal pairs are asymmetrical. Figure 3.18 illustrates two of these sc pairs and their idvs.

Table 3.12. R relations between sc pairs in the ICS-3 family

(a) scs from ic3 cyclic sets of *different* partitions				
R relation	**sc pair**			
1st-degree R_I:	3-2/3-3 5-10/5-16 6-z13/6-z23 6-z23/6-z50	3-2/3-7 5-10/5-25 6-z13/6-z49 6-z49/6-z50	3-3/3-11 5-16/5-32 6-z13/6-z50	3-7/3-11 5-25/5-32 6-23/6-z49
1st-degree R_{SYM-I}:	3-2/3-11 4-10/4-26	3-3/3-7 4-17/4-26	4-3/4-10 5-10/5-32	4-3/4-17 5-16/5-25
2nd-degree R_I:	4-3/4-26	4-10/4-17		

(b) scs from ic3 cyclic sets of *same* and *different* partitions				
R relation	**sc pair**			
1st-degree R_{SYM-I}:	3-10/3-2	3-10/3-3	3-10/3-7	3-10/3-11

Figure 3.18. Asymmetrical idvs between ICS-3 family sc pairs 4-3 and 4-28 (a) and 4-26 and 4-28 (b)

(a)						
4-3:	2	1	2	1	0	0
4-28:	0	0	4	0	0	2
idv:	+2	+1	-2	+1	0	-2

(b)						
4-26:	0	1	2	2	1	0
4-28:	0	0	4	0	0	2
idv:	0	+1	-2	+2	+1	-2

A variety of R relations exist in the ICS-2 family, as indicated in table 3.13. The trichordal and tetrachordal sc pairs drawn from the *same partition* category form R_2 relations (table 3.13a). Set class pairs from the *different partition* category form R_I and R_2 relations (table 3.13b), while R_{SYM-1} and R_{SYM-2} relations obtain across the partition categories (table 3.13c). Only scs 6-35 and 8-10 do not form R relations with any other scs of the same cardinality in the ICS-2 family.

Table 3.14 gives the R relations within the ICS-4 family. All the trichordal and tetrachordal scs within and between partition categories form 1st-degree R relations, except the 2nd-degree R2-related sc pair 3-12 and 3-6 (table 3.14d). Only the two pairs

Table 3.13. R relations between sc pairs in the ICS-2 family

(a) scs from ic2 cyclic sets of *same* partition		
R relation	**sc pair**	
1^{st}-degree R_2:	3-6/3-8	4-21/4-25

(b) scs from ic2 cyclic sets of *different* partitions				
R relation	**sc pair**			
1^{st}-degree R_1:	3-2/3-7	9-2/9-7		
1^{st}-degree R_2:	3-1/3-2 7-1/7-2	3-7/3-9 7-23/7-35	5-1/5-2 9-1/9-2	5-23/5-35 9-7/9-9
2^{nd}-degree R_1:	3-1/3-9	5-2/5-23	7-2/7-23	9-1/9-9
2^{nd}-degree R_2:	6-1/6-8	6-8/6-32		
3^{rd}-degree R_1:	4-1/4-23	8-1/8-23		
3^{rd}-degree R_2:	5-1/5-23	5-2/5-35	7-1/7-23	7-2/7-35
4^{th}-degree R_1:	5-1/5-35	6-1/6-32	7-1/7-35	

(c) scs from ic2 cyclic sets of *same* and *different* partitions				
R relation	**sc pair**			
1^{st}-degree R_{SYM-1}:	3-6/3-2	3-6/3-7	3-8/3-2	3-8/3-7
2^{nd}-degree R_{SYM-2}:	4-21/4-10			
6^{th}-degree R_{SYM-2}:	5-33/5-2	5-33/5-23		

of pentachordal and hexachordal scs from the contrasting *different partitions* categories fail to establish any R relations.

Finally, table 3.15 gives the R relations within the ICS-6 family. Each partition category generates a single trichord and single tetrachord, so R relations are only established between scs across partition categories. Set classes 3-10 and 4-28 from the *different partitions of unlike parity at ic3* category do not form any R relations.

Table 3.14. R relations between sc pairs in the ICS-4 family

(a) scs from ic4 cyclic sets of *different partitions of like parity*		
R relation	**sc pair**	
1^{st}-degree R_2:	3-6/3-8	4-21/4-25

(b) scs from ic4 cyclic sets of *different partitions of unlike parity*			
R relation	**sc pair**		
1^{st}-degree R_1:	3-3/3-4 4-7/4-17	3-3/3-11 4-7/4-20	3-4/3-11 4-17/4-20

(c) scs from ic4 cyclic sets of *different partitions of like* and *unlike parity*			
R relation	**sc pair**		
1^{st}-degree R_{SYM-1}:	3-8/3-3	3-8/3-4	3-8/3-11

(d) scs from ic4 cyclic sets of *same partition* and *different partitions of like parity*	
R relation	**sc pair**
2^{nd}-degree R_2:	3-12/3-6

Table 3.15. R relations in the ICS-6 family between sc pairs from ic6 cyclic sets of different partitions of like and unlike parity (at 1 ic apart)

R relation	**sc pair**
1^{st} degree R_{SYM-1}:	3-8/3-5
3^{rd} degree R_{SYM-1}:	4-25/4-9

Similarity relations between ICS families

Certain pairs of ICS families contain considerable sc invariance and similarity. The ICS-1 and ICS-5 families have ten invariant scs between them. Each remaining sc in one family forms an R_1 relation with a sc in the other family. In each instance the two variable icv entries are ic1 and ic5. The invariant scs between the ICS-2 and ICS-4 family derive from the *same partition* category in the former and the *different partitions of like parity* category in the latter. Of the variant trichordal scs between the two families, two pairs form R_1 relations and one pair forms an R_2 relation. Further, each trichordal sc in one family forms an R_{SYM-1} relation with two trichordal scs in the other family, with two of their four variable icv entries found in ic2 and ic4. The scs 4-10 and 4-17 are R_1-related, again with variable icv entries of ic2 and ic4. The remaining tetrachordal scs between the two families form R_{SYM-1}-related pairs, with two of their four variable entries also in ic2 and ic4.

The ICS-3 and ICS-6 families have two invariant scs, 3-10 and 4-28, which derive from the *same partition* category in the ICS-3 family and the *different partitions of unlike parity at ic3* category of the ICS-6 family. The remaining trichordal scs between the two families form R_{SYM-1} relations, with two of the four variable icv entries in ic3 and ic6. None of the tetrachordal scs form R relations, however, since their idvs are asymmetrical.

Although invariance and R relations exist sporadically in other combinations of ICS families, they do not account for all the scs in those pairings. The variable entries in the icvs of the R-related sc pairs correspond to the families to which they belong, thereby providing another means of establishing relations among ICS families. Whereas the TC property generates unifying links among scs within individual ICS families, the R-relations form such links within and across ICS families.

Modular equivalence between ICS families

Equivalence of sets typically refers to transpositional or inversional equivalence. Another form of set equivalence also exists between ICS families, one based on the mapping of pc sets into sets in a different modular universe. Mod-x arithmetic reduces the infinite range of pitches to a finite universe of x pcs. Hence a mod-12 universe comprises 12 pcs, a mod-6 universe comprises 6 pcs, a mod-4 universe comprises 4 pcs, and so on. Figure 3.19 illustrates how mod-12 pcs map into the pcs of mod-6, mod-4, mod-3, and mod-2 universes. The columns of mod-12 list the pcs and their corresponding mappings into the pcs of the other mod-x universes. The rows of mod-12 list the x-cycle partitions of the aggregate, each of which maps into a single pc in the corresponding mod-x universe.[18] The twelve pcs of mod-12 only map evenly into universes of mod-6, 4, 3, and 2 since these integers are all factors of 12. Although the integer 1 is also a factor of 12, mod-1 arithmetic maps all pcs into a single pc, 0, and as such is trivial. The mod-12 pcs will not map evenly into the remaining modular universes, such as mod-5.

Figure 3.19. Mod-12 pc mappings into pcs of other modular universes

mod-12 → mod-6			mod-12 → mod-4			mod-12 → mod-3			mod-12 → mod-2		
0,6	→	0	0,4,8	→	0	0,3,6,9	→	0	0,2,4,6,8,t	→	0
1,7	→	1	1,5,9	→	1	1,4,7,t	→	1	1,3,5,7,9,e	→	1
2,8	→	2	2,6,t	→	2	2,5,8,e	→	2			
3,9	→	3	3,7,e	→	3						
4,t	→	4									
5,e	→	5									

Similarly, the 224 pc scs in the mod-12 universe reduce to thirteen scs in mod-6, six scs in mod-4, four scs in mod-3, and three scs in mod-2.[19] Thus, for example, sc 4-7 (0145) maps into sc (0123) mod-6, sc (01) mod-4, sc (012) mod-3, and sc (01) mod-2.[20]

Table 3.16 gives the mappings of the mod-12 scs of the ICS-1 family into the other modular universes. The twenty-six scs in the ICS-1 family map into three mod-4 scs, one mod-2 sc, seven mod-6 scs, and two mod-3 scs. The mod-2 and mod-4 scs are combined in table 3.16a, while mod-3 and mod-6 scs are combined in table 3-16b.

[18] Cohn addresses this issue in the context of CYCLE homomorphisms (1991).

[19] These figures include the null set.

[20] Although the set {0,1,4,5} maps into the same pcs in mod-6, the set in normal order in mod-6 is {4,5,0,1}, and is thus (0,1,2,3) in prime form.

Table 3.16. Equivalence of mod-12 scs in the ICS-1 family in other modular universes

(a) mod-12 scs mapped into mod-2 and mod-4 scs

mod-2:	(01)							
mod-4:	(01)		(012)			(0123)		
mod-12:	3-3	3-4	3-1	3-2	3-5	4-1	4-9	
	4-7		4-3	4-8		5-1	5-7	
			5-3	5-6		6-1	6-z6	6-7
			6-z4			7-1	7-5	7-7
						8-1	8-6	8-9
						9-1	9-5	

(b) mod-12 scs mapped into mod-3 and mod-6 scs

mod-3:	(01)				(012)						
mod-6:	(01)	(013)		(0134)	(012)		(0123)		(01234)		(012345)
mod-12:	3-5	3-2	3-3	4-3	3-1	3-4	4-1	4-7	5-1	5-3	6-1
	4-9				4-8		5-6		6-z4		7-1
					5-7		6-z6		7-5		8-1
					6-7		7-7		8-6		9-1
							8-9		9-5		

The isomorphic relation between the ICS-1 and ICS-5 families extends to the mapping of their respective scs into scs of other modular universes. All the R_1-related scs between the two families map into the same scs in the other modular universes, as revealed in a comparison of table 3.16 (a and b) with table 3.17 (a and b).

Table 3.17. Equivalence of mod-12 scs in the ICS-5 family in other modular universes

(a) mod-12 scs mapped into mod-2 and mod-4 scs

mod-2:	(01)							
mod-4:	(01)		(012)			(0123)		
mod-12:	3-4	3-11	3-5	3-7	3-9	4-9	4-23	
	4-20		4-8	4-26		5-7	5-35	
			5-20	5-27		6-7	6-32	6-z38
			6-z26			7-7	7-14	7-35
						8-6	8-9	8-23
						9-5	9-9	

(b) mod-12 scs mapped into mod-3 and mod-6 scs

mod-3:	(01)				(012)						
mod-6:	(01)	(013)		(0134)	(012)		(0123)		(01234)		(012345)
mod-12:	3-5	3-7	3-11	4-26	3-4	3-9	4-20	4-23	5-27	5-35	6-32
	4-9				4-8		5-20		6-z26		7-35
					5-7		6-z38		7-14		8-23
					6-7		7-7		8-6		9-9
							8-9		9-5		

For each of the remaining ICS families, certain mappings constitute a "best fit" in other modular universes, in that the mappings correspond to the partition categories of the scs. For example, table 3.18a shows that in the ICS-2 family the scs derived from the *same partition* category all map into a single mod-2 sc (0) and a single mod-4 sc (02). This is because they all have entries of zero in all three odd-integer icv positions. The scs derived from the *different partitions* category map into one mod-2 sc and two mod-4 scs. On the other hand, the mod-3 and mod-6 mappings of the ICS-2 scs draw from both partition categories (table 3.18b).

Table 3.18. Equivalence of mod-12 scs in the ICS-2 family in other modular universes

(a) mod-12 scs mapped into mod-2 and mod-4 scs								
mod-2:	(0)		(01)					
mod-4:	(02)		(012)		(0123)			
mod-12:	same partition		different partitions					
	3-6	3-8	3-1	3-2	4-1	4-10	4-23	
	4-21	4-25	3-7	3-9	5-1	5-2	5-23	5-35
	5-33				6-1	6-8	6-32	
	6-35				7-1	7-2	7-23	7-35
					8-1	8-10	8-23	
					9-1	9-2	9-7	9-9

(b) mod-12 scs mapped into mod-3 and mod-6 scs											
mod-3:	(01)			(012)							
mod-6:	(02)	(013)	(0134)	(012)	(024)	(0123)	(01234)		(012345)		
mod-12:	3-8	3-2	4-10	3-1	3-6	4-1	5-1	5-2	6-1	6-8	6-32
	4-25	3-7		3-9	4-21	4-23	5-23	5-35	7-1	7-2	
					5-33				7-23	7-35	
					6-35				8-1	8-10	8-23
									9-1	9-2	
									9-7	9-9	

In the ICS-4 family, the scs derived from the *same partition* and *different partitions of like parity* categories map into one mod-2 sc, while those from the *different partitions of unlike parity* map into a second mod-2 sc. The mod-4 sc mappings establish an even closer correspondence with the three distinct partition categories. The scs from each of the three partitions map into three different mod-4 scs, as seen in table 3.19a. The mod-6 and mod-3 mappings draw scs from across the partition categories (table 3.19b).

Table 3.19. Equivalence of mod-12 scs in the ICS-4 family in other modular universes

(a) mod-12 scs mapped into mod-2 and mod-4 scs			
mod-2:	(0)		(01)
mod-4:	(0)	(02)	(01)
mod-12:	same partition	different partitions of like parity	different partitions of unlike parity
	3-12	3-6 3-8 4-21 4-25 5-33 6-35	3-3 3-4 3-11 4-7 4-17 4-20 5-21 6-20

(b) mod-12 scs mapped into mod-3 and mod-6 scs								
mod-3:	(01)			(012)				
mod-6:	(02)	(013)	(0134)	(012)	(024)	(0123)	(01234)	(012345)
mod-12:	3-8 4-25	3-3 3-11	4-17	3-4	3-6 3-12 4-21 5-33 6-35	4-7 4-20	5-21	6-20

Further, just as the ICS-1 and ICS-5 families map into the same scs in the four mod-x universes under discussion, the ICS-2 and ICS-4 families map into the same mod-2, mod-3 and mod-6 scs. They differ for two of the three mod-4 scs.

In the ICS-3 family, scs from the two partition categories map separately into two mod-3 scs, constituting a "best fit" mapping, as illustrated in table 3.20. The two scs from the *same partition* category map into sc (0) mod-3, while those from the *different partition* category map into sc (01) mod-3 (table 3.20a). The mappings into mod-2 and mod-4 draw from across the partition categories (table 3.20b).

Table 3.21 lists the mappings of the scs of the ICS-6 family. The scs map into two mod-2 and three mod-4 scs in accordance with the two categories of *different partitions of like* and *unlike parity* (table 3.21a). The partition categories of the scs map differently in the mod-3 and mod-6 universes; the scs from *different partitions of unlike parity at ic3* map into one mod-3 and mod-6 sc, while the remaining scs map into the other mod-3 and mod-6 scs (table 3.21b).

Table 3.20. Equivalence of mod-12 scs in the ICS-3 family in other modular universes

(a) mod-12 scs mapped into mod-3 and mod-6 scs							
mod-3:	(0)	(01)					
mod-6:	(03)	(013)		(0134)			
mod-12:	same partition	different partitions					
	3-10	3-2	3-3	4-3	4-10	4-17	4-26
	4-28	3-7	3-11	5-10	5-16	5-25	5-32
				6-z13	6-z23	6-z49	6-z50
				7-31			
				8-28			

(b) mod-12 scs mapped into mod-2 and mod-4 scs								
mod-2:	(01)							
mod-4:	(01)		(012)			(0123)		
mod-12:	3-3	3-11	3-2	3-7	3-10	4-10	4-28	
	4-17		4-3	4-26		5-10	5-25	
			5-16	5-32		6-z13	6-z23	6-z50
			6-z49			7-31		
						8-28		

Table 3.21. Equivalence of mod-12 scs in the ICS-6 family in other modular universes

(a) mod-12 scs mapped into mod-2 and mod-4 scs			
mod-2:	(0)	(01)	
mod-4:	(02)	(012)	(0123)
mod-12:	different partitions of like parity	different partitions of unlike parity	
	3-8	3-5	4-9
	4-25	3-10	4-28

(b) mod-12 scs mapped into mod-3 and mod-6 scs			
mod-3:	(0)	(01)	
mod-6:	(03)	(01)	(02)
mod-12:	different partitions of unlike parity at ic3	different partitions of unlike parity at ic1	different partitions of like parity
	3-10	3-5	3-8
	4-28	4-9	4-25

The scs in the ICS-3 and ICS-6 families all map into the same two mod-3 scs, with the invariant scs between the two families in one of these scs (0), and their variant scs in the other (01). The variant trichordal scs between the two ICS families share $R_{SYM\text{-}I}$ relations.

Thus, the equivalence of all the scs in other modular universes forges another bond among the scs within and among the ICS families.

* * *

This chapter has explored the nature of the cyclic sets, the bases on which Perle constructs his system of twelve-tone tonality. Imbrication of each cyclic set generates scs unfolding in palindromic or prograde formations. Because the scs resulting from each imbricated cyclic set share a number of structural properties, they are grouped together in associations identified as ICS families. This chapter provides a profile of each ICS family which specifically examines the scs' properties of transpositional combination, similarity relations, and equivalence in other modular universes. The isomorphism between the ic1 and ic5 cycles also exists between the scs of the ICS-1 and ICS-5 families in each of these properties.

The scs within each cyclic set are segments of symmetrical constructs, since the cyclic sets combine inversionally complementary ic cycles in alternation. As a result, all the scs of even cardinality possess the property of inversional symmetry, and all the scs of cardinalities greater than three (and some of the trichordal scs) possess the TC property as well. Recursive TC operations that employ as operands the segments from the cyclic sets and their generating cyclic ic create TC chains of all the scs within each ICS family, which are represented graphically in this study in the form of trees. Since the ic1 and ic5 cycles each exhaust the aggregate, all the scs within each of the ICS-1 and ICS-5 families are contained within a single TC tree. The remaining ICS families' scs are generated by several TC trees, corresponding to the various combinations of cyclic partitions. The isomorphic relation between the ICS-1 and ICS-5 families extends to their TC chains as well.

In the effort to establish icv similarity relations among the scs of the ICS families, the chapter contends that Forte's R_n relations by themselves are inadequate for the task, since they do not express similarity relations of a symmetrical nature, nor do they distinguish relative degrees of similarity. The study therefore introduces the R_{SYM} relation, which recognizes similarity between scs of the same cardinality based on symmetrical correspondence between pairs of variant icv entries. The study also employs Isaacson's IcVSIM to generate quantifiable measures of relative similarity between sc pairs in the

R_{SYM} relation and in Forte's R_n relations. Through the addition of the R_{SYM} relation most of the scs share a similarity relation with at least one other sc within the same ICS families. Further, many of the scs are also R_{SYM}-related or R_n-related to scs in different ICS families. Moreover, all the isomorphic sc pairs in the ICS-1 and ICS-5 families share the R_1 relation.

Finally, the distinct scs within each ICS family are shown to be equivalent in a smaller number of distinct scs in other modular universes. All the R_1-related scs between the ICS-1 and ICS-5 families map into the same scs in the other modular universes. Further, in the remaining ICS families, those mappings that correspond to the partition categories of the scs constitute a "best fit" in other modular universes.

Thus, through its investigation of the cyclic sets, this study demonstrates that the symmetrical nature of the interval cycles directly influences the structural properties and relationships of cycle-based formations.

Chapter Four

Analysis of Etude No.1 and Etude No.4 from *Six Etudes for Piano* by George Perle

This chapter returns to the context of twelve-tone tonality in its analytical applications of Perle's theoretical concepts to the first and fourth etudes from Perle's *Six Etudes for Piano*. Perle composed the *Six Etudes for Piano* between 1973 and 1976. The two etudes to be analysed here both adhere to a tripartite formal organization, but are contrasting in nature. The first etude has an improvisatory character, and is described by pianist Michael Boroskin as featuring "extremely rapid, staccato, pianissimo chords alternating between the hands, which scamper all over the keyboard" (1987, 14). A more introspective quality imbues the fourth etude, which demands intricate pedalling and dynamic sensitivity. This etude poses a particular challenge to the performer's rhythmic skills, in its shifting tempi, specific rubato indications, duple versus triple divisions of the beat, and metric modulation.

The two etudes are also contrasting in texture. The texture of Etude No.1 consists mostly of successive vertical dyads, with a few linear gestures interspersed. Etude No.4 evinces a more contrapuntal texture. Because so much of the music in the fourth etude unfolds linearly, it is useful to discuss its texture in terms of its component voices. Perle varies the texture through the systematic addition and subtraction of voices and through the combination of voices in alternating and interwoven pairs.

This chapter begins by discussing large-scale formal organization and abstract relationships in each etude, and then proceeds to explore compositional strategies in the succession of axis-dyad chords and sum tetrachords at the local level. The chapter continues by demonstrating the realization of these array segments at the musical surface, and by examining the results of other compositional techniques employed by Perle. These analyses will show how the principles of twelve-tone tonality serve not only as precompositional resources, but may also directly influence the compositional process. The chapter then offers additional insights into the etudes from the perspective of pc set theory, as codified by Allen Forte (1973), and with extensions to the theory developed by Robert Morris (1995b). These insights corroborate those gained from the context of twelve-tone tonality, and reveal other aspects of organization that would otherwise remain hidden. Thus, pc set theory contributes another facet to the analyst's understanding of the etudes.

The annotated scores

Appendices three and four contain annotated scores of Etude No.1 and Etude No.4. The annotations above the staves identify the formal units of the etudes, to be discussed below. The annotations below each staff represent the abstract dimension of the music: the array labels and segments. The concrete dimension is that of the music itself, in which the array segments are realized. A right-angle bracket identifies each array. In appendix four, for example, Etude No.4 begins with array p0i4/i9pe in m.1, and modulates to pti2/i7p9 in m.9. The series of integers following each array label are segments of the array's cyclic sets.[1] Extra spaces between pc integers define the limits of the array segments. Axis-dyad chords appear as hexachords divided into three-integer cyclic set segments.[2] In appendix four, m.1 of Etude No.4 comprises three axis-dyad chords. The upper cyclic set segments of each axis-dyad chord confirm the cyclic interval of 4, and the tonic sums of 0 and 4, while the lower cyclic set segments confirm the cyclic interval of 2, and the tonic sums of 9 and e.[3] These axis-dyad chords are derived from a difference alignment, as indicated by the vertically aligned italicized integers. In the passage from m.83 to m.89, however, the axis-dyad chords are derived from a sum alignment.[4] Array

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[1] The cyclic sets for all cyclic intervals (mod 12) are listed in appendix one.

[2] As noted in chapter two, Perle recognizes each pc occurrence as an independent member of an array segment, rather than as multiple instances of a single pc. Hence hexachordal, tetrachordal, and trichordal array segments may contain less than six, four, or three distinct pcs, respectively.

[3] Please see chapter two for a detailed explanation of how the axis-dyad chord reveals cyclic intervals and tonic sums.

[4] The analyst may choose to present the axis-dyad chords in either alignment. Perle states: "It is often

segments of variable size also appear, such as the pair of sum tetrachords at the end of m.2, and the octachordal segment in m.7, which may be considered as an extended axis-dyad chord. Occasionally only one cyclic set is active at the musical surface. Integer notation represents the active cyclic set in such instances, while dashes represent the inactive cyclic set. This notation occurs at the first beat of m.5, where cyclic set p0i4 is temporarily absent from the musical surface.

The following review of some of the applications of the principles of twelve-tone tonality will help clarify the annotations in appendices three and four, and the analyses in the rest of this chapter. The cyclic sets of an array serve as a precompositional resource free of any functional hierarchies. In constructing a cyclic set from an interval cycle which does not exhaust the aggregate, the composer may combine either the same or different partitions of the cyclic interval. The composer may then combine cyclic sets of the same or different cyclic intervals in the formation of the array. In the process of composition the composer may draw cyclic set segments from any part of the array, and move freely forward and backward in the array. The composer may modulate to any other array at any time, since there is no obligation to use the whole array.[5]

Modulation may occur either directly or through a pivot axis-dyad chord. Both types of modulation occur in Etude No.1 (appendix three). The notation at the beginning of m.2 shows two right-angle brackets joined together, indicating a modulation by a pivot axis-dyad chord.[6] A direct modulation occurs on beat 3 of the same measure, indicated by the single right-angle bracket around the new array.

In the annotated scores, the brackets underneath the bass staff indicate the segment of the musical surface that corresponds with the integers of the array segment. Although the abstract array segments themselves are ordered in terms of cyclic intervals and tonic

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possible to choose among several plausible readings in tracing a compositional statement to its pitch-class source in an array. Any axis-dyad chord may be derived from either a sum or a difference alignment" (1977b, 59).

[5] The modulatory procedures at the composer's disposal are discussed in chapter two.

[6] The pivot axis-dyad chord typically denotes a reinterpretative modulation (as discussed in chapter two).

sums, the segments are realized at the musical surface as an *unordered* collection of pitches. In most cases, each integer in the array segment represents a single pitch at the musical surface. Occasionally, however, duplicated integers may represent a single instance of a pitch. Conversely, a single integer may represent multiple realizations of a pc. In addition, a pc may be implied by the array segment although not actually present in the musical surface; the integer notation represents this in brackets. Thus the two-voice texture suggested by the abstract array may be realized in an infinite number of concrete musical settings.

A perusal of the brackets in the annotated scores shows that while the array segments generally correspond with the beat value in Etude No.4, they correspond mostly to groups of two and three pairs of dyads in Etude No.1. As well, array segments in both etudes frequently correspond with notational clues, such as phrasing slurs, beams, dynamics, and tempo changes, and in Etude No.4, with pedal markings. The remainder of this chapter will attempt to suggest principles that direct the compositional process in the context of twelve-tone tonality.

Analysis of Etude No. 1 from the perspective of twelve-tone tonality

Large-scale formal organization

Etude No.1 exhibits a three-part form, with its main sections labelled here as A (mm.1-10), B (mm.11-19), and C (mm.20-35). The content of the first two measures helps to articulate the form of the etude. This material returns at the beginning of the B section, (mm.11-13), although shifted metrically. It also returns at the beginning of the C-section (mm.20-23), here expanded in length by the addition of rests. As well, the B- and C-sections tranpose the pitch content of these measures by T_{P+2} and T_{P-2}, respectively.[7] The B-section continues with additional material drawn from the A-section, albeit in a truncated, modified form, whereas the C-section continues with new material. The last two measures of the etude are a reprise of the opening two measures of the A-section at its original pitch level.

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[7] This type of notation was introduced in chapter two (please see n.12).

Table 4.1 lists the etude's arrays and associated parameters in chronological order.[8] The table accounts for virtually all of the pitch material in the etude, with the exception of a passage that commences at the anacrusis to the third beat in m.9, and extends to the second beat of m.10.[9] The final column in table 4.1 lists the types of modulation between successive arrays. Modulation typically occurs either through two discrete axis-dyad chords or through a single axis-dyad chord that functions as a pivot between the two arrays. The reinterpretative modulations that utilize pivot axis-dyad chords occur between the first pair of arrays in each section, in m.2 (between the arrays of rows a-b), in m.12 (between the arrays of rows k-l), and in m.21 (between the arrays of rows s-t), as well as between the third- and second-last pairs of the etude's arrays, in mm.34 and 35 (between the arrays of rows y-z and rows z-aa, respectively). The reinterpretative modulation in m.5, however, occurs over two successive axis-dyad chords rather than through a pivot axis-dyad chord. The substitutional modulation in m.29 (between the arrays of rows w-x) involves an axis-dyad substitution between two overlapping axis-dyad chords.

Etude No.1 also employs a hybrid form of modulation that combines aspects of reinterpretative and substitutional modulation. In the first modulation of m.19 (between the arrays of rows p-q), the pcs of a sum tetrachord are reordered, as illustrated in the squares in figure 4.1a, while the remaining neighbour-dyad is replaced with another of the same interval. Another hybrid modulation may also be said to generate the new array of m.31 (row y), in that the outer pcs of the lower cyclic set segment are replaced with two others that form the same interval, as shown in figure 4.1b. Conversely, the etude also employs a direct form of modulation between two arrays which does not exhibit any of the characteristics of substitutional or reinterpretative modulation and does not utilize a pivot axis-dyad chord. Some examples of direct modulation are found in mm.3, 6, and 7.

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[8] In the top row of table 4.1 the abbreviations *IS*, *SM*, *SK*, *agg. sum*, and *ton* represent the parameters of interval system, master array of the synoptic mode, master array of the synoptic key, aggregate sum, and tonality, respectively.

[9] The pitches unfold crossing interval 9 and interval 1 cycles, and thus lie outside the domain of the preceding array of i7i8/p8p5.

Table 4.1. Arrays and associated parameters, Etude No.1

	array	bar	means of progression from preceding array	IS	SM	SK	mode	key	agg. sum	ton	modulation from preceding array
	A-section:										
a.	i1i2/i3i0	1	-	1,9	4	2	t,2	1,5	6	2	-
b.	i1i2/p0p9	2	invariant cyclic set	1,9	4	2	1,5	t,2	0	0	reinterpretation
c.	i9it/p8p5	2	transposition	1,9	4	2	1,5	2,6	8	0	direct
d.	ptpe/iei8	3	invariant tonic sums	1,9	4	2	e,3	6,t	4	0	direct
e.	ptpe/ptp7	5	invariant cyclic set	1,9	4	2	0,4	5,9	2	2	direct
f.	i9it/ptp7	5	invariant cyclic set	1,9	4	2	e,3	4,8	0	0	reinterpretation
g.	p8p9/ptp7	6	invariant cyclic set	1,9	4	2	t,2	3,7	t	2	direct
h.	p8p9/i9i6	7	invariant cyclic set	1,9	4	2	e,3	2,6	8	0	direct
i.	i7i8/p8p5	9	transposition	1,9	4	2	e,3	0,4	4	0	direct
j.	i5i6/i7i4	10	interval cycles	1,9	4	2	t,2	9,1	t	2	direct
	B-section:										
k.	i5i6/i7i4	11	-	1,9	4	2	t,2	9,1	t	2	-
l.	i5i6/p4p1	12	invariant cyclic set	1,9	4	2	1,5	6,t	4	0	reinterpretation
m.	i1i2/p0p9	13	transposition	1,9	4	2	1,5	t,2	0	0	direct
n.	p2p3/i3i0	13	invariant tonic sums	1,9	4	2	e,3	2,6	8	0	direct
o.	p0p1/i3i0	13	invariant cyclic set	1,9	4	2	9,1	0,4	4	0	direct
p.	p0p1/i1it	15	invariant cyclic set	1,9	4	2	e,3	t,2	0	0	direct
q.	ptpe/i1it	19	invariant cyclic set	1,9	4	2	9,1	8,0	8	0	hybrid
r.	i9it/iei8	19	invariant tonic sums	1,9	4	2	t,2	5,9	2	2	direct
	C-section:										
s.	i9it/iei8	20	-	1,9	4	2	t,2	5,9	2	2	-
t.	i9it/p8p5	21	invariant cyclic set	1,9	4	2	1,5	2,6	8	0	reinterpretation
u.	i7i8/p6p3	22	transposition	1,9	4	2	1,5	t,2	0	0	direct
v.	p8p9/i9i6	23	invariant tonic sums	1,9	4	2	e,3	2,6	8	0	direct
w.	i5i6/p8p5	25	invariant tonic sums	1,9	4	2	9,1	t,2	0	0	direct
x.	p2p3/i5i2	29	transposition	1,9	4	2	9,1	4,8	0	0	substitution
y.	p2p3/i3i0	31	invariant cyclic set	1,9	4	2	e,3	2,6	8	0	hybrid
z.	i1i2/i3i0	34	invariant cyclic set	1,9	4	2	t,2	1,5	6	2	reinterpretation
aa.	i1i2/p0p9	35	invariant cyclic set	1,9	4	2	1,5	t,2	0	0	reinterpretation
bb	i1i2/i3i0	35	invariant cyclic set	1,9	4	2	t,2	1,5	6	2	direct

Figure 4.1. Hybrid modulation between axis-dyad chords in mm.18-19 (a) and m.31 (b)

(a)	row p:	p0p1:	6	6	7	row q:	ptpe:	3	7	4
		i1it:	6	7	3		i1it:	7	6	4
(b)	row x:	p2p3:	7	7	8	row y:	p2p3:	7	7	8
		i5i2:	t	7	7		i3i0:	8	7	5

Despite the varied types of modulation, all the arrays of the etude belong to the single interval system of 1,9. This constancy of interval system has a direct impact on the more abstract parameters of synoptic mode and synoptic key. They too remain constant, since they reflect the relationship between the arrays' cyclic intervals. This corresponding consistency of interval system and synoptic arrays emphasizes the link between the surface and abstract levels of pc organization in twelve-tone tonality.

Conversely, the aggregate sums change with each modulation to a new array (except that of m.29), with corresponding fluctuations of tonality, mode, and key. All three sections begin and end in tonality 2; while the B-section's remaining arrays stay in tonality 0, the A- and C-section's remaining arrays vacillate between tonality 0 and 2.[10] Table 4.1 also shows considerable diversity among the modes and keys; nonetheless, a consistent relationship underlies these parameters. Although the etude utilizes five transpositionally related modes and ten transpositionally related keys, the component elements within each of the modes and keys show a difference of 4. This consistent difference is related to the fact that the various modulations preserve the cyclic intervals while changing the tonic sums. Since the cyclic intervals remain constant, so too do the intervallic relationships between the tonic sums within the arrays' cyclic sets. Hence the various parameters of an array are closely related, and changes made in any one parameter affect other parameters.

[10] All the arrays with aggregate sums of 0, 4, and 8 fall into tonality 0, while those of aggregate sums 2, 6, and t fall into tonality 2.

A specific pattern governs the progression among the first four arrays in each of the three sections, as is apparent in the fourth column of table 4.1. Motion between the first pair of arrays in each section (rows a-b, k-l, and s-t) involves one invariant cyclic set and one transposed cyclic set. Motion between the second pair involves a transposition of all four tonic sums. Motion between the third pair transposes two symmetrically positioned tonic sums and holds the other two symmetrically positioned tonic sums invariant. The remaining arrays in each section also utilize these means of progression, but not in the same order.

Some transpositional relationships exist between non-adjacent arrays as well. The arrays in rows d and f in the A-section, for example, display a T_{TSe} relationship. This transpositional relationship results from a two-stage process. Between rows d and e, the tonic sums of the left cyclic set are invariant while those of the right cyclic set are T_{TSe}-related. The situation is reversed between the arrays of rows e and f, which comprise invariant right cyclic sets and T_{TSe}-related left cyclic sets. In the succession of these three arrays, the middle array functions as a link between the two tranpositionally-related outer arrays. This two-stage process effects a smooth transition between two different arrays.[11]

The progression of arrays throughout Etude No.1 appears to be guided more by the systematic combinations of transposition and invariance than by a specific modulatory plan. As will be discussed below, the opposite situation obtains in the organization of Etude No.4.

Coherence in successive axis-dyad chords

Within an array, Perle moves from one axis-dyad chord to another by selecting trichordal segments from the component cyclic sets. Within a cyclic set segment, the neighbour notes and the axis note will always transpose by complementary values, since they derive from inversionally related interval cycles. In the first two axis-dyad chords of figure 4.2a, the neighbour notes in both cyclic set segments transpose by +3 while the

[11] The two-stage transpositional process is repeated between the arrays of rows f to h, as well as in the B-section, between the arrays of rows n-p and o-q (although the transposed tonic sums in this section are T_{TSt}-related).

axis notes transpose by -3. In this dissertation the symbol ± is used to denote the complementary transpositional values; hence the motion between the two axis-dyad chords is symbolized as ±3.[12] An axis-dyad chord's cyclic set segments do not have to be transposed by the same value, as illustrated in the succession of five axis-dyad chords from mm.5-6 in figure 4.2a. The second axis-dyad chord progresses to the third through a ±7/±8 transposition, while the third axis-dyad chord progresses to the fourth, and the fourth to the fifth, through a ±8/±7 transposition.

Figure 4.2. Complementary transpositional values between cyclic set trichordal segments in successive axis-dyad chords, mm.5-6 (a), and.mm.13-15 (b), Etude No.1

(a)	i9it:	3	6	4	±3	6	3	7	±7	1	8	2	±8	9	0	t	±8	5	4	6
	ptp7:	7	3	4	±3	t	0	7	±8	6	4	3	±7	1	9	t	±7	8	2	5
(b)	p0p1:	7	5	8	±9	4	8	5	±5	9	3	t	±4	1	e	2	±4	5	7	6
	i3i0:	7	8	4	±9	4	e	1	±4	8	7	5	±5	1	2	t	±5	6	9	3

Perle manipulates these alternating complementary transpositional values in the B-section. As noted above, the arrays in the opening measures of the B-section (mm.11-12) are T_{TS4} transpositions of those in mm.1-2, while the axis-dyad chords are T_2 transpositions of their counterparts in mm.1-2. The transpositional plan changes at the third beat of m.13, however (which corresponds to the first beat of m.3). Figure 4.3 illustrates how the upper and lower cyclic set segments in m.13's third and fourth axis-dyad chords exchange their complementary transpositional values relative to the corresponding axis-dyad chords in m.3. The B-section omits the subsequent melodic gesture of mm.3-4 and the following six vertical dyads of mm.4-5. In the ensuing succession of axis-dyad chords (from the end of m.13 to the beginning of m.15), the upper and lower cyclic set segments once again correspond to the five axis-dyad chords of the A-section (from the second axis-dyad chord of m.5 to the third of m.6). Yet, as

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[12] In this dissertation the symbol ± always denotes a *specific pattern* of transposition of alternating cyclic set segment elements by complementary values, rather than a free choice of + or - tranpositional values.

Figure 4.3. Exchange of upper and lower complementary transpositional values between corresponding pairs of axis-dyad chords, m.3 (a) and m.13 (b), Etude No.1

(a)	ptpe:	t 0 e	±3	1 9 2		
	iei8:	0 e 9	±8	8 3 5		
(b)	p2p3:	0 2 1	±8	8 6 9		
	i3i0:	2 1 e	±3	5 t 2		

seen in figure 4.2b above, Perle replaces the complementary transpositional values of the five axis-dyad chords in mm.5-6 with their inversionally related values in mm.13-15.

Perle frequently employs another technique in successions of axis-dyad chords, in which he transposes one cyclic set segment while holding the other invariant. This typically results in consistent patterns of secondary differences or sums. A striking example of this result occurs in the C-section of Etude No.1. Figure 4.4a shows the set of alternating secondary difference patterns that emerges in the succession of five axis-dyad chords beginning in m.27, with alternating trichordal segments held invariant between successive axis-dyad chords. In this progression, the non-invariant trichordal segments progress at $T_{\pm t}$. The final axis-dyad chord at m.29 leads into the new array in figure 4.4b through substitutional modulation (compare the last axis-dyad chord of figure 4.4a and the first of figure 4.4b). The axis-dyad chords of the new array again alternate pairs of invariant cyclic set segments and maintain the same sets of secondary difference patterns.

Figure 4.4. Secondary difference patterns and invariant trichordal segments in successive axis-dyad chords in a difference alignment, mm.27-31, Etude No.1

(a) m.27 to first chord of m.29:

i5i6:	*5 0 6*	→	*5 0 6*		*3 2 4*	→	*3 2 4*	*1 4 2*
p8p5:	*8 0 5*		*6 2 3*	→	*6 2 3*		*4 4 1*	→ *4 4 1*
diff:	3 0 e		1 t 9		3 0 e		1 t 9	3 0 e

(b) second chord of m.29 to penultimate chord of m.31:

p2p3:	*1 1 2*	→	*1 1 2*		*e 3 0*	→	*e 3 0*		*9 5 t*	→	*9 5 t*	*7 7 8*
i5i2:	*4 1 1*		*2 3 e*	→	*2 3 e*		*0 5 9*	→	*0 5 9*		*t 7 7*	→ *t 7 7*
diff:	3 0 e		1 t 9		3 0 e		1 t 9		3 0 e		1 t 9	3 0 e

Reinterpreting this *same* succession of axis-dyad chords in a *sum alignment* reveals a chain of axis dyad sums increasing by two, simultaneous with two chains of neighbour dyad sums decreasing by two in the cyclic chords, as illustrated in figure 4.5a.[13] At the modulation in m.29, the three chains begin again, completing their cycles of sums before the next modulation in m.31 (figure 4.5b).

Figure 4.5. Three chains of vertical dyad sum patterns and invariant trichordal segments in successive axis-dyad chords in a sum alignment, mm.27-31, Etude No.1

(a) m.27 to first chord of m.29:

i5i6:	*5* 0 *6*	→ *5* 0 *6*	*3* 2 *4*	→ *3* 2 *4*	*1* 4 *2*
p8p5:	5 *0* 8	3 *2* 6	→ 3 2 6	1 *4* 4	→ 1 *4* 4
sum:	t - -	8 - -	6 - -	4 - -	2 - -
sum:	- 0 -	- 2 -	- 4 -	- 6 -	- 8 -
sum:	- - 2	- - 0	- - t	- - 8	- - 6

(b) second chord of m.29 to penultimate chord of m.31:

p2p3:	*1* 1 *2*	→ *1* 1 *2*	*e* 3 *0*	→ *e* 3 *0*	*9* 5 *t*	→ *9* 5 *t*	*7* 7 *8*
i5i2:	1 *1* 4	e *3* 2	→ e *3* 2	9 *5* 0	→ 9 *5* 0	7 *7* t	→ 7 *7* t
sum:	2 - -	0 - -	t - -	8 - -	6 - -	4 - -	2 - -
sum:	- 2 -	- 4 -	- 6 -	- 8 -	- t -	- 0 -	- 2 -
sum:	- - 6	- - 4	- - 2	- - 0	- - t	- - 8	- - 6

Thus, through various combinations of transposed and invariant cyclic set segments, in either difference or sum alignments, Perle is able to create a cohesive succession of axis-dyad chords in the abstract dimension. The next section discusses how these axis-dyad chords may be realized at the musical surface.

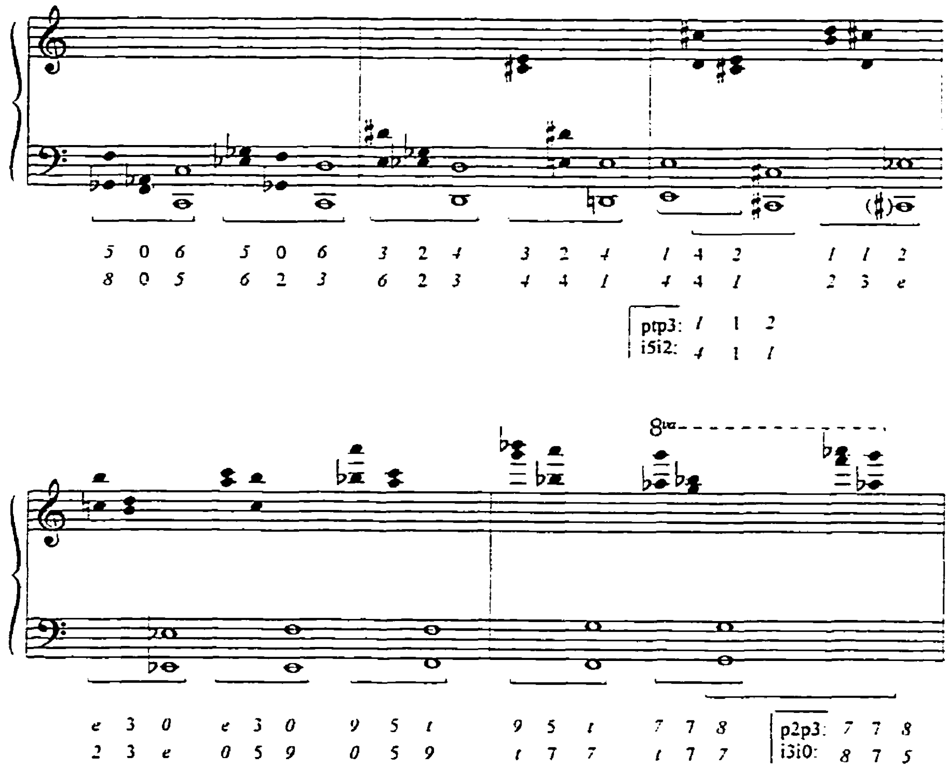
Realization of array segments

As stated above, array segments may be realized in any number of ways at the concrete musical surface. The score notation may reflect explicitly the organization of the array segments, as in the passage from the preceding discussion (figure 4.4a). In example 4.1, the notation and ordering of the pitches at the musical surface in mm.27-31 correspond with the structure of each axis-dyad chord in its difference alignment. Beginning in m.27,

[13] Sum and difference alignments are discussed in chapter two under the subheading "Secondary differences and sums."

the left hand's single sixteenth-note dyads of intervals 12 and 14 contain the axis-dyad notes of each axis-dyad chord, represented by open noteheads in example 4.1.[14] The closed noteheads in the example represent the collections of beamed sixteenth notes played by the right hand, which comprise the neighbour notes of each axis-dyad chord. The upper and lower pairs of pitches alternate the cyclic intervals of i5i6 and p8p5.

Example 4.1. Correspondence between axis-dyad chord structure and notation in mm.27-31, Etude No.1



[14] There is one exception: the first note played by the left hand in m.27 actually belongs to the axis-dyad chord's neighbour dyad.

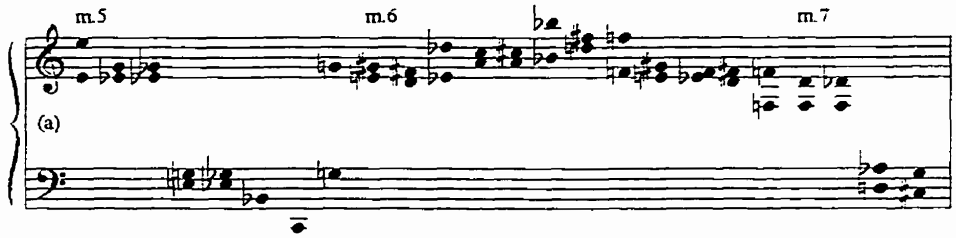
The relationship between the axis-dyad chords in mm.5-6 and mm.13-15 was described above as the replacement of the complementary transpositional values of the axis-dyad chords in mm.5-6 by their inversionally related values in mm.13-15. At the musical surface these passages comprise successions of vertical dyads. The inversional relationship is manifested as an exchange of upper and lower dyadic elements in mm.13-15, relative to those in mm.5-7.[15] This exchange occurs at the penultimate dyad of m.13, in correspondence with the fifth dyad of m.5, as illustrated in example 4.2. Further, Perle maintains the linear ic successions between the dyadic elements but inverts the melodic direction. Consequently, each pc in the B-section passage is inversionally related to its counterpart in the A-section (specifically, by T_eI). In the integer notation of this passage in figure 4.6, one corresponding pair of upper and lower dyadic elements between the two passages appears in boldface to facilitate comparison.

Example 4.3 shows m.25 of the C-section, in which a series of vertical dyads forms a transitional passage from array p8p9/i9i6 (of mm.23-24) to the ensuing array i5i6/p8p5 (at the end of m.25). Although array p8p9/i9i6 is still in effect in m.25, the passage exploits the tonic sums of both arrays, as illustrated in figure 4.7 (a and b). The four dyads in the second half of the measure are tritone transpositions of those in the first half.[16] In figure 4.7a, the tonic sums of 8 and 6 form a palindrome in the two halves in the *linear* dyads (as the surface realization of the sum tetrachord array segments). At the same time, the tonic sums of 9 and 5 occur in alternation in the *vertical* dyads' sums (figure 4.7b). Moreover, the disposition of the pcs themselves creates a formation of nested palindromes of dyadic sums (figure 4.7c). These formations are generated by the tritone transposition of each dyadic element. Although the transpositional operation results in new pcs within each dyad, it also preserves each dyadic sum. Thus Perle may create formations that retain the same sums but which introduce new pc material.

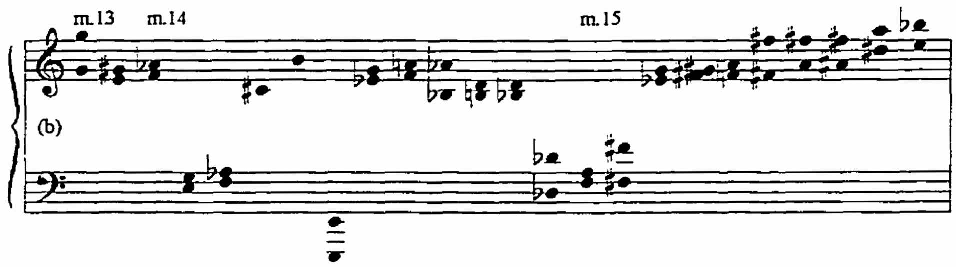
[15] While mm.13-15 correspond to mm.5-6 in the abstract dimension of axis-dyad chord, the correspondence expands to include the first four dyads of m.7 at the surface.

[16] The final dyad in the measure is a member of the first axis-dyad chord in the new array, and as such is not included in the transitional passage.

Example 4.2. Corresponding passages in mm.5-7 of A-section (a) and mm.13-15 of B-section (b), Etude No.1



upper pcs:	4	7	6	7	6	-	-	7	8	6	1	0	1	t	6	5	8	5	6	5	2	1	8	7
lower pcs:	4	3	3	4	3	t	0	7	4	2	3	9	9	t	2	5	4	3	2	5	5	5	2	1



upper pcs:	7	8	8	7	8	1	e	4	7	9	8	2	2	1	9	6	7	8	9	6	6	6	9	t
lower pcs:	7	4	5	4	5	-	-	4	3	5	t	e	t	1	5	6	3	6	5	6	9	t	3	4

Figure 4.6. Inversion of pcs and contour, and exchange of upper and lower dyadic elements, in mm.5-7 (a) and mm.13-15 (b), Etude No.1

ics:		3	1	1	1		1		1	2	5	1	1	3	4	1	3	3	1	1	3	1	5	1
intervals:		-9	-1	-e	-1		+13		+1	-2	+7	-1	+1	+9	-4	-1	-9	-3	+1	-1	-3	-1	-5	-1
upper pcs:	**4**	**7**	**6**	**7**	**6**	**-**	**-**	**7**	**8**	**6**	**1**	**0**	**1**	**t**	**6**	**5**	**8**	**5**	**6**	**5**	**2**	**1**	**8**	**7**
lower pcs:	4	3	3	4	3	t	0	7	4	2	3	9	9	t	2	5	4	3	2	5	5	5	2	1
intervals:		-1	0	-e	-1	-5	-t	+19	+9	-2	+1	+6	0	+1	+4	-9	-1	-1	-1	-9	0	0	-3	-1
ics:		1	0	1	1	5	2	5	3	2	1	6	0	1	4	3	1	1	1	3	0	0	3	1

ics:		1	0	1	1	5	2	5	3	2	1	6	0	1	4	3	1	1	1	3	0	0	3	1
intervals:		-e	0	-13	+1	+5	+t	-31	+27	+2	-1	-6	0	-1	-4	+9	+1	+1	+1	+9	0	0	+3	+1
upper pcs:	7	8	8	7	8	1	e	4	7	9	8	2	2	1	9	6	7	8	9	6	6	6	9	t
lower pcs:	**7**	**4**	**5**	**4**	**5**	**-**	**-**	**4**	**3**	**5**	**t**	**e**	**t**	**1**	**5**	**6**	**3**	**6**	**5**	**6**	**9**	**t**	**3**	**4**
intervals:		-3	+1	-13	+1		-25		+35	+2	-7	+1	-1	-9	+4	+1	+9	+3	-1	+1	+3	+1	+5	+1
ics:		3	1	1	1		1		1	2	5	1	1	3	4	1	3	3	1	1	3	1	5	1

Example 4.3. Transitional passage of sum tetrachords, m.25, Etude No.1

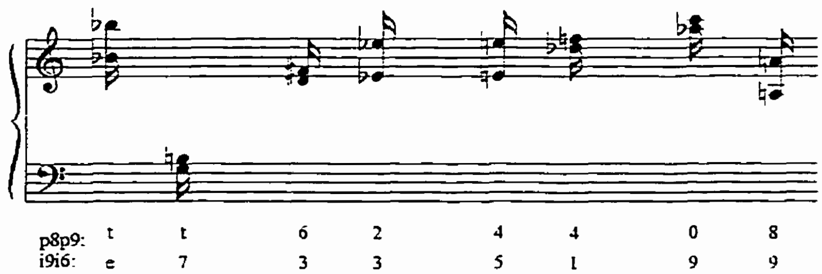
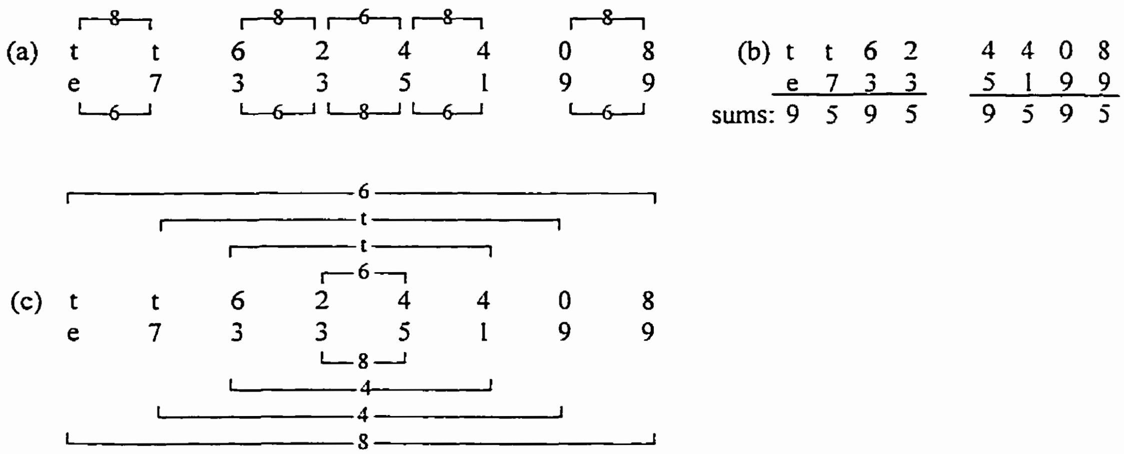



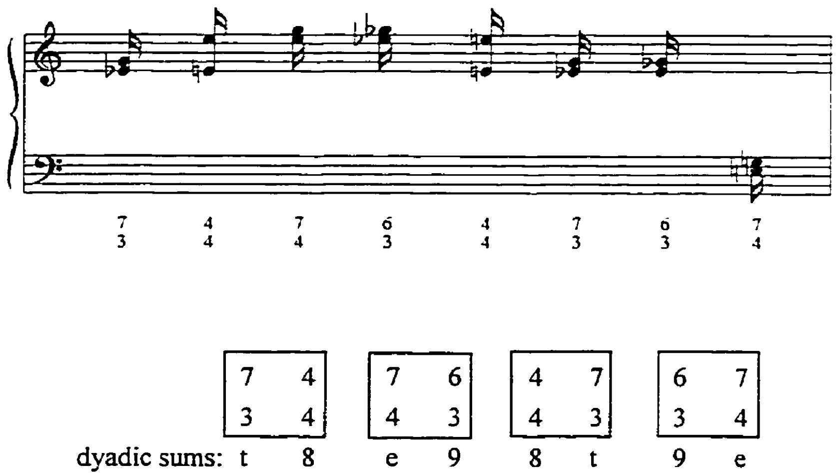
Figure 4.7. Tonic sum duplication in vertical and linear dyads, m.25, Etude No.1



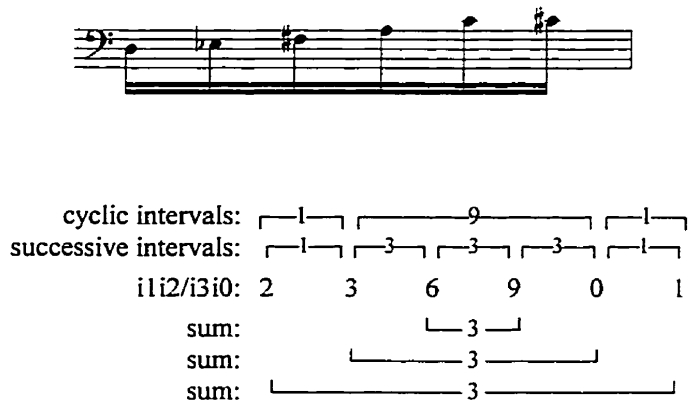
Rotation of vertical dyads in m.5 results in another symmetrical formation, as illustrated in example 4.4. The first eight dyads in this measure divide into two halves each consisting of four discrete tetrachords. The dyads in each tetrachord in the first half simply switch places in the corresponding tetrachords in the second half. This rotation reverses the ordering of the dyadic sums in corresponding tetrachords.

Every melodic gesture in Etude No.1 embeds a specific formation of dyadic sums. Example 4.5 shows how in the opening gesture of m.1, the first six pitches unfold in a symmetrical succession of intervals <1-3-3-3-1>. In addition, they present the array's cyclic intervals 1 and 9, and the cyclic set sums 3 and 3 (i1+i2 and i3+i0), also in symmetrical arrangements.

Example 4.4. Dyadic rotation in alternating tetrachords, m.5, Etude No.1



Example 4.5. Linear gesture, m.1, Etude No.1



Example 4.6 gives the melodic gesture spanning mm.3-4. Here the discrete dyads form an alternating pattern of sums e, t, and 9, which represent tonic sums culled from the current array ptpe/iei8 and the array i9it/ptp7, the goal of the two-stage transpositional process.[17] In addition, the sums formed by symmetrically positioned pairs of dyads creates a 7-t-7 palindromic pattern, adumbrating the tonic sums in the next array.

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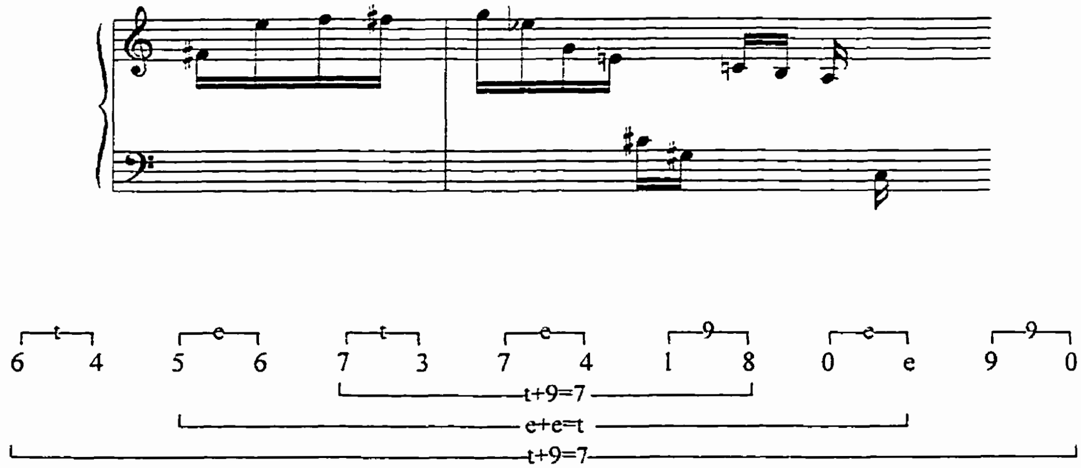
[17] This process was described earlier in this chapter, under the subheading "Large-scale formal organization."

Example 4.6. Dyadic sum patterns in melodic gesture of array ptpe/iei8, mm.3-4, Etude No.1



The third melodic gesture of the etude occurs in m.8 (example 4.7); its symmetrical design is shown in figure 4.8. The eight successive pcs form two tetrachords of sum 5 whose linear dyads are symmetrically positioned, and whose transpositional levels are complementary (figure 4.8a). Further, the pcs themselves are disposed so as to form nested dyads of sum 7 and sum t (figure 4.8b).

Finally, m.9 contains a melodic gesture of crossing interval cycles embedded in vertical dyads. Example 4.8 illustrates how, beginning at the eighth dyad of m.9, the two interval cycles on which the arrays of the etude are based unfold linearly in two textural voices. This becomes evident only at m.10. The cycles are embedded in the vertical dyads in m.9, and cross voices at the penultimate dyad of the measure. This is shown in example 4.8b, where the crossed cycles (closed noteheads) are realigned in the integer notation to elucidate their motion. These interval cycles lie outside the domain of the prevailing array i7i8/p8p5, yet they share a connection with the axis-dyad chords on either side (open noteheads). The immediately preceding axis-dyad chord (the third axis-dyad chord of m.9, given in example 4.8a) in its sum alignment produces secondary sums of exclusively odd integers. The ensuing series of vertical dyads containing the linear

Example 4.7. Melodic gesture, m.8, Etude No.1

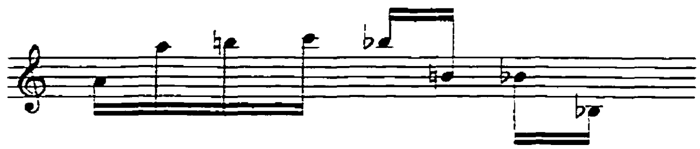
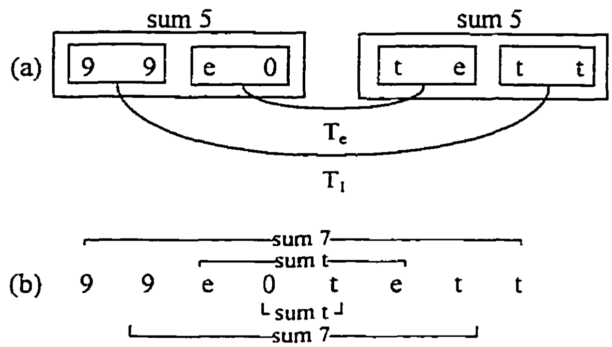


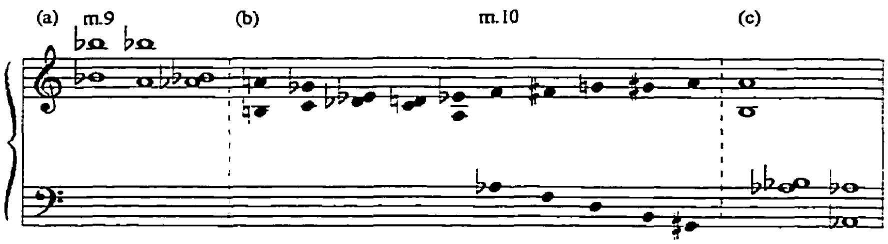
Figure 4.8. Dyadic and tetrachordal sum pairs in melodic gesture, m.8, Etude No.1



interval cycles produce five even sums followed by five odd sums (example 4.8b). The switch from even to odd sums coincides with the crossing of the interval cycles, after which the interval-9 cycle shifts to another of its partitions. The interval-1 cycle skips an element (pc 4), thereby changing the dyadic sum formed by the two cycles from even to odd integers. The cycles continue to their goal, an axis-dyad chord in the new array i5i6/i7i4. In the difference alignment of this axis-dyad chord the dyadic sums revert back to even integers (example 4.8c).

The preceding analysis has aimed to show that the principles of twelve-tone tonality function in the background as precompositional structures, but may also directly influence the foreground, in the realization of array segments at the musical surface. Thus the abstract dimension of twelve-tone tonality may intersect with the concrete dimension, or the two dimensions may maintain their independence, thereby providing the composer with rich resources and considerable freedom in the compositional process.

Example 4.8. Crossing interval-1 and interval-9 cycles, mm.9-10, Etude No.1



	(a) m.9			(b) m.9					m.10						(c) m.10		
i7i8:	9	t	t	9	6	3	0	9	5	6	7	8	9	i5i6:	9	8	t
p8p5:	8	9	e	e	0	1	2	3	8	5	2	e	8	i7i4:	e	8	8
sums:	5	7	9	8	6	4	2	0	1	e	9	7	5		8	4	6
	(axis-dyad chord)			(crossing interval-9 and interval-1 cycles)											(axis-dyad chord)		

Analysis of Etude No. 4 from the perspective of twelve-tone tonality

Large-scale formal organization

Etude No.4 also divides into three main sections, labelled in the notated score of appendix four as A, B, and A', each section occupying twenty-seven bars. Between each pair of sections lies a linking passage. Further emphasizing Perle's attention to symmetrical design, each of the main sections also divides into three subsections, as given in table 4.2.

Table 4.2. Tripartite formal organization, Etude No.4

A (mm.1-27)			link (mm.28-31)	B (mm.32-58)			link (mm.58-62)	A' (mm.63-89)		
a	b	c		d	e	f		a'	b'	c'

The A-section consists of three subsections, a, b, and c. Each of these subsections comprises a *subject* of two measures, followed by contrasting material, identified as an *episode*. This subject recurs throughout the etude, either in exact repetition, or in transposed, inverted, or otherwise modified form. Each unit under discussion will be identified by its formal description, followed by the main section and subsection to which it belongs (such as *subject A-a*). The three statements of the subject in the A-section appear in a transpositional pattern of T_{P0}, T_{P-1}, and T_{P-2}. As table 4.3 shows, the first and third statements repeat before leading into the episodic material.[18] The second statement does not repeat, however; this is the first instance of truncation, a device used throughout the etude. The material identified as subject A-c in mm.16-17 represents a modified statement of subject material.[19] An exact transposition of the subject resumes at T_{P-2} in mm.18-19, and leads to a repeat without modifications. In connecting the A- and B-section, the linking passage of mm.28-31 inverts the subject material and exchanges treble and bass content.

Table 4.3. Subsections of the A-section, Etude No.4

A-section					
a (mm.1-9)		b (mm.10-15)		c (mm.16-27)	
subject mm.1-2 T_{P0} — repeated subject mm.3-4	episode mm.5-9	subject mm.10-11 T_{P-1}	episode mm.12-15	extended subject mm.16-19 T_{P-2} — repeated subject mm.20-21	episode mm.22-27

[18] Only subject A-a's repeat is indicated by repeat signs; in every other instance the repeat is written out. Hence to conform to the symmetrical arrangement of proportions of the sections these initial repeating measures receive their own measure numbers.

[19] This modification represents the first instance of *mutation*, a technique whereby array segments or passages from the musical surface (or both) are modified by a combination of processes, such as transposition, truncation, reordering, and so on. Mutation is discussed in detail below, under the subheadings "Coherence in successive axis-dyad chords" and "Realization of array segments."

The B-section also consists of three subsections, labelled here as *phrase groups* B-d, B-e, and B-f, as seen in table 4.4. Each group contains three loosely related phrases of varying lengths, with none forming such tight relations as found between subject and episode or antecedent and consequent phrases. But as will be discussed below, the three phrase groups maintain symmetrical relationships within some of their twelve-tone tonal parameters, such as their interval systems, synoptic arrays, and aggregate sums.

Table 4.4. Subsections of the B-section, Etude No.4

B-section								
phrase group B-d (mm.32-38)			phrase group B-e (mm.39-47)			phrase group B-f (mm.47-58)		
phrase 1 mm.32-33	phrase 2 mm.34-36	phrase 3 mm.37-38	phrase 1 mm.39-40	phrase 2 mm.41-43	phrase 3 mm.44-47	phrase 1 mm.47-50	phrase 2 mm.51-55	phrase 3 mm.55-58

The phrase groups of the B-section do show some connections with the A-section, however. The third phrase of phrase group B-d (mm.37-38) recalls subject material from the A-section. As well, the first two measures in each of the three phrases in phrase group B-f (mm.47-48, 51-52, and 55-56) recall the inverted subject found in the linking passage between the A- and B-sections (mm.28-30). The linking passage between the B- and A'-sections also borrows the inverted subject of the A-section, and then diverges into a multi-layered gesture (mm.58-62).

The A'-section begins with a literal transposed restatement of the A-a and A-b subsections at T_{P2} and T_{P1}, as indicated in table 4.5. But the expected A'-c' subsection does not follow; rather, the subject and episode of the A'-b' subsection repeat in truncated, modified form. The delayed A'-c' subsection then follows with the subject at T_{P0} in m.83. Again the expected subsections do not appear (in this case, a repeat of subject A'-c' followed by episode A'-c'). Instead, the subject material is systematically modified and truncated until the final measure.

Table 4.5. Subsections of the A'-section, Etude No.4

A'-section						
a' (mm.63-71)		b' (mm.72-82)				c' (mm.83-89)
subject mm.63-64 T_{P2} — repeated subject mm.65-66	episode mm.67-71	subject mm.72-73 T_{P1}	episode mm.74-77	repeated subject mm.78-79	repeated episode mm.79-82	subject mm.83-84 T_{P0} — repeated subject mm.85-86 mm.87-88 m.89

Table 4.6 lists the arrays and associated parameters of Etude No.4 in chronological order.[20] The formal designs of Etudes Nos.1 and 4 are similar in their three-part organization, and by the fact that in both etudes the A- and B-sections conclude with arrays that open the next sections. The two etudes differ, however, in the overall means of progression among the arrays. In Etude No.1 the progression among arrays is achieved primarily through the transposition and invariance of tonic sums, whereas in Etude No.4 progression adheres to a systematic modulatory plan.

Table 4.6 highlights the stability within each parameter of the A-section.[21] Substitutional modulation governs motion between successive arrays, resulting in tranpositional relationships between their corresponding tonic sums. The interval system is invariant throughout the A-section (see column 5 of table 4.6), and so therefore are the master arrays of the synoptic mode and synoptic key (columns 6 and 7). The mode also

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[20] Although George Perle graciously shared with me the array labels for Etude No.4, the analytical assertions made in this study about the etude are entirely my own.

In table 4.6, the three phrase groups of the B-section are separated by horizontal lines. Because the array in row i ends phrase group B-d and begins phrase group B-e, it is listed twice in the table. In the same way, the array in row k ends phrase group B-e and begins phrase group B-f; hence row k is also listed twice. The duplicate listing shows more clearly the symmetrical relationships among the parameters of the arrays in the three phrase groups. These symmetrical relationships are discussed in detail below.

[21] Subject A-c actually begins in m.16 in the array listed in row c of table 4.6. But this array is merely transitional, delaying the onset of the subject statement's array until m.18, as listed in row d. This event is discussed in detail below.

Table 4.6. Arrays and associated parameters, Etude No.4

	array	bar	means of progression from preceding array	IS	SM	SK	mode	key	agg. sum	ton	modulation from preceding array
	A-section:										
a.	p0i4/i9pe	1	transposition	4,2	2	6	3,5	e,1	0	0	-
b.	pti2/i7p9	9	transposition	4,2	2	6	3,5	7,9	4	0	substitution
c.	iep3/p8it	15	transposition	4,2	2	6	3,5	9,e	8	0	substitution
d.	p8i0/i5p7	18	transposition	4,2	2	6	3,5	3,5	8	0	substitution
e.	p0i4/i9pe	24	transposition	4,2	2	6	3,5	e,1	0	0	substitution
f.	p8i0/i1p3	27	sum transposition	4,2	2	6	7,9	e,1	0	0	reinterpretation
	B-section:										
g.	p8i0/i1p3	32	-	4,2	2	6	7,9	e,1	0	0	-
h.	p2p1/p4i8	33	invariant tonic sums	e,4	5	3	t,5	t,5	3	1	reinterpretation
i.	pti2/i5p7	37	invariant tonic sum	4,2	2	6	5,7	5,7	0	0	reinterpretation
i.	pti2/i5p7	37	invariant tonic sum	4,2	2	6	5,7	5,7	0	0	reinterpretation
j.	iei2/i3p5	42	invariant tonic sums	3,2	1	5	8,9	4,5	9	1	reinterpretation
k.	pti2/i3p5	45	invariant tonic sums	4,2	2	6	7,9	3,5	8	0	reinterpretation
k.	pti2/i3p5	45	invariant tonic sums	4,2	2	6	7,9	3,5	8	0	reinterpretation
l.	i5pe/i5it	51	invariant tonic sums	6,5	1	1	0,1	3,4	7	1	reinterpretation
m.	p4p5/iep3	54	invariant tonic sums	1,4	3	5	5,2	7,4	e	1	reinterpretation
n.	p4i8/i1p3	61	invariant tonic sums	4,2	2	6	3,5	7,9	4	0	reinterpretation
	A'-section:										
o.	p4i8/i1p3	63	-	4,2	2	6	3,5	7,9	4	0	-
p.	p2i6/iep1	71	transposition	4,2	2	6	3,5	3,5	8	0	substitution
q.	i3p7/p0i2	77	transposition	4,2	2	6	3,5	5,7	0	0	substitution
r.	p0i4/i9pe	82	transposition	4,2	2	6	3,5	e,1	0	0	substitution
s.	p0p3/ptpe	86	invariant tonic sums	3,1	2	4	2,4	e,1	0	0	reinterpretation
t.	p0i2/iepe	88	invariant tonic sums	2,0	2	2	1,3	e,1	0	0	reinterpretation
u.	p0p1/p0pe	89	invariant tonic sums	1,e	2	0	0,2	e,1	0	0	reinterpretation

remains invariant throughout the A-section until the linking passage at m.27 (row f), wherein it shifts by T_{M4} as a result of the reinterpretative modulation (columns 8 and 12). This modulation also has a corresponding effect of $T_{TS\pm8}$ on the tonic sums of the array (column 2). Although the A-section utilizes four different keys, the interval between the component elements in the keys is preserved in each case. Finally, in the arrays of rows a-e, the aggregate sums change in increments of 4 (column 10); hence there is no change to the overall tonality.[22]

[22] Arrays whose aggregate sums differ by 4 have transpositionally related axes of symmetry composed of either two identical even or odd integers, or of an even and odd integer that differ by 1. Arrays with transpositionally related axes of symmetry belong to the same tonality. Please see chapter two for a detailed discussion of the three categories of tonality.

At first glance, the B-section shows considerable diversity in all parameters. A closer examination reveals that most of the fluctuations unfold in a palindromic fashion, with each phrase group containing a closed circle of relations. Phrase group B-d (table 4.6, rows g, h, and i) shows such a relationship in its interval systems, with corresponding relationships in the master arrays of the synoptic mode and synoptic key. Palindromic patterns also appear in the phrase group's parameters of aggregate sum and tonality. While the phrase group's modes, keys, and arrays do not display palindromic patterns, they do adhere to a quasi-systematic transpositional plan. Although the successive modes in rows g-h and h-i exhibit non-uniform levels of transposition between their corresponding component elements, the *outer* modes (rows g and i) show a T_{Mt} relationship between their corresponding component elements. Similarly, the successive keys in rows g-h and h-i also display non-uniform levels of transposition, although the outer keys show a relationship of T_{K6} between their corresponding elements.[23] Finally, while the successive arrays in phrase group B-d also show non-uniform levels of transposition between their corresponding tonic sums, the *outer* arrays in each phrase group exhibit a uniform transpositional relationship between the tonic sums of their corresponding cyclic sets. Specifically, between the first two arrays of phrase group B-d (rows g and h) the corresponding tonic sums are related by four different transpositional values: T_{TS6} and T_{TS1} in the left cyclic set tonic sums, and T_{TS3} and T_{TS5} in the right cyclic set tonic sums. But between the first and last arrays of the same phrase group (rows g and i), the corresponding tonic sum relationships are T_{TS2} in the left cyclic set and T_{TS4} in the right cyclic set.

The arrays of phrase group B-e display palindromic relationships within the parameters of interval system, master arrays of synoptic mode and synoptic key, and tonality (see table 4.6, rows i, j, and k). The aggregate sum of the array in row k does not return to a value of 0; still, its value of 8 still compels a reversion to tonality 0. A transpositional relationship obtains within the modes, keys, and tonic sums of this phrase

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[23] These procedures bear some resemblance to the two-stage transpositional process utilized in Etude No.1, described above in the analysis of Etude No.1 under the subheading "Large-scale formal organization."

group, similar to those in the previous phrase group: the component elements of the outer modes are related by T_{M2}, while those of the outer keys are related by T_{Kt} (rows i and k). In the outer arrays, the tonic sums of the left cyclic set are T_{TS0}-related, while those of the right cyclic set are T_{TSt}-related.

Phrase group B-f (rows k-m) and the subsequent linking passage (rows m-n) together establish quasi-palindromic patterns. The parameters of interval system, master arrays of synoptic mode and synoptic key, and tonality all display a departure from and return to their original values (compare rows k, l, m, and n). As in the preceding two phrase groups, transpositional relationships obtain between the first and last modes, keys, and tonic sums of the arrays of phrase group B-f. The *outer* modes and keys are transpositionally related by T_{M8} and T_{K4}, respectively (compare rows k and n). Between the outer arrays of the phrase group, the tonic sums of the left cyclic set are T_{TS6}-related, while those of the right cyclic set are T_{TSt}-related.

The A'-section marks a return to the A-section's stability in its interval system, master arrays of synoptic mode and synoptic key, mode, tonality, and progression by substitutional modulation (as shown in table 4.6, rows o to r). In the final subsection of the etude (subject A'-c', spanning mm.83-89) the arrays undergo a symmetrical modification of their tonic sums. While one pair of oppositely aligned tonic sums remains invariant the other pair of oppositely aligned sums is variable, and is systematically transposed by values of $T_{TS\pm 1}$ (see rows r, s, t, and u).[24] As a result, the key, aggregate sum, and tonality remain invariant.

Because the variable pairs of tonic sums are transposed by complementary values, the successive arrays display a symmetrical relationship between the corresponding variable tonic sums. Their complementary transpositions affect their arrays' interval systems by changing the size of the component cyclic intervals. In turn, the changes in the cyclic intervals cause corresponding changes to the mode and the master array of the synoptic key.

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[24] The oppositely aligned tonic sums include the first tonic sum in the left cyclic set and the second tonic sum in the right cyclic set as the invariant pair, and the second tonic sum in the left cyclic set and the first tonic sum in the right cyclic set as the systematically transposed or variable pair. Thus, in array p0i4/i9pe of row r, the invariant pair of tonic sums is p0 and pe, and the variable pair of tonic sums is i4 and i9.

As in Etude No.1, Perle's use of dynamics, rests, pedal, tempo, and meter, as well as his notational subtleties also underscore the formal organization of Etude No.4.[25] Each subsection of the etude begins with a dynamic marking that differs from that of the preceding subsection, thus defining each of the formal units.[26] As well, the three separate dynamic markings at the beginning of each statement of the subject underscore the subject's multi-voice texture.

Rests also define the sections and subsections of the etude. In the A-section, rests conclude each of the episodes, demarcating self-contained units of subject and episode. In the B-section, rests separate not only the three phrase groups, but also the three phrases within the phrase groups. The two linking passages between the main sections (mm.28-31 and mm.58-62) are preceded by rests. Finally, rests isolate most of the truncated passages in the A'-section.

The tempo changes also contribute to the formal organization of the etude, as illustrated in table 4.7. In the A-section fluctuations of tempo occur in the A-a and A-c subsections, corresponding with subject and episodic statements, while the A-b section proceeds without tempo changes. In the B-section, tempo changes mark the beginnings of the first two phrase groups and the last phrase of the second phrase group. The A'-section repeats the tempo changes of the A-section until m.79, at which point the A'-section diverges from the A-section. All of the tempo changes in the etude are related according to simple ratios, as listed in the final column of table 4.7.

Thus, through the systematic progression between arrays in each section in conjunction with the parameters of dynamics, rests, and tempo markings, Perle defines the sections and subsections of the etude, thereby generating a clearly structured formal design.

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[25] In addition, the dynamic markings, pedalling indications, and rests often coincide with discrete array segments.

[26] Contrasting dynamics also delineate the phrases within the phrase groups of the B-section.

Table 4.7. Tempo changes between successive formal units, Etude No.4

bar	formal unit	metronome marking	relationship to Tempo I
	A-section:		
1	subject A-a	♩ = 120	Tempo I
7	episode A-a, end of first half	♪ = 120	1:2
8	episode A-a, beginning of second half	♩ = 120	Tempo I
18	subject A-c, end of mutated repeat	♪ = 120	1:2
19	subject A-c, resumption of transposed subject	♩ = 80	2:3 (Tempo II)
20	subject A-c, repeat	♩ = 120	Tempo I
29	linking passage	♩ = 96	4:5
	B-section:		
32	phrase group B-d, beginning	♪. = 160, ♩ = 120	Tempo I
38	phrase group B-d, near end	♪. = 160	Tempo I
38	phrase group B-d, end	♪. = 120	3:4
39	phrase group B-e, beginning of first phrase	♪ = 160	2:3 (Tempo II)
43	phrase group B-e, end of second phrase	♪. = 160	Tempo I
55	phrase group B-f, end of second phrase	♩ = 120	Tempo I
62	linking passage, end	♩ = 80	2:3 (Tempo II)
	A'-section:		
63	subject A'-a'	♩ = 120	Tempo I
69	episode A'-a', end of first half	♪ = 120	1:2
70	episode A'-a', beginning of second half	♩ = 120	Tempo I
79	subject A'-b', end of mutated repeat	♩ = 80	2:3 (Tempo II)
80	episode A'-b', beginning of mutated repeat	♩ = 120	Tempo I
86	subject A'-c', beginning of mutated repeat	♩ = 90	3:4

Coherence in successive axis-dyad chords

Etude No.4 makes extensive use of a process I will call *mutation*, whereby a new gesture is formed from a systematically modified version of an earlier gesture or gestures. Mutation occurs throughout this etude in a variety of contexts. In the abstract dimension of array segments, mutation typically involves some combination of array segment reordering, neighbour- or axis-dyad replacement, multiple transposition levels, truncation, or extension.

The first instance of mutation in Etude No.4 occurs at the modified statement of subject A-c in m.16. The two axis-dyad chords in this measure represent a fusion of dyads from the corresponding axis-dyad chords of the two previous subject statements in mm.1 and 10. That is, the two axis-dyad chords of m.16 have the same axis dyads as m.1 and the same neighbour dyads as those in m.10. These dyads are enclosed in rectangles in figure 4.9.

Figure 4.9. Mutation of subject axis-dyad chords of mm.1 and 10 in m.16, Etude No.4

m.1:							m.10:							m.16:					
t	2	2		2	t	6	9	1	1		1	9	5	9	2	1	1	t	5
2	7	4		7	2	9	1	6	3		6	1	8	1	7	3	6	2	8

Mutation also occurs in the B-section, between phrase groups B-d and B-e. Figure 4.10 shows how the five axis-dyad chords of mm.39-40 derive from those of mm.32-33. The upper cyclic set segments of the five axis-dyad chords in mm.39-40 are related by T_1 to the corresponding cyclic set segments in mm.32-33, while the lower cyclic set segments are similarly related by T_2.

Figure 4.10. Mutation of phrase group B-d axis-dyad chords (mm.32-33) in phrase group B-e (mm.39-40), Etude No.4

mm.32-33:							mm.39-40:					
p8i0:	448	357	71e	62t	266	$T_1 \rightarrow$	pti2:	559	468	820	73e	377
i1p3:	t30	496	769	587	85t	$T_2 \rightarrow$	i5p7:	052	6e8	98e	7t9	t70

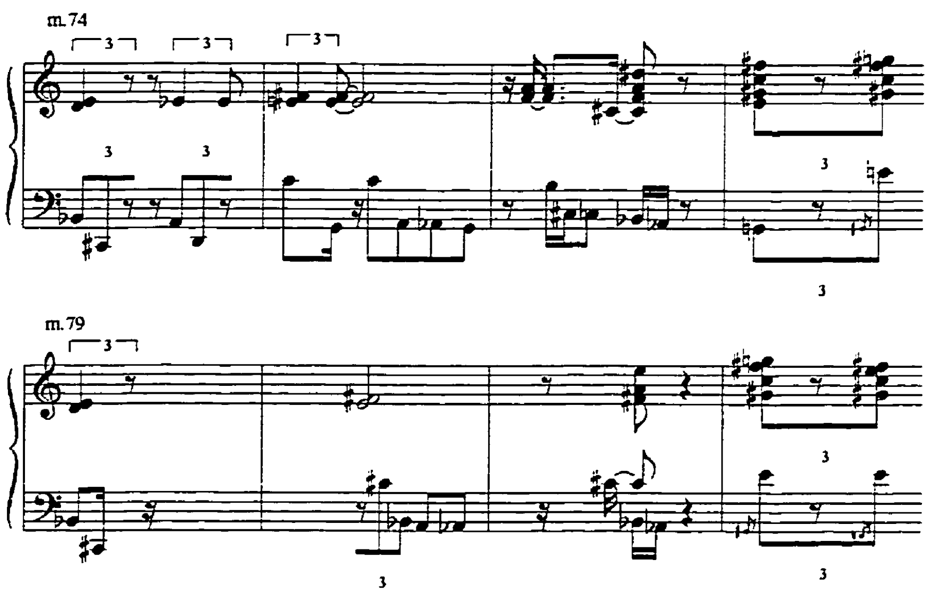
In the A'-section, a mutated repeat of the second subject and episode at mm.78-82 delays the third statement of the subject until m.83. The mutation of the repeated subject A'-b' is similar to the conflation of m.1 and m.10 in the mutated subject of m.16 (figure 4.9), but involves a succession of four axis-dyad chords here, rather than two. Figure 4.11 illustrates how, in the mm.78-79 passage, the first and fourth axis-dyad chords retain the axis dyads of their counterparts in mm.63-64, and the neighbour dyads from those of mm.72-73. Conversely, the third axis-dyad chord in the mm.78-79 passage retains the neighbour dyads of its m.63 counterpart, and the axis dyad from its m.72 counterpart. The second axis-dyad chord in the mm.78-79 passage comprises an unsystematic mixture of pcs from the second axis-dyad chords in both m.63 and m.72, with the pcs reordered in m.78 to conform to the prevailing array. The first sum tetrachord in m.79 takes its pcs from the left vertical dyad of the corresponding sum tetrachord in m.64, and from the right vertical dyad of the corresponding sum tetrachord in m.73.

Figure 4.11. Mutation of repeated subject A'-b', mm.78-79 (compared to mm.72-73 and mm.63-64), Etude No.4

<u>mm.63-64:</u>						
p4i8:	0 4 4	4 0 8	5 e 9	4 0 8	t t	4 4
i1p3:	4 9 6	9 4 e	4 9 6	5 8 7	1 0	t 3
<u>mm.72-73:</u>						
p2i6:	e 3 3	3 e 7	4 t 8	3 e 7	9 9	3 3
iep1:	3 8 5	8 3 t	3 8 5	4 7 6	0 e	9 2
<u>mm.78-79:</u>						
i3p7:	e 4 3	8 7 0	5 t 9	3 0 7	t 9	- -
p0i2:	3 9 5	8 4 t	4 8 6	4 8 6	1 e	- -

Although the final sum tetrachord of m.79 completes the subject statement, this sum tetrachord actually derives from the first sum tetrachord of episode A'-b' (m.74), and as such initiates a mutated repeat of episode A'-b' (mm.79-82). Example 4.9 compares the first statement of episode A'-b' (mm.74-77) with the mutated repeat (mm.79-82). In the corresponding integer notation, the sum tetrachord's linear dyads of m.74 have been reordered as vertical dyads in m.79 (example 4.9a). The truncated repeat omits the next two sum tetrachords in episode A'-b' (at the end of m.74 and beginning of m.75). The repeat resumes in m.80 with a mutation of the final axis-dyad chord of m.75 (example 4.9b). The axis-dyad chord in m.80 retains the axis dyad from that of the axis-dyad chord of m.75, but transposes the latter's neighbour dyads in m.75 by T_{P1}. The opposite situation obtains in m.81, which derives from the second half of m.76 (the first axis dyad chord of m.76 also having been omitted in the repeat). The axis-dyad chord of m.81

retains the neighbour dyads of the corresponding axis-dyad chord in m.76, but transposes the axis dyad by T_{PI} (example 4.9c). Finally, the order of the two axis-dyad chords in m.77 is reversed in m.82 (example 4.9d). The first axis-dyad chord of m.82 is the same as the second axis-dyad chord of m.77. The second axis-dyad chord in m.82 retains the same neighbour dyads as the first axis-dyad chord of m.77, but replaces the axis dyad with another of the same interval.

Example 4.9. Mutated array segments in the repeat of episode A'-b', Etude No.4

(a)			(b)			(c)			(d)						
m.74:			m.75:			m.76:			m.77:						
p2i6:	2	4	8	6	0	9	5	1	8	6	0	i3p7:	8	7	0
iep1:	1	t	7	4	9	8	3	t	4	7	6	p0i2:	4	8	6
m.79:			m.80:			m.81:			m.82:						
i3p7:	1	2	9	6	1	9	6	1	8	7	0	p0i4:	8	4	0
p0i2:	t	4	8	4	t	8	4	t	4	8	6	i9pe:	4	5	6

Section A'-c' comprises a complete statement of the subject at T_{P0} followed by three mutated repeats, which are reflected in the systematic changes in the tonic sums of the arrays. Figure 4.12 illustrates the succession of arrays beginning in m.83, where Perle decreases the second tonic sum in each of the upper cyclic sets, and increases the first tonic sum in each of the lower cyclic sets. These alterations to the tonic sums are manifested in the corresponding axis-dyad chords through the process of mutation.

Figure 4.12. Systematic alterations of tonic sums in section A'-c', mm.84-89, Etude No.4

m.83:	m.86:	m.88:	m.89:
p0 **i4**	p0 **i3**	p0 **i2**	p0 **p1**
i9 pe	**pt** pe	**ie** pe	**p0** pe

The effects of mutation in the axis-dyad chords are apparent when they appear in a *sum alignment*, as in figure 4.13.[27] The subject is first stated in its entirety in mm.83-84 at the transpositional level of the original subject of mm.1-2 (figure 4.13a). But each subsequent statement involves systematic changes to the axis-dyad chords, first to those in just the latter half of the subject in the second statement (figure 4.13b), and then to those in both halves in the third statement (figure 4.13c). The first half of the subject is then omitted, replaced by one more modified repeat of the latter half (figure 4.13d).

In the second statement of subject A'-c', the axis dyad of m.86 is related by $T_{\pm 1}$ to its counterpart in m.84 (compare the fourth axis-dyad chord of figure 4.13a with the fourth axis-dyad chord of figure 4.13b). Flipping the neighbour dyad (5/4) of this axis-dyad chord in m.86 provokes a reinterpretative modulation (figure 4.13b). The sum tetrachords which follow are mutated repeats of their counterparts in m.84. Each sum tetrachord in m.86 holds one neighbour dyad invariant and replaces the other neighbour dyad with one related by $T_{\pm 1}$.

In the third statement of the subject at m.87, the two remaining axis-dyad chords each replace one of their respective neighbour dyads with another of the same sum (figure 4.13c). Again at m.88, the axis dyad is related by $T_{\pm 1}$ to its counterpart in m.86, and again

[27] A sum alignment presents the transposed and invariant pcs as neighbour dyads and sum tetrachords, respectively, whereas a difference alignment positions the transposed pcs in opposite neighbour dyads and the invariant pcs in all three vertical dyads. The annotated score in appendix four gives a sum alignment from m.83 to the end of the etude.

Figure 4.13. Mutation in A'-c' section, mm.83-89, Etude No.4

(a)		m.83:				m.84:		
	p0i4:	t 2 2	2 t 6	3 9 7		2 t 6	8 8	2 2
	i9pe:	4 7 2	9 2 7	4 7 2		5 6 3	e t	8 1
		m.85:				m.86:		
(b)		t 2 2	2 t 6	3 9 7		1 e 5		
		4 7 2	9 2 7	4 7 2		6 5 4		
					p0p3:	1 e 4	8 7	2 1
					ptpe:	6 5 5	e e	8 2
			m.87:			m.88:		
(c)		- - -	2 t 5	3 9 6		2 t 5		
		- - -	9 2 8	4 7 3		5 6 4		
					p0i2:	2 t 4	8 6	2 0
					iepe:	5 6 5	e 0	8 3
						m.89:		
(d)		- - -	- - -	- - -		3 9 5		
		- - -	- - -	- - -		4 7 4		
					p0p1:	3 9 4	8 5	0 0 1
					p0pe:	4 7 5	e 1	e 0 0

it flips one of its neighbour dyads in a reinterpretative modulation. The following two sum tetrachords each replace a neighbour dyad with another $T_{\pm 1}$-related dyad. The final mutated statement omits the first half of the subject and its three axis-dyad chords, resuming with the fourth axis-dyad chord related by $T_{\pm 1}$ to its counterpart in m.88. This final axis-dyad chord flips one of its neighbour dyads to effect the final reinterpretative modulation. Only the first of the two sum tetrachords follows, with a neighbour dyad replaced by another $T_{\pm 1}$-related dyad.

The mutated subject concludes with a tonic axis-dyad chord.[28] Although tonic axis-dyad chords appear periodically throughout the etude, they are a significant feature of the

[28] In a tonic axis-dyad chord, the axis dyad pcs duplicate pcs in one of the trichordal cyclic set segments, and the axis dyad sums duplicate one of the array's tonic sums. Due to the high degree of pc and tonic sum duplication, the tonic axis-dyad chord is the only array segment Perle accords hierarchical superiority in twelve-tone tonality. Please see chapter two for a more detailed discussion of tonic axis-dyad chords.

subject: all four axis-dyad chords in the non-mutated statements of the subject are tonic axis-dyad chords. In the first mutated subject statement in mm.16-17 of the A-section, the mutated portion of the subject involves non-tonic axis-dyad chords; the tonic status of subject axis-dyad chords returns with the final axis-dyad chord of m.17, adumbrating the resumption of the transposed portion of the non-mutated subject. In a similar way, in the final subsection of the etude the mutated portion of the subject statements involve mostly non-tonic axis-dyad chords.[29] Hence the conclusion of the etude in a tonic axis-dyad chord marks a return of one of the distinctive features of the subject.

In Etude No.1 coherence between successive axis-dyad chords and between formal sections is achieved primarily by establishing consistent secondary difference and sum patterns (through combinations of transposed and invariant cyclic set segments), and by exchanging and inverting complementary transpositional values between axis-dyad chords. While these events occur in Etude No.4, the process of mutation proves to be the primary guiding force in achieving coherence between axis-dyad chords within and between formal units.

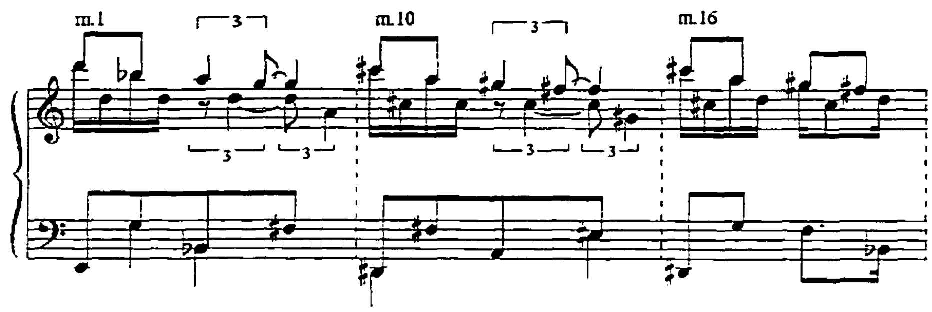
Realization of array segments

The musical surface of Etude No.4 also manifests the effects of mutation. Example 4.10 and the corresponding integer notation show the A-section's mutated subject A-c at m.16 (derived from the previous subject statements in mm.1 and 10).[30] Although the upper treble voice of m.10 is unchanged in m.16, the lower treble voice alternates its pitches C♯ and D from m.10 and m.1. The bass dyads comprise pitches from m.10 and m.1 in alternation (separately stemmed in the example).

[29] The first axis-dyad chord in m.87 is a tonic axis-dyad chord, the lone exception among the mutated axis-dyad chords of this subsection.

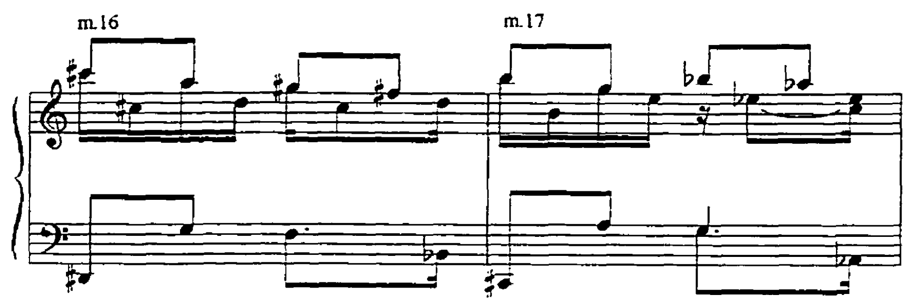
[30] Figure 4.9 illustrates the effects of mutation on the axis-dyad chords of these measures.

Example 4.10. Mutation of individual voices of mm.1 and 10 in m.16, Etude No.4



	m.1:	m.10:	m.16:
upper treble:	2 t 9 7	1 9 8 6	1 9 8 6
lower treble:	2 2 2 9	1 1 1 8	1 2 1 2
bass:	4 7 t 6	3 6 9 5	3 7 5 t

Example 4.11. Mutation of m.16 in m.17, Etude No.4



m.16: m.17:

upper treble: 1—9 (T_{P-2}) e—7
8—6 (T_{P+2}) t—8

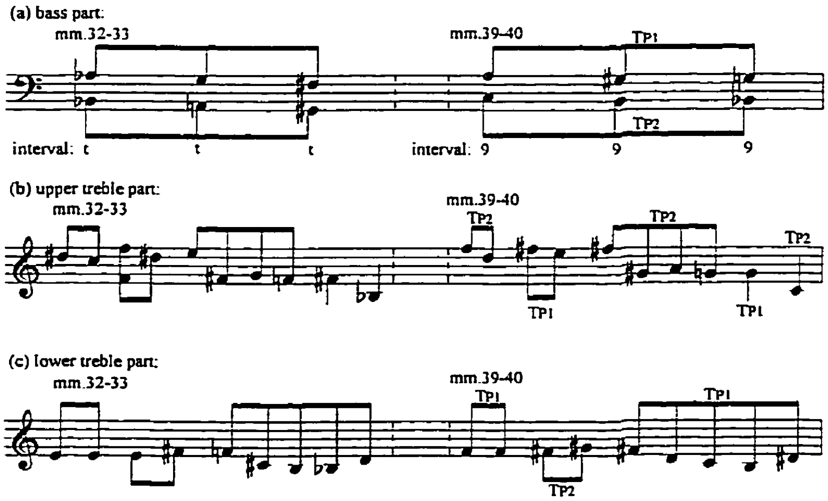
lower treble: 1—2 ($T_{P\pm2}$) e—4
1—2 ($T_{P\pm1}$) 0—3

bass: 3—7 ($T_{P\pm2}$) 1—9
5—t ($T_{P\pm2}$) 7—8

In turn, the content in m.17 is a mutation of that in m.16, as seen in example 4.11 above. The upper treble voice's dyads in m.17 are related by T_{P-2} and T_{P+2} to their counterparts in m.16. The lower treble voice's dyads expand outward by $T_{P\pm2}$ and $T_{P\pm1}$ in m.17, while both bass dyads expand by $T_{P\pm2}$.

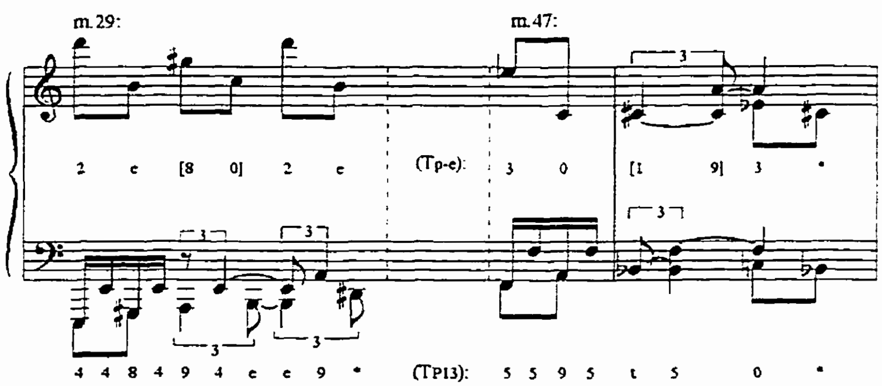
The process of mutation connects the diverse phrases in the three phrase groups of the B-section. Three different effects obtain between the first phrases of phrase groups B-d and B-e (mm.32-33 and 39-40, respectively), as illustrated in example 4.12. The example uses upper and lower beams to show correlations between segments transposed by corresponding values. First, the vertical dyads are a semitone smaller in the bass in mm.39-40 than their dyadic counterparts in mm.32-33. Example 4.12a demonstrates how the decrease in the bass part from interval t dyads to interval 9 dyads results from the transposition of the individual upper and lower bass voices by different values: the upper bass voice by T_{P1}, the lower bass voice by T_{P2}. Second, the upper and lower treble voices of mm.32-33 are each mutated in mm.39-40 by transpositions of discrete pitch segments by T_{P1} and T_{P2} (example 4.12b and c). Finally, the durational values of mm.32-33 are augmented in mm.39-40, strengthening the connection between the passages related by mutation.

Example 4.12. First phrase of phrase group B-e (mm.39-40), derived from first phrase of phrase group B-d (mm.32-33) through mutation, Etude No.4



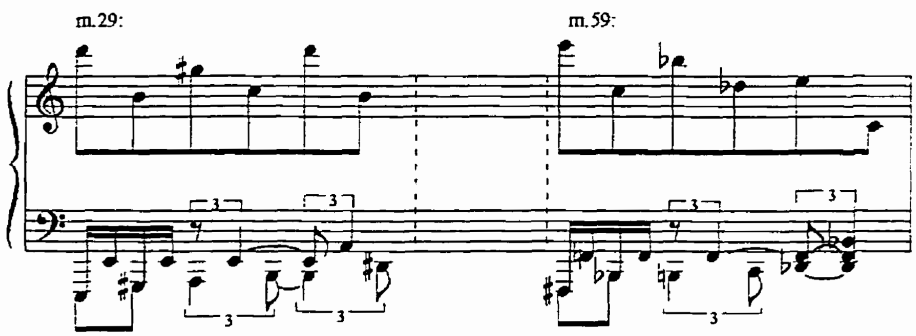
Mutation also connects the first phrase of phrase group B-f (mm.47-48) with the inverted subject in the linking passage in the A-section (mm.29-30). As shown in example 4.13 and the corresponding integer notation, the treble part of mm.47-48 transposes the treble part of mm.29-30 at $T_{P\text{-}e}$, although the order of the pitches in the second dyad is reversed, and only the $E^{\flat}$ of the third dyad reappears. The bass part of mm.47-48 transposes the bass part of mm.29-30 at T_{P13} until the penultimate pitch of m.48, at which point the bass dyad expands by $T_{P\pm2}$ (not shown in the example). The integer notation gives the pcs of both passages, which display a T_1 relationship. But due to the pitch transposition of the treble part by $T_{P\text{-}e}$ and the bass part by T_{P+13}, the spatial distances between the two parts contract instead of remaining invariant.

Example 4.13. First phrase of phrase group B-f (mm.47-48), derived from linking passage (m.29) through mutation, Etude No.4



The opening of the linking passage between the B- and A'-sections (mm.59-60) also derives from a mutation process. This linking passage actually originates in the linking passage between the A- and B-sections at mm.29-30, as shown in example 4.14 and the corresponding integer notation. The upper and lower treble parts in m.29 are transposed by differing values in m.59, resulting in the treble part's three vertical dyads in m.59 being a semitone larger than their counterparts in m.29 (see example 4.14a). As well, the third vertical dyad is projected downward by an octave. Hence the first two pitches in the upper treble voice in m.29 are transposed by T_{P2} in m.59 while the final pitch is transposed by $T_{P\text{-}t}$. Similarly, the first two pitches in the lower treble voice in m.29 are transposed by T_{P1} while the third pitch is transposed by $T_{P\text{-}e}$. In the bass part, the vertical dyads are a semitone smaller in m.59 than their counterparts in m.29 (example 4.14b). This decrease is due to the transposition in m.59 of the upper and lower bass voices in m.29 by T_{P1} and T_{P2}, respectively.

Example 4.14. Opening of linking passage between B- and A'-sections (m.59) derived from opening of linking passage between A- and B-sections (m.29), Etude No.4

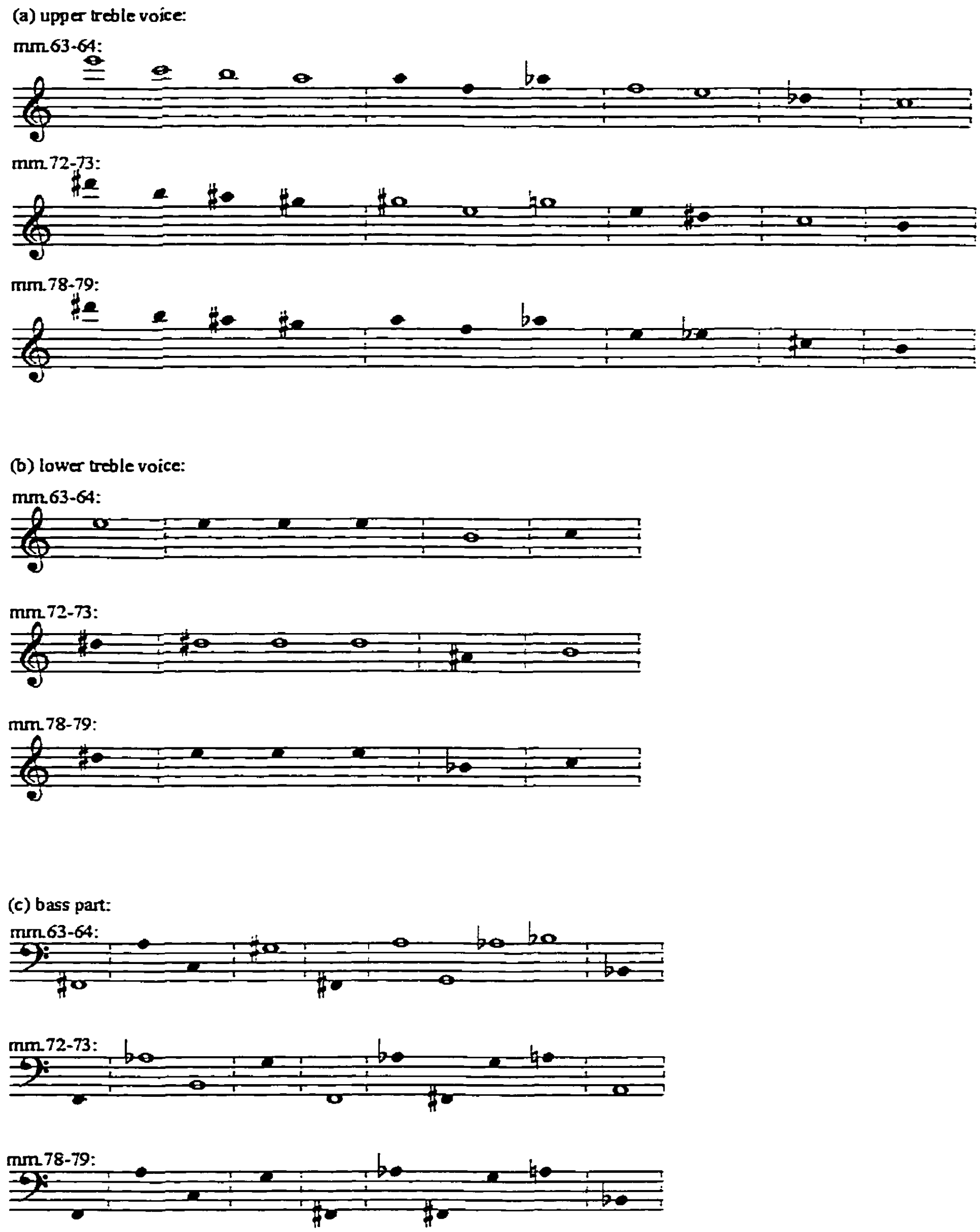


(a)	m.29:					m.59:		
upper treble pcs:	2	8	2	T_{P2}, T_{P2}, $T_{P\text{-}t}$	→	4	t	4
lower treble pcs:	e	0	e	T_{P1}, T_{P1}, $T_{P\text{-}e}$	→	0	1	0
intervals:	15	8	15			16	9	16

(b)	m.29:							m.59:				
upper bass pcs:	4	4	4	4	9	T_{P1}	→	5	5	5	5	t
lower bass pcs:	4	8	9	e	e	T_{P2}	→	6	t	e (0)	1	1
intervals:	12	8	7	5	t			e	7	6	4	9

As discussed above, a mutated repeat of the A'-section's second subject and episode beginning at m.78 delays the statement of subject A'-c' until m.83. In this mutated repeat, the treble and bass parts derive their pitches from the first and second statements of the subject in mm.63-64 and mm.72-73. Example 4.15 represents these derived pitches as closed noteheads. The dashed barlines isolating these pitches in the example highlight the mutation's systematic patterns of derivation. In example 4.15a, the upper treble voice

Example 4.15. Mutated repeat of subject A'-b' (derived from subject statements in mm.63-64 and mm.72-73), mm.78-79, Etude No.4

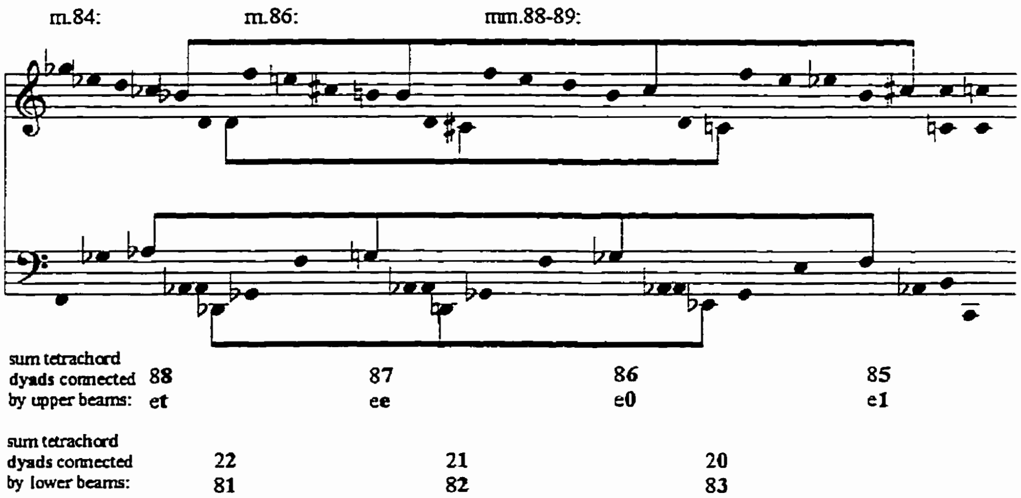


of mm.78-79 derives its pitches from the two statements in mm.63-64 and mm.72-73 in alternation; the upper treble voice in m.78-79 takes the first four pitches from m.72, then the next three pitches from mm.63-64, then the next two from m.73, and so on. This mutation process thus establishes a pattern of alternating invariant pitches of <4-3-2-1-1>. In a similar way, the lower treble voice and the bass voice alternate between pitches from the two corresponding passages in a pattern of <1-3-1-1> and <1-2-1-1-4-1>, respectively (examples 4.15b and c). This mutation process resembles that found in the third subject statement of the A-section, where the conflation of the subject statements in mm.1 and 10 result in the mutated subject in m.16.

Finally, in the final section of the etude (mm.83-89), the musical surface exhibits the systematic mutation of the sum tetrachords that conclude each repeated subject statement.[31] The first and second sum tetrachords are realized in mm.84, 86, and 88 as linear dyads in on the second and third beats, respectively. The systematic replacements of the *vertical* dyads in the sum tetrachords are manifested in the music as the progression of the corresponding pitches within the *linear* treble and bass dyads. Example 4.16 highlights these progressions in the beamed notes. The notes connected by the upper beams in the treble and bass staves represent pitches from the first sum tetrachord while the notes connected by the lower beams represent pitches from the second sum tetrachord. In the score, the first pitches in each of the bass's linear dyads (on the second beats of mm.84, 86, 88, and 89) form a long-range descending melodic motion of A♭-G-G♭-F, while the second pitches in each of the treble dyads form a long-range ascending melodic motion, of B♭-B♮-C-C♯. In the abstract dimension, these two passing motions are represented as the replacement of the vertical dyad of the *first* sum tetrachord by another of the same sum in each mutated subject repeat. The notes connected by the upper beams in the treble and bass staves show these passing motions. The *second* sum tetrachord in each pair is similarly realized at the surface in the linear dyads of the third beat of mm.84, 86, and 88: the second pitches in each of the bass and treble dyads form long-range melodic motions of D♭-D♮-E♭ and D-C♯-C♮, respectively. The notes connected by the lower beams show these passing motions.

[31] This discussion focuses on the second half of each subject statement in mm.83-89 due to the nature of the mutation process in this passage. The mutated array segments of the second half of the subject are realized systematically at the musical surface, but as figure 4.13 shows, the first half of the subject is truncated in the third statement at m.87, then omitted from the fourth statement (m.89).

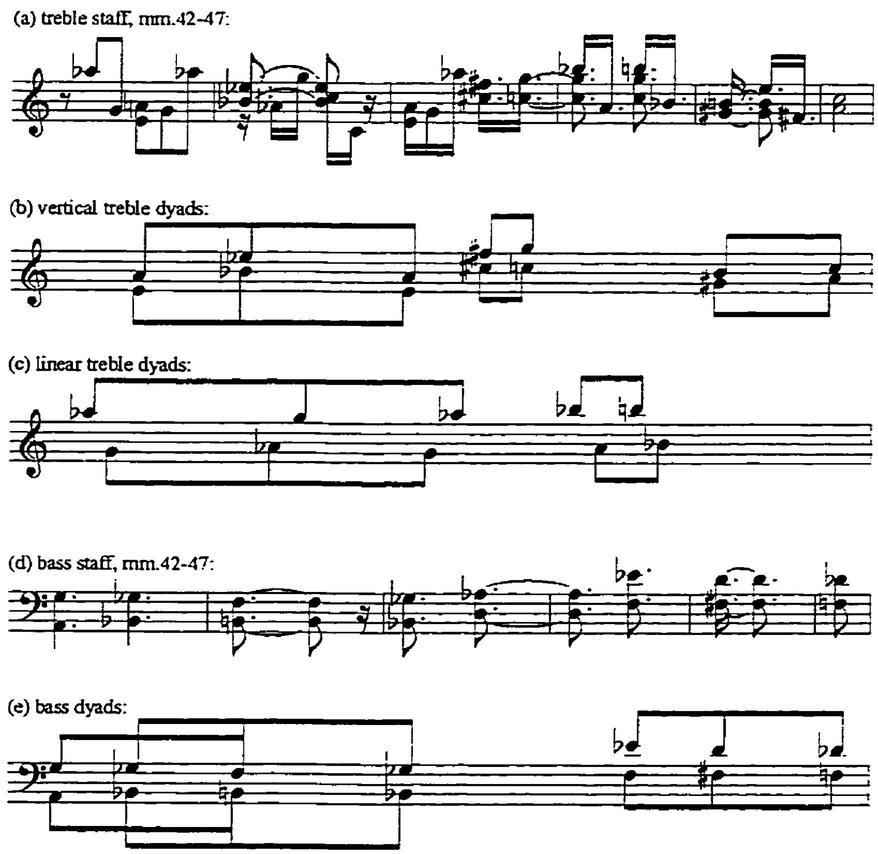
Example 4.16. Realization of sum tetrachords as passing motions in mutated repeated subject statements of A'-c' subsection, mm.84, 86, 88, and 89, Etude No.4



Etude No.4 features textural contrast throughout, but most notably in the B-section, where the texture often comprises a complex mixture of linear and vertical dyads which form independent passing and neighbour gestures. Example 4.17 illustrates a passage that crosses the boundary of the second and third phrases in phrase group B-e, extending from mm.42-47. In example 4.17a, the treble part combines both sustained vertical dyads and linear dyads, both of which contain embedded gestures. The progression of the sustained vertical dyads in example 4.17b generates two parallel linear neighbour gestures. These are followed by two pairs of dyads of contrasting motion: the first pair's voices unfold in contrary motion (preserving the same dyadic sum) while the second pair's voices unfold in parallel motion (preserving the same dyadic interval). The linear dyads in example 4.17c contain complementary neighbour gestures of the same component pcs, followed by a pair of dyads whose parallel motion preserves the same dyadic interval.

Example 4.17d shows the vertical dyads in the bass in mm.42-47. The first four vertical dyads contain a pair of passing gestures in contrary motion (dyads one to three) overlapped by a pair of neighbour gestures (dyads two to four). (example 4.17e). These neighbour gestures coincide with those of the treble vertical dyads in example 4.17b. The

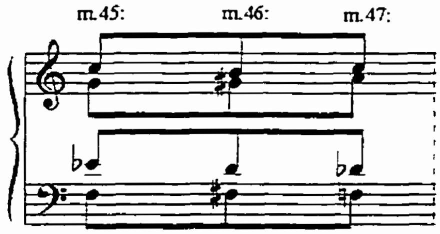
Example 4.17. Embedded passing and neighbour gestures in vertical and linear dyads, mm.42-47, Etude No.4



bass part concludes with three vertical dyads which simultaneously embed both a passing and a neighbour gesture. Example 4.18 illustrates how the final three sustained treble dyads of the passage (see example 4.17b) also form this combination of gestures, by inverting the 7/0 dyad of m.44. Thus, the bass and sustained treble dyads in mm.44-47 share a symmetrical relationship: the outer voices form a pair of neighbour gestures, while the inner voices form a pair of passing gestures in contrary motion.

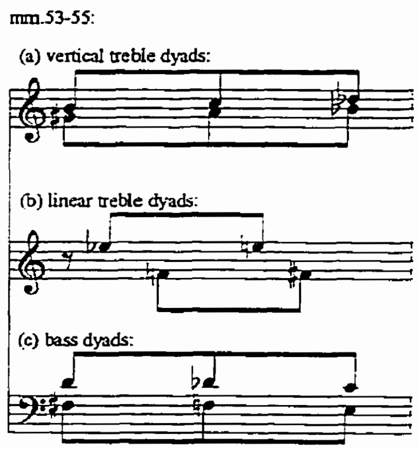
Similarly, phrase group B-f (mm.47-58) also contains embedded gestures in its textural combination of linear and vertical dyads. Each of the three phrases in phrase group B-f has similar opening measures (mm.47, 51, and 55-56, respectively), but then the second and third phrases diverge from the first in length and content. These extensions of the two latter phrases contain the embedded gestures. In the extension of

Example 4.18. Corresponding gestures embedded in treble and bass parts, mm.44-47, Etude No.4



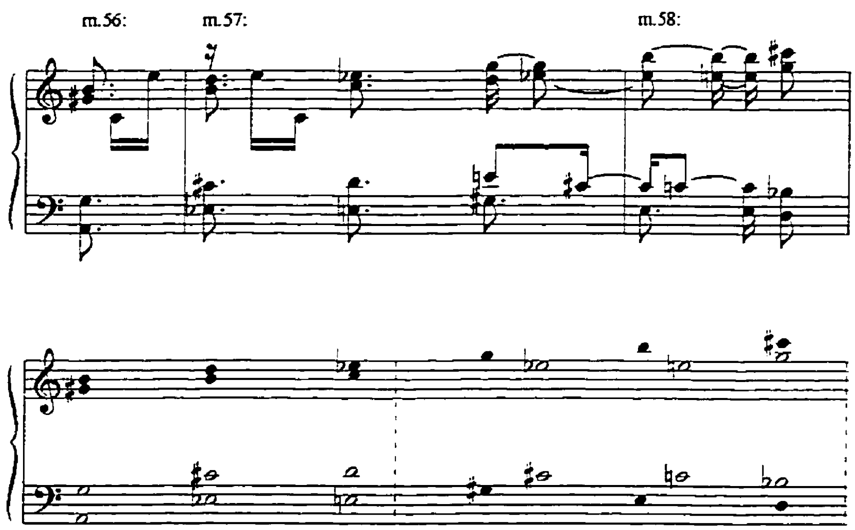
the second phrase (mm.53-55) the treble part alternates its vertical and linear dyads. The vertical dyads embed two passing gestures in parallel motion and thus preserve the same interval (example 4.19a), while the two linear dyads also proceed in parallel motion, preserving the same dyadic interval (example 4.19b). The bass part's vertical dyads contain two passing gestures (example 4.19c). These gestures unfold simultaneously with and in contrary motion to those embedded in the treble vertical dyads (example 4.19a).

Example 4.19. Embedded passing gestures, mm.52-55, Etude No.4



The extension of the third phrase of phrase group B-f begins on the last dotted eighth value of m.56. Example 4.20 shows how the component pitches in the treble part (represented in closed noteheads) proceed in parallel motion, forming three vertical dyads of the same interval, as indicated in the corresponding integer notation.[32] Simultaneously, the component pitches of the bass part (represented in open noteheads) also proceed in parallel motion, forming dyads of the same interval. At the third dotted-eighth value of m.57, however, the component pitches no longer move simultaneously within their respective treble and bass parts. Instead, only the outer voices move together (upper treble and lower bass, represented by closed noteheads); they lead to the end of the phrase in contrary motion forming three dyads of the same sum. Meanwhile, the inner voices move in alternation with one another in contrary motion (lower treble and upper bass, represented by open noteheads). Two of the three resulting dyads maintain the same sum.

Example 4.20. Pairs of voices moving in parallel and contrary motion, mm.56-58, Etude No.4



upper treble:	e	2	3	upper treble:	7	e	1
lower treble:	8	e	0	lower bass:	8	4	2
interval:	3	3	3	sum:	3	3	3

upper bass:	7	1	2	lower treble:	3	4	7
lower bass:	9	3	4	upper bass:	1	0	t
interval:	t	t	t	sum:	4	4	(5)

[32] The example excludes from consideration the two linear dyads of C-E in the treble, mm.56-57, as they do not contribute to the progression under discussion.

The preceding analysis has aimed to show more intricate techniques of array realization than were employed in Etude No.1. Although the system of twelve-tone tonality is highly structured, it does not limit the infinite possibilities of realization for the composer.

Set-theoretical perspectives

Pitch-class set theory is a useful tool for revealing both surface and structural relationships in Perle's music composed in the twelve-tone tonality system. The following discussion examines passages from Etude No.1 and Etude No.4 from the perspective of pc set theory to complement the revelations from the perspective of twelve-tone tonality. Specifically, the set-theoretical approach highlights distinguishing features of the etudes: the palindromic formations in Etude No.1, and the distinct formal units and mutated passages in Etude No.4. Moreover, the set-theoretical inquiries contribute additional insights into these etudes.

Two pc sets of the same sc are considered equivalent on the basis of transposition or inversion, as established by Allen Forte (1973). Yet two equivalent pc sets may bear no resemblance to one another in their realization at the musical surface. Hence their equivalence is primarily an abstraction. Robert Morris (1995b) extends the notion of equivalence to include pitch realizations of pc sets. He has introduced three contextually derived measures of equivalence relations which establish finer means of relating sets within a sc: the *PSC*, *PCINT*, and *FB* classes. Morris provides examples of each of these equivalences in various pitch realizations of given scs, which are reproduced below in example 4.21.[33]

The first of Morris's equivalence relations is *PSC* (for pitch set-class), in which pitch sets of the same sc are considered equivalent if they have identical spacing between their adjacent pitch elements when ordered from lowest to highest.[34] Example 4.21a gives

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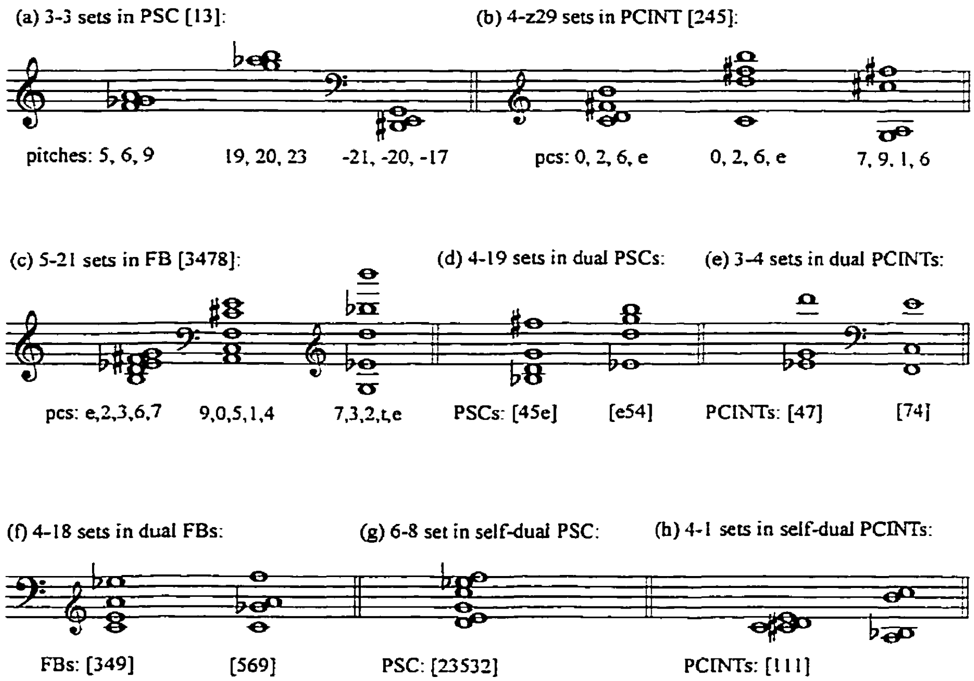
[33] Examples 4.21a and c are reproductions of Morris's examples 6 and 10a (1995b); the remaining examples 4.21b and 4.21d-h derive from Morris's related discussions on pp.213-214, 217-218, and 220.

[34] Integers are used to represent pitches as well as pcs throughout this discussion and in the accompanying examples. With regard to pitches, integer 0 represents middle C; other pitches are measured in semitones above and below middle C. Hence the integers -10, -9, and -6 represent the pitches D, E♭, and F♯ immediately below middle C. This practice follows those of Rahn (1980) and Morris (1987, 1995b).

three pitch realizations of sc 3-3; they are all members of the same PSC-class because they contain identical spacing from their lowest to their highest pitches. PSC-classes derive their names from the intervals between their adjacent pitch elements. Hence these three sets of sc 3-3 belong to PSC [13].

Example 4.21. Pitch realizations of sets from the same scs that are members of the same PSC-class (a), PCINT-class (b), FB-class (c), dual PSC-class (d), dual PCINT-class (e), dual FB-classes (f), self-dual PSC-class (g), and self-dual PCINT-class (h)



The second relation is identified as *PCINT* (for pc INT equivalence).[35] In this relation, pitches are expressed as pcs; sets of the same sc are considered equivalent if they have identical spacing between adjacent pc elements when listed in the most compressed order. Example 4.21b gives three pitch realizations of sc 4-z29, with the pitches listed as pcs. All three sets display the same succession of intervals between their adjacent pcs. Since the PCINT-class derives its name from the interval succession between adjacent pc elements, these three sets are members of PCINT [245].

[35] Morris defines INT as the list of a pc-segment's "successive adjacent ordered-intervals" (1995b, 211).

Morris's third equivalence relation is *FB* (for figured bass). Pitches in this relation are also expressed as pcs; sets of the same sc are considered FB-equivalent if their pcs form the same collection of intervals with a designated "bass" (the lowest pitch in the pitch realization of the sc).[36] The FB relation differs from the PCINT relation in that it does not measure the intervals between adjacent intervals in an ordered set of pcs. Rather, it measures the intervals of a partially ordered pc set, between the lowest pitch (expressed as a pc) and each of the remaining pcs. In example 4.21c, the three pitch realizations of sc 5-21 are members of the same FB-class; in each realization the pitches expressed as pcs form the same collection of intervals with the "bass" (intervals 3, 4, 7, and 8). The FB-class derives its name from these intervals; hence the three pitch realizations of sc 5-21 in example 4.21c belong to FB-class [3478].

Morris also introduces the related concept of the *dual*, in which the interval succession within a given PSC or PCINT appears in reverse order. Example 4.21 (d and e) illustrates the dual PSC classes [45e] and [e54] and the dual PCINT classes [47] and [74], respectively. Morris uses the formula (k-n) to find the dual of an FB-class, with k as the largest integer in the FB-class, and the variable n as each of the other integers in the class. In the final step, Morris replaces the 0 (resulting from k-k) with k itself. Thus, in determining the dual of FB [349], the steps are: (9-4=5), (9-3=6), and (9-9=0, with 0 then replaced by 9). Hence the dual of FB [349] is [569], as shown in example 4.21f (1995b, 220). Finally, certain PSC and PCINT classes are *self-duals*, since their adjacent intervals lie in a symmetrical order (examples 4.21g and h).

Together, these equivalence measures form a continuum of associations between pairs of sets within a sc. The PSC relation determines equivalence between two realizations of a sc based on the intervallic succession between their component pitches, and as such is the most concrete measure. The PCINT relation determines equivalence between two realizations based on the intervallic succession between their pitches expressed as pcs, and is thus a more abstract measure than the PSC. The FB relation is the most abstract

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[36] In designating a "bass" note, Morris states: "One pc occurs before all the others in the set. The concept of 'before' is interpreted in pitch as lower. Thus the pcs that follow the first pc can be realized in any register of pitch space as long as they are higher than the first pc's realization" (1995b, 218).

measure of the three relations, in that it determines equivalence based on collections of intervals formed by unordered pcs in relation to a designated pc. Since they are dependent on contextual realization of members of a sc, these three equivalence measures establish more limited definitions of equivalence than do the standard measures of transpositional and inversional equivalence. As such, the PSC, PCINT, and FB relations may be used either to strengthen or diminish the assertion of equivalence between two pitch realizations of sets from the same sc.

Etude No.1

For the set-theoretical analyses of both etudes, the beamed notes in most of the musical examples are specifically intended to connect members of pc sets of various cardinalities within the same passage. The beams do not represent durational values (unless otherwise indicated), nor do they indicate hierarchical structural levels. Instead, the outermost beams connect pitches belonging to the same set, while each inner beam connects pitches belonging to the same subset.

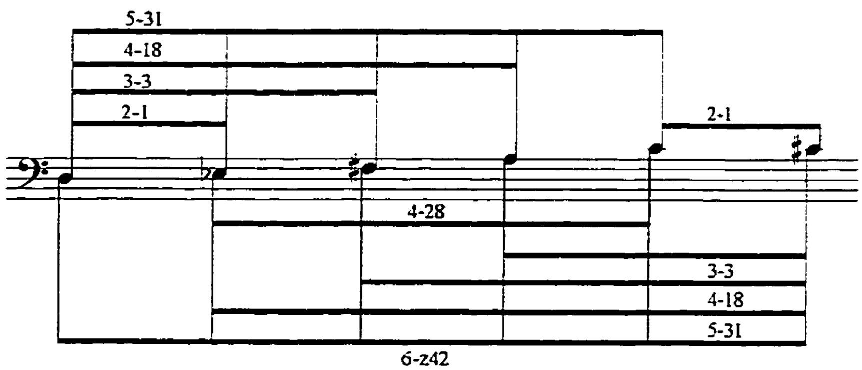
Principles of primary segmentation (Forte 1973, 83) guide the segmenting practices in the following analyses of the etudes. Groups of pcs are segmented as sets on the basis of clear thematic elements, simultaneities, adjacencies, distinct linear textures, and register. In addition, other such notational clues as rests, dynamics, tempo changes, and pedal markings assist in identifying groups of pcs as sets.

In the earlier discussion of array segments' realizations at the musical surface, examples 4.5, 4.6, and 4.7 showed how the melodic gestures of Etude No.1 embed symmetrical formations of dyadic sums. The linear dyads of m.25 also unfold the array's tonic sums in a palindromic arrangement (figure 4.7a). This symmetrical feature is greatly enhanced when seen through the lens of pc set theory, which reveals that Etude No.1 is rich in palindromic formations of sets.

The opening gesture of the etude comprises set 6-z42, as shown in example 4.22a. The particular ordering of the pitches creates a palindromic arrangement of pairs of subsets 3-3, 4-18, and 5-31. Further strengthening the symmetrical relationships, the two sets in each sc of this segmentation are inversionally related by the same value (T_3I), and are also

Example 4.22. Subsets of 6-z42 set in palindromic formation (a) and in Morris's equivalence relations (b), m.1, Etude No.1

(a)



(b)

set	pitches	pcs	PSC	PCINT	FB
3-3 3-3	-10,-9,-6 -3,0,1	2,3,6 9,0,1	13 31	13 31	14 34
4-18 4-18	-10.-9,-6,-3 -6,-3,0,1	2,3,6,9 6,9,0,1	133 331	133 331	147 367
5-31 5-31	-10,-9,-6,-3,0 -9,-6,-3,0,1	2,3,6,9,0 3,6,9,0,1	1333 3331	1333 3331	147t 369t

members of dual PSC, PCINT, and FB classes (example 4.22b). In addition, the symmetrical positioning of the 4-28 tetrachord between two 2-1 dyads makes explicit the cyclic intervals of 1 and 9 (of the underlying array i1i2/i3i0).

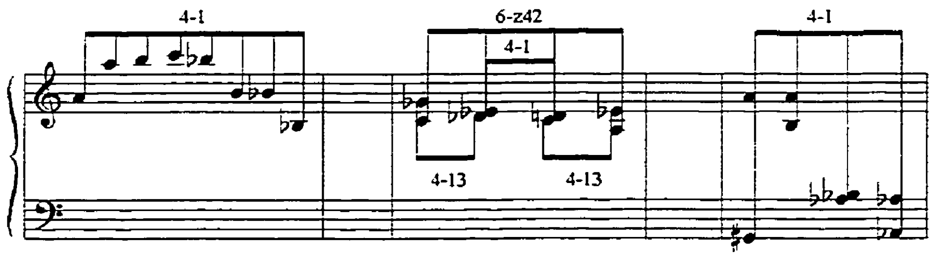
Example 4.23a illustrates another palindromic formation of scs, found in the passage extending from the linear gesture at beat two of m.8, through the crossing interval cycles of mm.9-10, to the end of the A-section at m.10.[37] In the middle of the crossing interval cycles lies the central set of the palindrome, hexachord 6-z42 (the same sc as found in the opening gesture), which itself contains a nested palindrome of tetrachords. The two 4-13 sets flanking the central 4-1 set are inversionally related by the same value (T_3I), and also

[37] Pcs not belonging to sets in the palindromic formation have been omitted from the example, with their omission indicated by empty measures.

belong to dual PSC, PCINT, and FB classes (example 4.23b). Hence the palindromic formation within the 6-z42 set of m.9 is similar to that of m.1. In addition, both 6-z42 sets comprise the same pcs, but in a systematically different ordering (compare examples 4.22a and 4.23a). That is, the pcs in the inner tetrachord and outer dyads of m.9 are the reverse of those in m.1. The 4-1 set at the center of the 6-z42 set in m.9 contains pcs 0, 1, 2, and 3. These pcs appear as the outer interval-1 dyads in the 6-z42 palindrome of m.1. As well, the interval-6 dyads immediately outside the 4-1 set in m.9 contain the pcs 0-6 and 9-3. These pcs together constitute a member of sc 4-28, the same collection of pcs that lies at the center of the 6-z42 set of m.1. Thus, by symmetrically repositioning the pcs in m.9, the configuration turns "inside out" to form a different palindrome of scs, and establishes a close relationship between the two passages.

Example 4.23. Sets in palindromic formations (a) and in Morris's equivalence classes (b), mm.8-10, Etude No.1

(a)



(b)

set	pitches	pcs	PSC	PCINT	FB
4-13	0,1,3,6	0,1,3,6	123	123	136
4-13	-3,0,2,3	9,0,2,3	321	321	356

In the larger palindrome extending from m.8 to m.10, the two 4-1 tetrachords surrounding the 6-z42 hexachord are related by the transpositional value T_e. Because of their varied realization in the score, however, they are not related further by any of Morris's equivalence classes. This weakens somewhat the conjecture of a larger palindromic organization of sets, in contrast to the strong assertion of palindromic organization of the 6-z42 hexachord at the center of the passage.

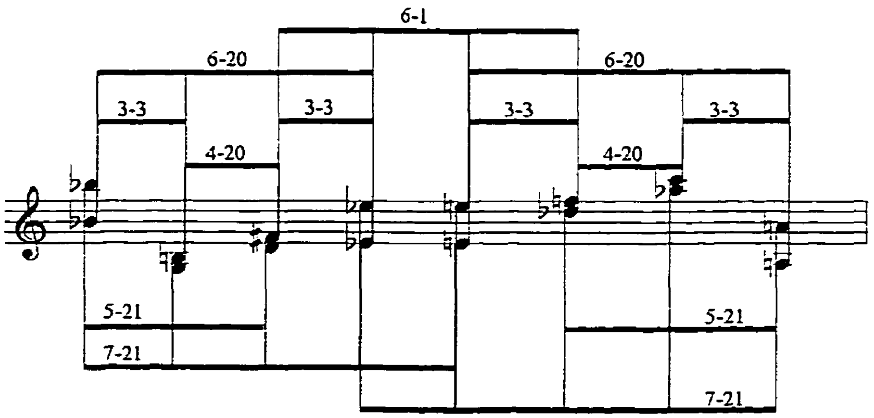
From the perspective of twelve-tone tonality, figure 4.7 above illustrated how the dyads in m.25 unfold the tonic sums of the array in a symmetrical fashion. As well, the passage divides into two halves, with the dyads of the second half related by T_6 to those in the first half, both vertically and horizontally (example 4.3). A set-theoretical segmentation confirms these relationships in slightly different terms, as shown in example 4.24. First, the symmetrically unfolding dyads create a palindrome of hexachords, with a pair of literally complementary 6-20 sets surrounding a central 6-1 set (example 4.24a). Each 6-20 set contains a nested palindrome of 3-3 sets flanking a central 4-20 set. Two overlapping 7-21 sets also span the measure, each containing the other's literal 5-21 complement.

The set pairs in each half of the passage are not only transpositionally related (by T_6), they are also inversionally related (by T_7I). Further, the set pairs in each half are related according to Morris's equivalence classes, as indicated in example 4.24b. The ordering of the 3-3 sets emphasizes these multiple relations. In a prograde ordering, the first 3-3 sets in each half form a transpositionally related pair, as do the last 3-3 sets. In a symmetrical ordering, the first and fourth 3-3 sets form an inversionally related pair that belong to dual PSC and PCINT classes, as do the second and third sets. Moreover, the first pair of 3-3 sets belongs to dual FB classes. The 4-20 and 6-20 set pairs belong to self-dual equivalence classes, while the 5-21 and 7-21 set pairs belong to dual PSC and PCINT classes. Thus, from the perspectives of both twelve-tone tonality and pc set theory, m.25 exhibits close relations among its pitch elements, in both prograde and symmetrical configurations.

In the earlier discussion of the passage extending from mm.27-31, the twelve-tone tonal analysis revealed three successive streams of axis-dyad chords with alternating secondary difference patterns concurrent with axis-dyad chords showing three simultaneous streams of secondary sum patterns (figures 4.4 and 4.5). In addition, the neighbour dyads and axis dyads of the axis-dyad chords were shown to correspond with the left- and right-hand notation indications. The set-theoretical perspective also

Example 4.24. Sets in palindromic formations (a) and in Morris's equivalence classes (b), m.25, Etude No.1

(a)



(b)

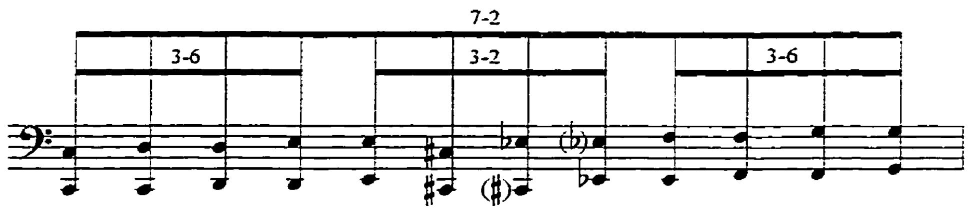
set	pitches	pcs	PSC[38]	PCINT	FB
7-21 7-21	-5,-1,2,3,4,6,10,15,16,22 -3,3,4,9,13,15,16,17,20,24	7,e,2,3,4,6,t,3,4,t 9,3,4,9,1,3,4,5,8,0	431124516 615421134	431124516 615421134	34789e 34678e
6-20 6-20	-5,-1,2,3,6,10,15,22 -3,4,9,13,16,17,20,24	7,e,2,3,6,t,3,t 9,4,9,1,4,5,8,0	4313457 7543134	4313457 7543134	3478e 3478e
5-21 5-21	-5,-1,2,6,10,22 -3,9,13,17,20,24	7,e,2,6,t,t 9,9,1,5,8,0	4344v v4434	43440 04434	347e 348e
4-20 4-20	-5,-1,2,6 13,17,20,24	7,e,2,6 1,5,8,0	434 434	434 434	47e 47e
3-3 3-3 3-3 3-3	-5,-1,10,22 2,3,6,15 4,13,16,17 -3,9,20,24	7,e,t,t 2,3,6,3 4,1,4,5 9,9,8,0	4ev 139 931 ve4	4e0 139 931 0e4	34 14 19 3e

highlights the systematic organization underlying these measures, uncovering an intricate series of superimposed palindromes undetected in the twelve-tone tonal analysis. This organization is more evident by examining the left- and right-hand segments separately.

[38] Analogous to the abbreviations 't' and 'e' for the integers 10 and 11, the abbreviation 'v' is used to represent the integer 12 in the PSC column of this table.

Example 4.25 shows how the series of pcs in the left-hand dyads forms a 7-2 heptachord, which embeds a palindrome of trichords 3-6 and 3-2. The sets in the 3-6 pair are both transpositionally and inversionally related. Their particular pitch realization further emphasizes their membership in the same equivalence classes.

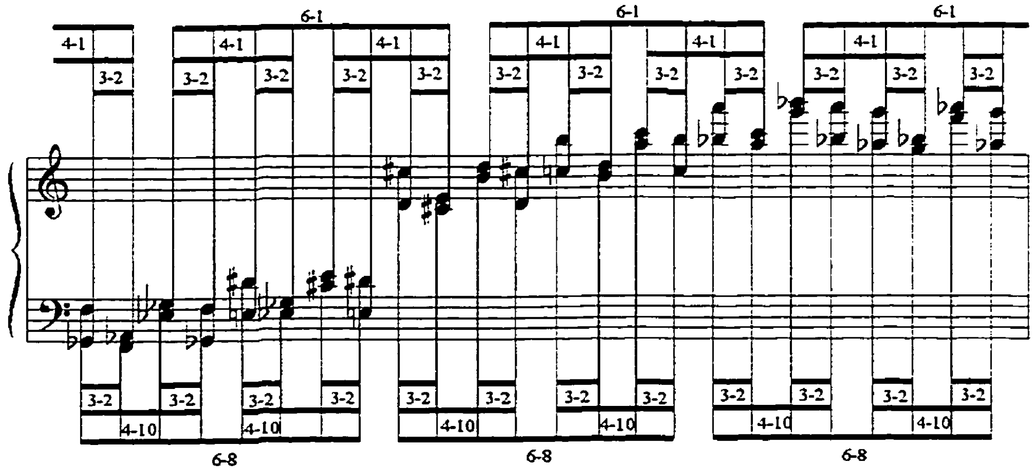
Example 4.25. Segmentation of left-hand line, mm.27-31, Etude No.1



As illustrated in example 4.26, the right-hand progression comprises a series of vertical dyads, pairs of which combine to form discrete 3-2 trichords. The adjacent trichords are inversionally related and belong to dual equivalence classes. Pairs of 3-2 trichords combine to form alternating, overlapping 4-10 and 4-1 tetrachords. Since scs 4-10 and 4-1 are inversionally symmetrical scs, all of the sets within the 4-10 and 4-1 scs are both transpositionally and inversionally related. In addition, since the component 3-2 trichords are inversionally related, their combination into tetrachords generates 4-10 and 4-1 tetrachords belonging to self-dual PSC and PCINT classes. In turn, pairs of these tetrachords combine to form alternating, overlapping 6-8 and 6-1 hexachords. (The final "partial" 6-1 collection is completed by the tetrachord at the beginning of the passage through a wrap-around procedure, suggested by the extended beams at the beginning and end of example 4.26.) Each hexachord thus comprises a palindrome of tetrachordal and trichordal subsets. Like the tetrachords, the 6-8 and 6-1 hexachords are also inversionally symmetrical, and so all instances of each hexachordal sc in this example are transpositionally and inversionally related, and belong to self-dual PSC and PCINT classes. Example 4.26b gives the equivalence relations of the first pair of each of the sets in the passage.

Example 4.26. Segmentation of right-hand line, mm.27-31, Etude No.1

(a)

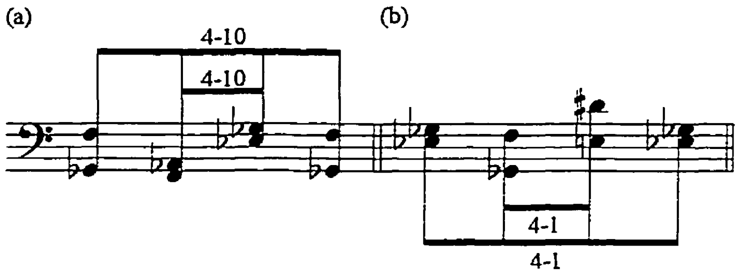


(b)

bar	set	pitches	pcs	PSC	PCINT	FB
27 27	3-2 3-2	-19,-18,-16,-7 -18,-9,-7,-6	5,6,8,5 6,3,5,6	129 921	129 921	13 9e
27 28	4-10 4-10	-19,-18,-16,-9,-7,-6 -9,-8,-6, 1, 3, 4	5,6,8,3,5,6 3,4,6,1,3,4	12721 12721	12721 12721	13t 13t
27-28 28-29	4-1 4-1	-18,-9,-8,-7,-6, 3 -8, 1, 2, 3, 4, 13	6,3,4,5,6,3 4,1,2,3,4,1	91119 91119	91119 91119	9te 9te
27-28 29-30	6-8 6-8	-19,-18,-16,-9,-8,-7,-6, 1, 3, 4 1, 2, 4, 11,12,13,14,21,23,24	5,6,8,3,4,5,6,1,3,4 1,2,4,e,0,1,2,9,e,0	127111721 127111721	127111721 127111721	138te 138te
27-29 29-30	6-1 6-1	-18,-9,-8,-7,-6, 1, 2, 3, 4, 13 2,11,12,13,14,21,22,23,24,33	6,3,4,5,6,1,2,3,4,1 2,e,0,1,2,9,t,e,0,9	911171119 911171119	911171119 911171119	789te 789te

The tetrachordal structures are also palindromic. First, each of the tetrachords is flanked by juxtaposed 3-2 trichords, as discussed above. In addition, within each 4-10 structure (which actually comprises six notes) the inner core of the 4-10 tetrachord is enclosed by identical vertical ic1 dyads, as illustrated in example 4.27a. An analogous situation obtains in the 4-1 structure, in which the inner core of the 4-1 tetrachord is surrounded by identical vertical ic3 dyads (example 4.27b).

Example 4.27. Representative tetrachordal structures in right-hand line of mm.27-31, Etude No.1



Thus, through set-theoretical analysis palindromic formations emerge as a distinctive feature of Etude No.1. Although the twelve-tone tonal analysis alluded to this feature, the tools of pc set theory were required to make these these relationships visible.

Etude No.4

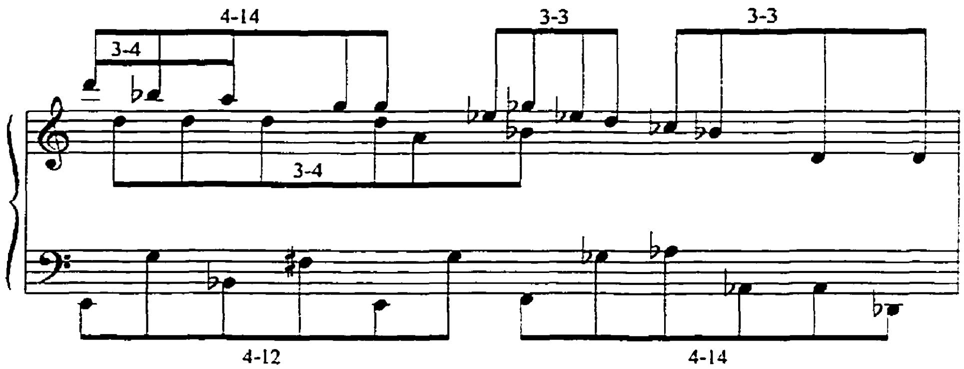
Etude No.4 does not feature palindromic formations to the same extent as does Etude No.1. Rather, the fourth etude is distinguished by the mutated repeats of many of its formal units. Pc set theory helps clarify the effects of mutation. In some cases, the process of mutation holds a set invariant in a passage while changing or omitting other sets; in other cases, mutation changes the content of a set while still retaining the sc identity; and in still other cases, mutation creates an entirely new sc. These effects will be demonstrated below.

First, however, set theory proves useful in defining the fourth etude's formal units. My segmentation reveals a small inventory of sets in each subsection. Example 4.28a shows one such segmentation of mm.1-2, in which two members of sc 4-14 serve as a frame around subject A-a. That is, the subject opens with a 4-14 set in the upper treble voice, and ends with another 4-14 set in the bass. Although the two sets are inversionally related, they are realized quite differently in the score; yet their equivalence is strengthened by the fact that both belong to dual FB classes, as illustrated in example

4.28b. Moreover, members of sc 4-14 frame all statements of the subject, whether exact transpositions or mutations. The 3-4 set in the lower treble voice echoes the 3-4 subset in the upper treble. The equivalence of these 3-4 sets is unmistakable: both share the same pcs and hence belong to the same equivalence classes. The treble concludes with two discrete members of sc 3-3, whose realization in the score limits their equivalence to one of transposition only.

Example 4.28. Sets in subject A-a (a) and in Morris's equivalence classes (b), mm.1-2, Etude No.4

(a)

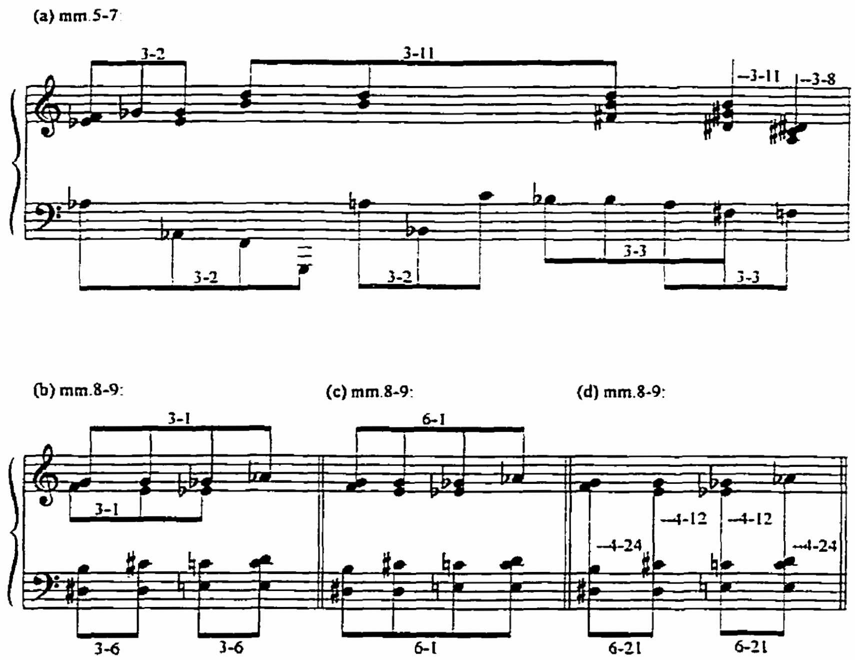


(b)

set	pitches	pcs	PSC	PCINT	FB
4-14 4-14	19, 21, 22,26 -23,-19,-16,-6,-4	7, 9, t, 2 1, 5, 8, 6, 8	214 43t2	214 43t2	237 457
3-4 3-4	21, 22, 26 9, 10, 14	9, t, 2 9, t, 2	14 14	14 14	15 15
3-3 3-3	14, 15, 18 2, 10, 11	2, 3, 6 2, t, e	13 81	13 81	14 89

Episode A-a is also defined by a small inventory of set pairs. Example 4.29a shows the first half of this subsection (mm.5-7), which opens with a pair of discrete 3-2 sets in the bass followed by a pair of overlapping 3-3 sets. The sets within each pair are inversionally equivalent, but in the latter pair the sets also belong to dual equivalence

Example 4.29. Set pairs in episode A-a (a, b, c, d) and in Morris's equivalence classes (e), mm.5-9. Etude No.4



(e)

ex.	set	pitches	pcs	PSC	PCINT	FB
4.29a	3-2	-29,-19,-16,-4	7,5,8,8	t3v	t30	1t
4.29a	3-2	-14,-3,0	t,9,0	e3	e3	2e
4.29a	3-3	-6,-3,-2	6,9,t	31	31	34
4.29a	3-3	-7,-6,-3	5,6,9	13	13	14
4.29a	3-11	6,11,14	6,e,2	53	53	58
4.29a	3-11	3,8,11	3,8,e	53	53	58
4.29b	3-1	6,7,8	6,7,8	11	11	12
4.29b	3-1	3,4,5	3,4,5	11	11	12
4.29b	3-6	-9,-1,1	3,e,1	82	82	8t
4.29b	3-6	-8,0,2	4,0,2	82	82	8t
4.29c	6-1	3,4,5,6,7,8	3,4,5,6,7,8	11111	11111	12345
4.29c	6-1	-9,-8,-1,0,1,2	3,4,e,0,1,2	17111	17111	189te
4.29d	6-21	-9,-1,1,4,5,7	3,e,1,4,5,7	82312	82312	1248t
4.29d	6-21	-8,0,2,3,6,8	4,0,2,3,6,8	82132	82132	248te
4.29d	4-24	-9,-1,5,7	3,e,5,7	862	862	248
4.29d	4-24	-8,0,2,8	4,0,2,8	826	826	48t
4.29d	4-12	-9,1,4,7	3,1,4,7	t33	t33	14t
4.29d	4-12	-8,0,3,6	4,0,3,6	833	833	28e

classes, as evident in example 4.29e.[39] The two 3-11 sets in the treble are equivalent under transposition as well as Morris's three equivalence classes.

Example 4.29 (b, c, and d) gives three different segmentations of mm.8-9, of the second half of the episode A-a. Each of these segmentations produces pairs of sc members equivalent under either transposition or inversion. In example 4.29b, the treble part contains two linear 3-1 trichords, while the bass contains two 3-6 trichords. The trichords in each pair are transpositionally related and belong to the same PSC, PCINT, and FB equivalence classes. In example 4.29c, the treble and bass divide into two 6-1 hexachords. Although the two sets are transpositionally and inversionally related, they are not equivalent in any of Morris's equivalence classes. Example 4.29d shows collections formed by treble and bass pitches combined. Although the 6-21 hexachords are inversionally related, they are not entirely equivalent in Morris's classes; the third and fourth intervals are reversed, as seen in the PSC and PCINT columns of example 4.29e. This results in differing FB classes as well. Within the pair of 6-21 hexachords lies a palindrome of 4-24 and 4-12 tetrachords. Example 4.29e shows that while the 4-24 sets are transpositionally related, their respective second and third intervals are reversed. The 4-12 sets are inversionally related, and differ in the size of their lowest interval. If, however, the final two pitches in the bass had occurred two octaves higher instead (with the E as pitch 16 instead of -8), the sets in all three pairs of scs in this last segmentation would belong to dual PSC and PCINT classes. But this realization is merely hypothetical. Of the three segmentations of mm.8-9 in example 4.29, only example 4.29b produces pairs of sets that are equivalent in both concrete and abstract dimensions, as equivalent pitch sets and pc sets.

The linking passage between the B- and A'-sections also comprises a small inventory of sets. Following the exchange of subject material between treble and bass in m.59, the treble and bass parts then divide into six different voices, which are notated separately in example 4.30a. Voices 1-5 are named according to their relative position on the two staves, from the top of the upper staff to the bottom of the lower staff. Voice 6 comprises

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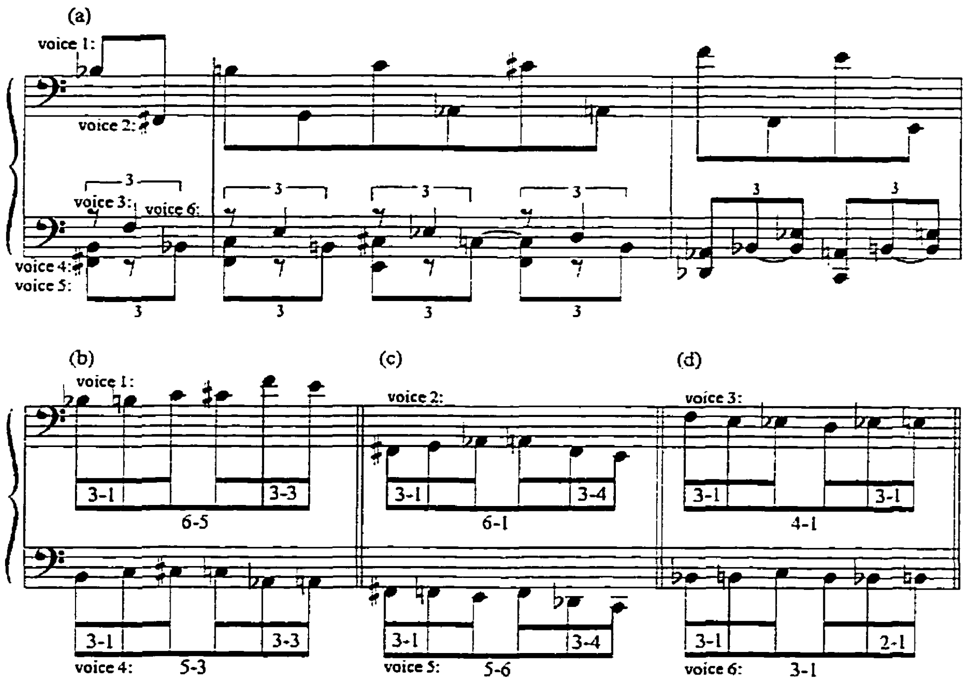
[39] The 3-2 set in the treble is only equivalent under transposition with the first bass 3-2 set, and under inversion with the second bass 3-2 set.

Example 4.30. Notated six-voice texture (a), realigned according to sc correspondence (b, c, d), mm.60-62, Etude No.4



the last eighth notes in the first four triplets, and the second eighth notes in the final two triplets.

All six voices unfold members of different scs, and all but voice 6 contain two discrete trichordal subsets. These trichordal subsets group the six individual voices into related pairs. The six voices open with 3-1 subsets, as illustrated in example 4.30 (b, c, and d). All six 3-1 sets are transpositionally and inversionally equivalent, and belong to the same self-dual equivalence classes. Example 4.30b shows how voices 1 and 4 continue with 3-3 trichords, which are inversionally related and belong to dual equivalence classes. In example 4.30c, voices 2 and 5 continue with 3-4 trichords, which are tranpositionally related and belong to the same equivalence classes. In example 4.30d, however, voices 3 and 6 continue with differing collections (sets 3-1 and 2-1 respectively). Although these collections do not form analogous relationships established by the other voice pairs, they do bring an elegant conclusion to the B-section in that they represent miniature passing and neighbour motions, which occur throughout this entire section.

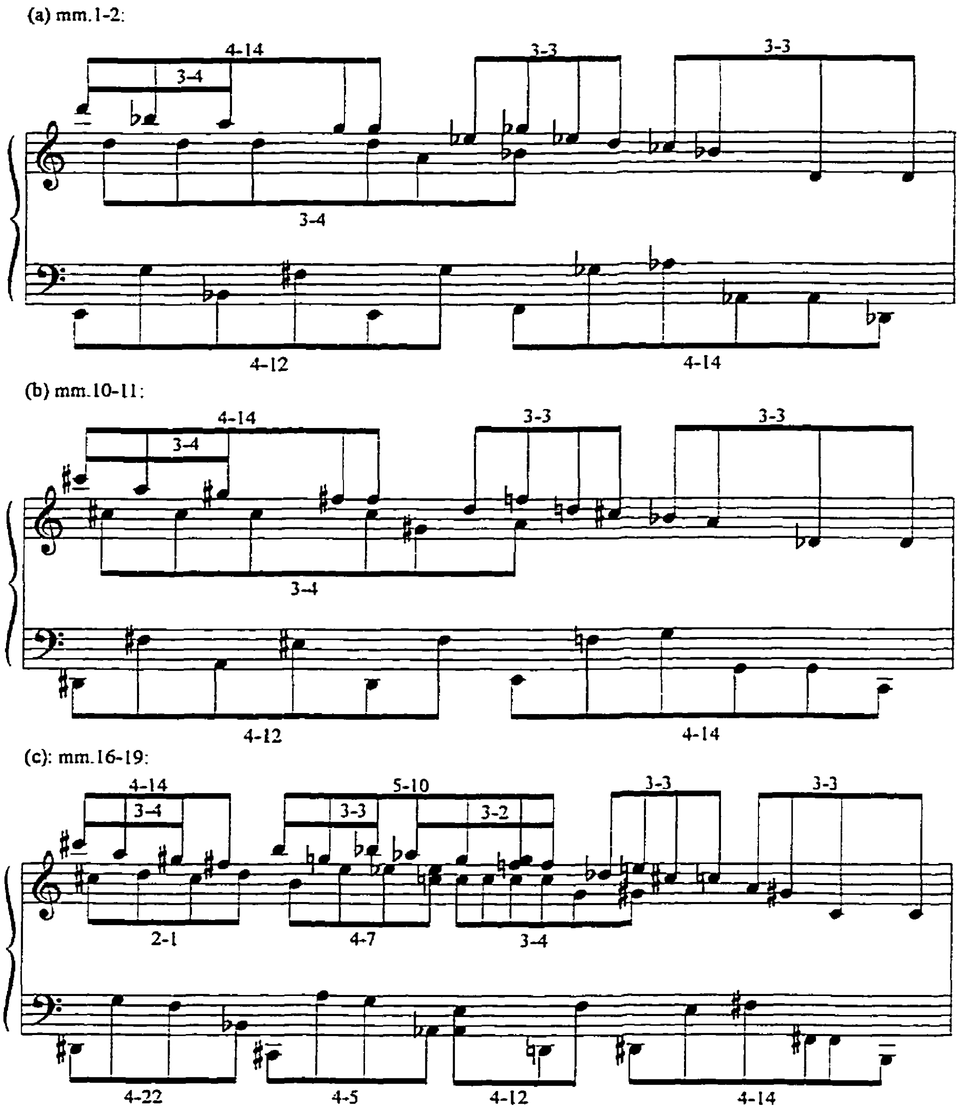
The process of mutation is a distinctive feature of Etude No.4, as discussed above from the perspective of twelve-tone tonality. As previously defined, mutation involves the formation of a new gesture from a systematically modified version of an earlier gesture or gestures, and may involve reordering of pitch or pc elements, multiple transpositional levels, truncation, or extension. The set-theoretical perspective provides another view of the varying effects of mutation. In some passages, mutated sets change into sets of different scs; in other passages, mutated sets remain in the same scs, despite the change of pc content. In still other passages, both effects may occur, either separately or simultaneously.

Example 4.31 illustrates the effects of mutation on the third statement of the subject in the A-section (mm.16-17, given in example 4.31c).[40] In order to conform to the transpositional pattern established between the first two statements of the subject (T_{P-1}), the upper treble voice in m.16 should proceed at T_{P-2} of the opening statement. Instead, the upper treble voice restates scs 4-14 and 3-4, scs associated with the previous statements of the subject, at the same transpositional level of subject A-b (m.10). The two members of sc 3-3, also associated with the subject, do follow at T_{P-2}, but are delayed by an interpolated 5-10 set. This pentachord juxtaposes a 3-3 trichord with a 3-2 trichord, the latter of which is the T_{P-1} transposition of the last three notes of the preceding 4-14 set. From the onset of this 3-2 trichord, the subject proceeds uninterrupted in the upper treble voice at the expected transpositional level of T_{P-2}

Mutation also delays the scs associated with the subject in the lower treble (3-4) and the bass voices (4-12 and 4-14). Instead, both voices in m.16 alternate pitches drawn from their counterparts in m.1 and m.10. This alternation results in a 2-1 dyad which expands into a 4-7 tetrachord in the lower treble voice, and in tetrachord 4-22 in the bass, whose discrete dyads expand outward to form a member of sc 4-5. The delayed scs from the subject then follow at T_{P-2}, although the bass's 4-12 tetrachord is abbreviated. The repeat of the subject proceeds in mm.20-21 at T_{P-2} in its original form.

[40] Example 4.11 and the accompanying discussion describes this passage in the context of array segment realization.

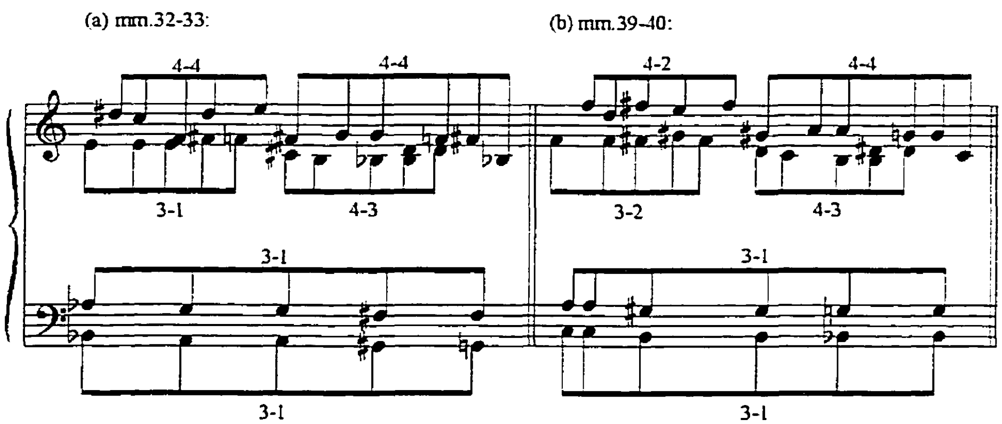
Example 4.31. Statements of subject A-a in mm.1-2 (a), subject A-b in mm.10-11 (b), and mutated subject A-c in mm.16-19 (c), Etude No.4



Example 4.32 illustrates that the opening of phrase group B-e is a mutated repeat of the opening of phrase group B-d. In example 4.12 above, each of the individual voices in mm.39-40 was shown to transpose the pitches of its counterpart in mm.32-33 at alternating levels of T_{P2} and T_{P1}. The transpositions occur simultaneously only in the voices of the bass part. Thus, although the mutation results in differing vertical dyads in the bass (from interval t to interval 9), the linear trichords remain in the same 3-1 sc. In

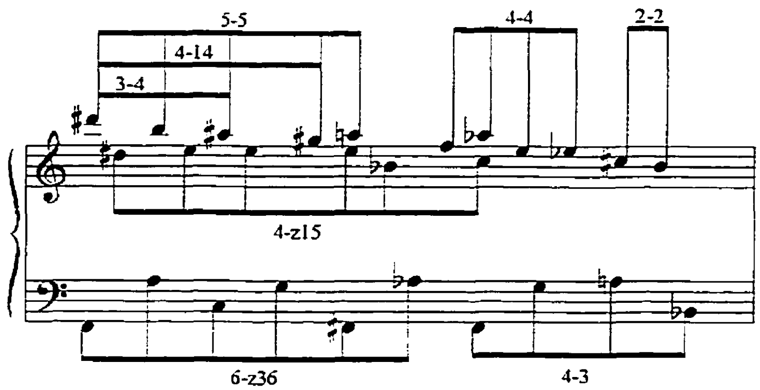
the treble part the transpositional levels do not alternate simultaneously between the voices; they also do not conform to the boundaries of the pc set segmentations. Nonetheless, the mutation process causes similar results in both voices. The first sets in both voices in m.39 belong to different scs than those of m.32; the second sets belong to the same scs as their counterparts in m.33, and are equivalent under transposition and Morris's three equivalence classes.

Example 4.32. First phrase of phrase group B-d, mm.32-33 (a) and its mutated repeat in the first phrase of phrase group B-d, mm.39-40 (b), Etude No.4



In a similar process, the mutated repeat of the A'-section's second subject and episode at m.78 involves the derivation of pitches from mm.63-64 and 72-73 in alternation, although in non-transposed form (see example 4.15 and accompanying discussion). The patterns of alternation do not correspond with the pc set boundaries in the bass and lower treble voices; thus the mutation process results in the formation of different scs, as shown in example 4.33. Specifically, the bass's 4-12 and 4-14 tetrachords mutate into 6-z36 and 4-3 collections respectively, while the lower treble's 3-4 trichord mutates into a 4-z15 collection. The upper treble voice, however, retains its familiar 4-14 set and 3-4 subset, since it repeats the first four pitches from its counterpart in m.72. In preceding statements of the subject in the upper treble voice, the fifth pitch is tied to the fourth in the upper treble voice. But in this mutated repeat the fifth pitch is transposed (in that it derives from m.63 rather than m.72), thereby creating a member of a new sc, 5-5, which acts as a

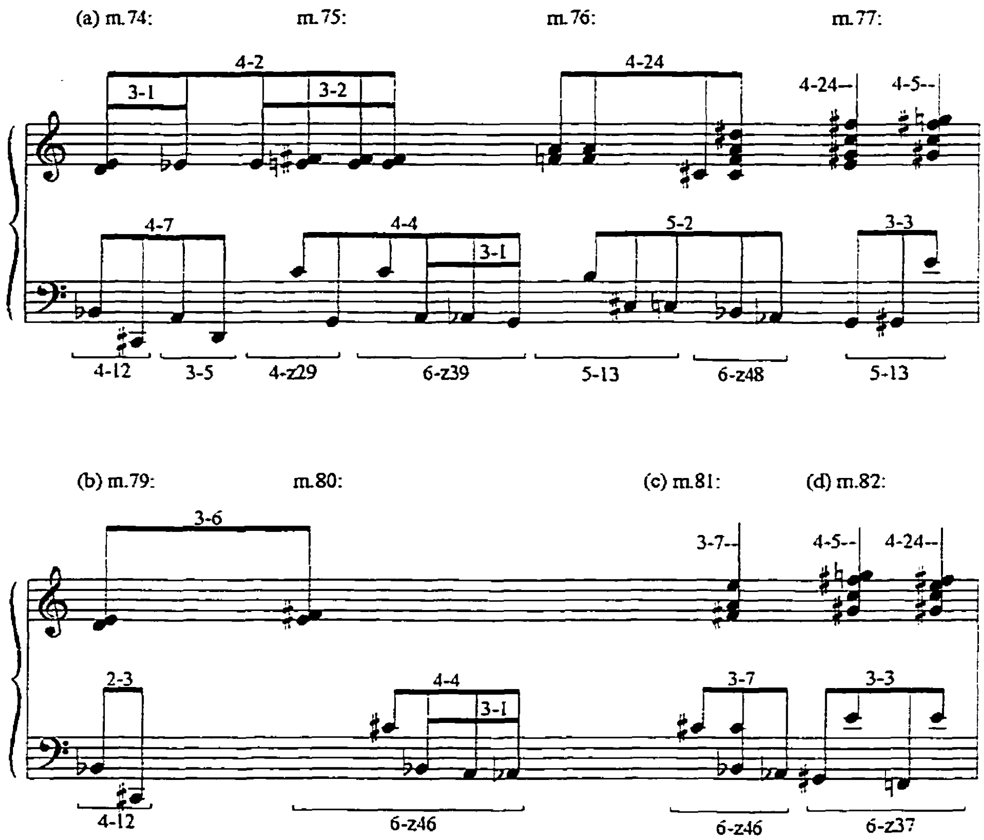
Example 4.33. Mutated repeat of subject A'-b', mm.78-79, Etude No.4



superset to the 4-14 and 3-4. As the pattern of alternation continues, the remaining segments in the upper treble voice also mutate into other scs: the 4-4 and 2-2 collections replace the expected pair of 3-3 trichords.

Example 4.34 compares episode A'-b' of mm.74-77 with its mutated repeat in mm.79-82. The mutation involves a variety of processes: truncation, transposition and invariance in combination, and reordering. First, example 4.34b shows how the pitch material in mm.79-80 is a truncated version of that in mm.74-75 (example 4.34a). The treble 4-2 tetrachord in m.74 contracts into 3-6 trichord in m.79 with the omission of the $E^{\flat}$. The bass retains the first two pitches and omits the next four, resulting in a 2-3 dyad instead of the original 4-7 tetrachord. As well, the 4-12 tetrachord that encompasses both treble and bass remains intact, although the 3-5 and 4-z29 collections disappear with the omission of the bass notes. The subsequent 4-4 set and 3-1 subset are retained in the bass, although transposed by T_{PI}. This transposition changes the combined treble and bass collection (from the 6-z39 hexachord in m.75 to the 6-z46 hexachord in m.80).

Example 4.34. Episode A'-b', mm.74-77 (a) and its truncated repeat, mm.79-82 (b, c, d), Etude No.4



In its mutated repeat of m.76, m.81 omits the first three bass notes and the treble dyads, as illustrated in example 4.34c. In addition, Perle repositions the treble part's C♯ in m.76 to the bass part in m.81; this results in a member of sc 3-7 in the treble. This trichord comprises only one invariant pitch (A) from its counterpart in m.76; the other two pitches (F and D♯) are transposed by T_{P1} in m.81. At the same time, the bass part's 5-2 pentachord from m.76 contracts into another member of sc 3-7 in m.81.[41] Further, the collection encompassing both treble and bass at the end of m.76 also changes from hexachord 6-z48 to 6-z46, due to the partial transposition of the treble verticality. Thus, the mutation process results in two discrete 6-z46 hexachords in mm.80-81; five of the six pitches are invariant, while the sixth pitches belong to the same pc (pc4).

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[41] The two 3-7 sets are equivalent only under transposition.

Example 4.34d shows that the individual treble and bass scs of m.77 are retained in m.82. The treble tetrachordal sets occur in reverse order, however, with the lowest pitch E in m.77's set 4-24 transposed up an octave in m.82's set 4-24. The bass 3-3 set is replaced by an inversionally equivalent 3-3 set which holds two of the three pitches invariant. The component pitches are reordered so that the invariant pitches occur before the new pitch, to simulate the reordering of the sets in the treble part. The single changed pc in m.82 results in a new superset (6-z37) encompassing the invariant treble and bass scs.

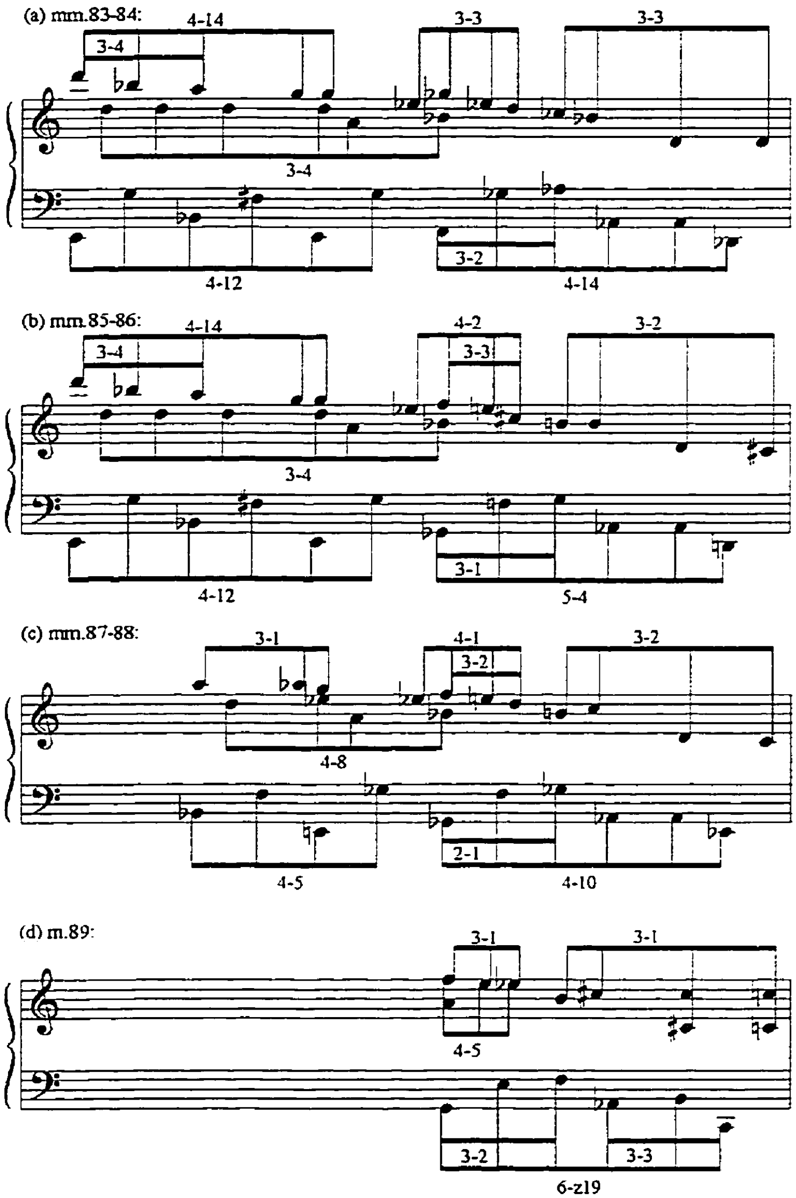
In the final subsection of Etude No.4 (mm.83-89), the mutation of subject A'-c' involves truncated, modified repeats of transposed and invariant pitch collections in a systematic combination. From the perspective of twelve-tone tonality, these processes are described above in both abstract and concrete dimensions: figure 4.13 interprets the passage as the reinterpretative modulation of axis-dyad chords followed by sum tetrachords with substituted neighbour dyads, while example 4.16 realizes the altered sum tetrachords as long-range passing gestures. The pc set segmentation of the same passage in example 4.35 below shows the formation of new scs with each mutated repeat of the subject. The new scs result from the combination of invariant and variant pitches: in each new sc at least one pitch is retained from its corresponding sc in the immediately preceding statement, while the remaining pitches are transposed by some combinations of T_{P1} and T_{P-1}.[42]

The treble part shows the systematic effects of mutation in its succession of trichordal sets in the second half of each subject statement (mm.84, 86, 88, and 89). Each statement unfolds a pair of trichords either from the same sc, or from different scs that share similarity relations.[43] The succession in the bass part involves a more varied collection of scs. The difference in their cardinalities precludes any equivalence or similarity in terms

[42] The mutation process generates new scs with each mutated repeat in this final passage, with one exception: the mutation of the final treble trichord in m.86 generates an inversionally related member of the same sc 3-2 in the corresponding treble trichord in m.88.

[43] The scs 3-3 and 3-2 (between mm.84 and 86, and mm.86 and 88) are R_1-related; both of the sc pairs 3-2 and 3-1 (mm.88 and 89) and 4-2 and 4-1 (mm.86 and 88) are R_2-related.

Example 4.35. Subject A'-c', mm.83-84 (a) and its mutated repeats in mm.85-86 (b), mm.87-88 (c), and m.89 (d), Etude No.4



of R relations among the scs, yet they can be shown to be fairly similar to the original sc 4-14 by utilizing the IcVSIM measure.[44]

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[44] The IcVSIM values for each pair are: 4-14 / 5-4 (0.943), 4-14 / 4-10 (0.816), and 4-14 / 6-z19 (0.957).

Hence, despite the seeming diversity in this passage created by the mutation process, the scs form a relatively homogeneous collection in several dimensions: in the concrete realization at the musical surface, and in the abstract dimensions of array segments and pc scs. Moreover, mutation has been shown to be a significant compositional factor throughout this etude; pc set theory offers different yet complementary insights into its multifaceted features.

* * *

Perle composed his *Six Etudes for Piano* within the system of twelve-tone tonality; he clearly did not avail himself of the tools of pc set theory in their composition. On the contrary, Perle spurns pc set theory, stating: "My rejection of Forte's system . . . is based, as I've said, on the fact that I find the system irrelevant to my experience as a composer, to my perceptions as a listener, and to my discoveries as an analyst" (1990b, 152). Perle's objections are manifold: he disapproves of the important status accorded the z-relation, stating that identical interval vectors is not a sufficient or relevant measure of relatedness between sets. In a similar vein Perle rejects the concept of abstract complementation, due to the implausibility of its aural perception. Perle's most strenuous objections are directed at the recognition of equivalence of pc sets solely on the basis of transposition or inversion, regardless of ordering, register, contour, or repeated notes. But Morris's equivalence classes are designed to address this contentious issue, allowing the analyst to distinguish degrees of equivalence based on pitch realizations of pc sets of the same sc in specific contexts.

Perle's objections to more abstract, less audible relationships could well be directed at his own theory of twelve-tone tonality, aimed toward the more hierarchical levels of structure such as the synoptic arrays or the three types of tonality. Yet in an earlier article, Perle himself expresses a less judgmental attitude: "An analysis is an interpretation, and the concept of a single 'correct' analysis is as fallacious as the concept of a single 'correct' performance. There is no way to categorize and delimit absolutely and unequivocally the collection of notes that we are to take as compositionally articulated musical entities, or to define, in the relations that we assert among these collections, the precise boundary between reasonable and 'top-knot' [i.e. spurious] connections" (1982, 377). This statement weakens Perle's later arguments against the more abstract relationships of pc set theory, and paves the way for the analyses undertaken in the present chapter.

It is my belief that the two theories need not be mutually exclusive. On the contrary, pc set theory may prove to be a potent analytical ally. An analysis which considers several perspectives may offer a variety of insights into the work, strengthening some assertions and spawning others. This was the aim of the present chapter. Although detailed and compelling examinations of Etudes No.1 and No.4 may be generated entirely within the context of Perle's theory, the set-theoretical insights complement, supplement, and enhance the conclusions reached from the perspective of twelve-tone tonality. Together, the two perspectives allow the analyst to develop a more profound understanding of the music.

Chapter Five

Conclusions

The propagation of the theory of twelve-tone tonality

The foundations of the present study are laid in chapter one, beginning with the discussion of Perle's analytical studies of twentieth-century works. Perle sees himself as continuing in a tradition established by composers who employ interval cycles and inversional symmetry. Other theorists have also pursued this approach to analysis, as summarized in the chapter. As well, chapter one presents some investigations of interval cycles from a theoretical rather than analytical perspective, specifically in the work of Robert Morris (1987) and John Clough (1979-80), thereby laying the groundwork for the independent investigation of the properties of cyclic sets in chapter three. Finally, chapter one reviews other representations of Perle's theory of twelve-tone tonality, which chapter two takes as its point of departure.

The exposition of the theory of twelve-tone tonality in this study differs in a number of ways from Perle's own exposition. Chapter two reorganizes the topics, presenting them in an entirely different order. The chapter omits some topics, shortens some explanations while expanding others, and consolidates still others. Chapter two also includes the latest additions to the theory, as presented in the second edition of *Twelve-Tone Tonality.*[1]

In unfolding his theory Perle makes distinctions within certain topics; I have omitted many of these since they did not prove essential to the present study. These include *interval-type* and *sum-type* axis-dyad chords (1977b, 119-22), as well as *tonic* sum tetrachords (110-22) and *tonic* cyclic chords (69-72, 110-22).[2] Chapter two of the present

[1] These latest additions include the topics of *consonant and dissonant figuration* (1996, 229-34), *background and foreground sums and cycles* (235-40), and *synoptic arrays* (192-97).

[2] Interval-type and sum-type axis-dyad chords duplicate cyclic intervals and tonic sums, respectively, in

study also omits discussion of *tonic* and *resultant set forms* (47-50) and *derived sets* (69-72).[3] In the preface to the second edition of *Twelve-Tone Tonality* Perle comments that these topics are part of a "comprehensive and logical exposition of the system, [but] do not . . . play a role in the conscious compositional process" (1996, xiv). Also excluded from consideration in this study is the topic of *triadic arrays*, which are formed from three cyclic sets rather than two. Perle devotes a chapter of *Twelve-Tone Tonality* to triadic arrays (1977b, 152-161), but notes that "presumably it would be possible to work out the implications of triadic sets in terms of analogies with dyadic sets, with their respective sum and difference tables, master modes and master keys, tonic and resultant set forms [i.e., tonic and resultant cyclic sets], derived sets, and so on. Compositional experience to date with triadic sets has not led in the direction of this intimidating prospect" (155).

In the first edition of *Twelve-Tone Tonality* Perle also distinguished between *master modes* (87-94) and *master keys* (95-106), which refer to difference and sum alignments of arrays within the master array (99). Chapter two of this dissertation does not present the concept of master arrays at this level of detail, however, since the distinction does not affect the master array designation, nor is it a factor in the analyses of the etudes in chapter four.

Perle presents many of the more abstract topics in his theory from the perspective of *difference scales* and *sum scales*, configurations that align two transpositionally related semitonal scales to yield the same vertical dyadic intervals or two inversionally related semitonal scales to yield the same vertical dyadic sums, respectively (80-81, passim). As described in chapter two of this study, when cyclic sets are combined in arrays they produce secondary differences or sums in their vertically aligned dyads. Perle shows how these dyads originate in difference or sum scales. Perle then demonstrates how different

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their vertical dyads. Tonic sum tetrachords and tonic cyclic chords duplicate both tonic sums and cyclic intervals.

[3] Cyclic sets that when combined in an array generate tonic axis-dyad chords are called tonic set forms (cyclic sets). Cyclic sets that generate non-tonic axis-dyad chords when combined in an array are called resultant set forms (cyclic sets). Derived sets are the generators of axis-dyad chords formed from different cyclic intervals. Derived sets may also be distinguished as tonic derived sets and resultant derived sets.

arrays may belong to the same modes, keys, and master arrays on the basis of shared origins in difference- or sum-scale alignments. Perle also discusses the implications for modulation between arrays by shifting the component scales within the difference or sum scales. Hence Perle uses difference scales and sum scales to present the same material from a different orientation. For purposes of clarity and brevity the present study omits this alternate perspective of twelve-tone tonality.

Perle makes very few changes to the original material in his second edition of *Twelve-Tone Tonality.* Instead, he elects to add eleven new chapters at the end of the first edition (as chapters 31-41). Some of these later chapters are intended as "postscripts" to earlier chapters, in that they develop or expand the earlier topics. For example, while Perle discusses modulation in the first edition (1977b, 36-39, 55-58, 123-132) it is not until the second edition that he formally identifies and presents the two modulatory methods as *modulation through substitution* and *modulation through reinterpretation* (1996, 198-223). In addition, Perle introduces the topic of *synoptic key* to balance that of *synoptic mode* (1996, 195-97), which had been labelled *master array* in the first edition (1977b, 99). This dissertation explores in detail the topic of synoptic key beyond Perle's own discussion; it establishes principles governing symmetrical consistency across all the parameters of different arrays within a single synoptic key. This topic has not been investigated in any of the other expositions of twelve-tone tonality. It is pursued in the present study since it is a significant feature in the design of Etude No.4, as discussed in chapter four.

Perle revises his view of mode and key in the postscript chapter 33 of the second edition of *Twelve-Tone Tonality.* In the earlier chapters 9 and 12 mode and key are determined as the difference between an array's secondary differences and as the sum of an array's secondary sums, respectively (1977b, 29-30, 45-46). In the discussion of mode and key in chapter 33, however, Perle shifts the emphasis to the more fundamental relationship between the array's cyclic sets (1996, 183-185).

Perle's discussions of topics in the postscript chapters sometimes conflict with related discussions in earlier chapters. In chapter 6 Perle first explains how to name cyclic sets using the p- and i-designations (1977b, 21). Although he gives precise instructions on the assignment of the prefixes p- and i-, Perle states that the order in which they are presented is inconsequential; hence, p0p7 and p7p0 refer to the same interval-7 cyclic set.[4] In chapter 33, however, Perle asserts a specific ordering of the tonic sums so as to generate the cyclic intervals consistently (1996, 183).

A more serious conflict emerges in Perle's revised discussion of his conception of tonality in the second edition. In chapter 28 of both editions Perle determines tonality in each cyclic set of an array based on the individual cyclic set sums (1977b, 145-46; 1996, 140-41). In the newly added chapter 33 of the second edition, he determines the tonality of the array as a whole, according to the aggregate sum of the tonic sums (1996, 190-91). These two approaches lead to differing designations of tonality. In his discussion in chapter 28 of the final passage in Etude No.4, Perle follows the first approach (see appendix four, mm.83-89, of the present study); he states that while the tonic sums in the arrays change systematically, the pairs of oppositely aligned sums in the arrays each retain sums 11 and 1. He concludes that the oppositely aligned tonic-sum pairs are each in Tonality 1. The analysis in chapter four of this dissertation follows Perle's second definition (as given in chapter 33) in determining the tonality of each array in the passage; the aggregate sums of the arrays all retain the sum 0, thereby placing them all in Tonality 0. Although Perle acknowledges in the preface of the second edition that much of the material is restated and reformulated, he does not note any contradictions that arise, nor does he substantially revise the earlier chapters to correspond more closely with the related postscript chapters.

Chapter three steps outside the context of twelve-tone tonality in its theoretical exploration of the nature of cyclic sets. An imbrication process segments the cyclic sets of each ic into assocations of sc families called *ICS families*. The scs within each of the six ICS families share a number of properties due to the origin of the scs as segments of the symmetrical configurations of the cyclic sets: common cyclic origin, similar icvs, inversional symmetry in scs of even-integer cardinalities, transpositional combination in scs of cardinalities greater than three, and equivalence in other modular universes.

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[4] In chapter two, n.6, I explain why I disagree with Perle's contention that ordering of the tonic sums in the array name is immaterial.

In addition, chapter three views the property of transpositional combination from another perspective, as a symmetrical expansion of a pc set segmented from the cyclic set. This study employs as operands a sc belonging to a specific ICS family and its generating cyclic ic in each TC operation.

In the discussion of similarity relations among the scs in the ICS families, chapter three also presents Forte's R_1 and R_2 relations from a different perspective: scs in either of these similarity relations possess a *symmetrical* relationship between the variant entries in their icvs. The chapter accordingly introduces another category of similarity relation, the *R_{SYM} relation*, which broadens the definition of similarity to include scs with pairs of variant entries in symmetrical relationships in their icvs. The particular ics containing these variant entries in the icvs correspond to the ICS families to which they belong, thereby providing another means of establishing relations within and among ICS-families.

The chapter also recognizes another form of pc set equivalence between ICS families, based on the mapping of scs into scs in other modular universes. All the R_1-related scs between the ICS-1 and ICS-5 families map into the same scs in the same modular universes, thereby extending the isomorphic relation between the two families. For each of the remaining ICS families, those mappings that correspond to the partition categories of the scs constitute a "best fit" in other modular universes.

Chapter three demonstrates the close associations among the scs resulting from imbrication of the cyclic sets. In particular, the chapter highlights the isomorphic relations that obtain between the scs of the ICS-1 and ICS-5 families in each of the properties of inversional symmetry, transpositional combination, R similarity relations, and equivalence in other modular universes. Thus, the symmetrical structure of the interval cycles has far-reaching implications, affecting relationships formed between and among cyclic-based configurations and establishing compelling relationships in several dimensions. Chapter three has explored these relationships from a purely theoretical perspective; conversely, chapter four returns to the context of Perle's theory of twelve-tone tonality in its analyses of the first and fourth etudes from Perle's *Six Etudes for Piano*. Hence chapters three and four considered together form a bifurcated structure, representing respectively the speculative and analytical branches of thought in this study.

The analytical methods in chapter four differ in substantial ways from those used by other writers. Both Carrabré (1993) and Antokoletz (1992, 434-47) provide analyses of excerpts of Perle's twelve-tone tonal compositions. Carrabré's short analyses of numerous works by Perle complement his chronological exposition of the theory. Headlam analyzes the third movement of Perle's *Piano Concerto I* in varying levels of detail (1995), while Rosenhaus aims to show the large-scale harmonic motion in each of the four movements in Perle's *Wind Quintet No.4* (1995). The analyses in the present study show both local and long-range structure and account for every pitch in each of the two etudes. The analyses differentiate between the abstract dimension of arrays and array segments and their concrete realization at the musical surface. The study also suggests some strategies that may have guided the compositional process. Further, the analyses view the etudes from two different perspectives, those of twelve-tone tonality and of pc set theory. While Headlam also turns to pc set theory, he uses it simply to classify note groups as pc sets for easier reference. Conversely, the analyses in the present study actively employ set-theoretical techniques and conventions, coupled with Morris's extensions of the concept of equivalence within a sc (1995b). This multifaceted analytical approach highlights distinctive features of the etudes. These include the many palindromic formations in Etude No.1 and the mutated passages in Etude No.4.

The analyses in chapter four also demonstrate that although the etudes were composed entirely within the context of twelve-tone tonality, they display traditional formal designs. Both etudes are tripartite in structure with clearly defined sections and subsections, and both utilize conventional techniques of motivic development of the twelve-tone tonal material. In addition, Perle's use of transposition, inversion, retrograde, rotation, and symmetrical progressions reflect his analytical studies of other composers' works, thus strengthening his claim of continuing in a lineage of compositional tradition.

Critical reception of Perle's theory of twelve-tone tonality

Upon its publication Perle's book *Twelve-Tone Tonality* was greeted with mixed reviews. The negative criticisms mostly address three main issues: intelligibility of the text, guidelines for analysis, and musical communication or audibility of relations. Most of the reviewers consistently criticize the text for its dense, terse prose. Jonathan Dunsby complains of the lack of "narrative flow" and any "substantial connective commentary" (1979, 364). Reviewers comment that the definitions are sometimes ambiguous and often inappropriately placed. Martha MacLean states that, as a result, readers may form unfounded assumptions, forcing them to backtrack and reassess earlier information (1979, 123). Stefan Ehrenkreutz argues that definitions should be self-sufficient; in *Twelve-Tone Tonality* they are often difficult to understand outside the given context (1979, 34). Bo Alphonce notes also that Perle does not sufficiently warn the reader that he uses established terms in unconventional ways (1982, 181). Most reviewers agree that Perle's "Index to Basic Definitions" is woefully inadequate, suggesting that a full index and glossary would have been very helpful. Although Perle utilizes boldface font to highlight key points throughout the text, some reviewers think the practice was overused and frequently extended to relatively superficial statements while more crucial information remained in plain text (Dunsby, 1979, 364; Ehrenkreutz, 1979, 34). Mark DeVoto appears to stand alone in finding the text "simply and clearly written." Although he acknowledges the theory itself is not easily comprehended, DeVoto attributes the source of the difficulty to Perle's use of relatively new dodecaphonic terms in unfamiliar ways (1978, 295).

Perle responds to some of these criticisms in the second edition of *Twelve-Tone Tonality*. In reply to a reviewer's statement that Perle's book is "very tough reading," Perle argues that

> it is not, however, a book for 'reading' in any ordinary sense, any more than a traditional harmony textbook would be. I address myself to the concerns of both composers and theorists, but these do not always coincide. The composer working in the traditional major-minor system did not commence with a Schenkerian graph and proceed to elaborate therefrom a composition from its *Urlinie* through successive middleground and foreground details, ultimately deriving, in the final stages of creation, the principal motives and rhythmic patterns (1996, xiii).

This response neatly sidesteps the original charge; it ignores the fact that theoretical writing should be as clear as possible in order to communicate the ideas therein, despite the topic's complexity. The present study, therefore, takes the goals of intelligibility and accessibility as justification for the omission, expansion, and reordering of the tenets of twelve-tone tonality in chapter two.

Several of the reviewers focus on the need for clear guidelines in determining arrays and array segments. Alphonce observes that Perle does not address directly the issue of segmentation (1982, 203-4). MacLean also questions the basis of succession between axis-dyad chords in a single array, since Perle does not avail himself of the conventional means of harmonic connection, such as invariant subsets, complement relations, and total intervallic content (1979, 125). Chapter four of the present study suggests a number of potential strategies for connecting array segments within a single array. Ehrenkreutz discusses what he deems another problematic area, that of pitch symmetry relations. Ehrenkreutz writes: "Perle's use of pitch symmetry is further complicated because his system is, for the most part, abstractly defined. Among other things, this means that pitch and interval relations are defined in terms of PC's and IC's" (1979, 35). Chapter four of the present study addresses this conflict as well: the chapter distinguishes between abstract entities and events and their concrete realization, it illustrates pitch and interval relations in the musical examples, and it employs Morris's contextual equivalence relations in its set-theoretical observations.

The third category of negative criticism concerns the audibility of the relations stressed by Perle within twelve-tone tonality. Similar to Ehrenkreutz's observations regarding pitch symmetry, Alphonce writes that once Perle extends inversional symmetry beyond pitches to pcs, the relation is no longer audible. Alphonce directly addresses Perle's stated goal of establishing a normative rather than reflexive precompositional reference (1982, 202-5). Alphonce acknowledges that structures do not need to be audible in order to serve as foundations on which to erect other structures closer to the surface. But if these substructures are intended to serve as elements of a normative precompositional system then the listener must be able to recognize the patterns that trigger the frame of reference. Alphonce doubts that this is possible:

> From the analytical point of view it [*Twelve-Tone Tonality*] is one giant hypothesis. Either it takes for granted that twelve-tone tonality is available as a shared precompositional norm, or it assumes the existence of extraordinary powers of instant pattern recognition in the human brain. The latter assumption offers a challenge to Articifial-Intelligence research; the former needs testing in actual listening. My impression is that this system, despite its impressive coherence and aesthetic attractiveness, does not in itself guarantee an optimal balance between the two principles of precompositional norm and reflexive reference (205).

On the other hand, the same reviewers find many aspects of *Twelve-Tone Tonality* to be praiseworthy. The book contains a wealth of musical examples accompanied by provocative commentary. Perle's theory is hailed as a rich and innovative system, exhaustive in its detail. MacLean acknowledges Perle's effort to appeal to composers, theorists, analysts, and historians (1979, 123). Dunsby concludes that the theory shows "outstanding intellectual vision" (1979, 366). Ehrenkreutz views the theory as an example of the possibilities that exist for constructing a system that may rival tonality in its hierarchical structure and referential qualities (1979, 33). Perhaps the most effusive praise is bestowed by Alphonce, who writes that "the precompositional system Perle suggests represents a major intellectual achievement and reveals insights into and a mastery of twelve-tone relationships that anyone might envy. The mathematical model that represents his pitch-class system exhibits symmetries of breathtaking depth and serene beauty" (1982, 180).

Suggestions for further study

Many of the findings in this study raise questions that invite further investigation, both within and outside the domain of twelve-tone tonality. As discussed above, the question remains unanswered as to whether and to what degree twelve-tone entities and relations are audible, thereby suggesting a potential line of inquiry for researchers in the field of perception and cognition.

Another issue involves the role of intuition in the development of the various facets of twelve-tone tonality in the compositional process. In the chapter "Composing with Symmetries" in *The Listening Composer*, Perle asserts that he never thinks in terms of background structures and models while composing (1990a, 141). In *Twelve-Tone Tonality* Perle provides an analysis of the first movement of his *Modal Suite* for piano. Although he describes the movement in terms of its array, secondary difference alignments, and tonic and non-tonic axis-dyad chords, Perle states in a footnote that he did not consciously exploit these properties during the compositional process, since the piece was written over three decades earlier, in 1940. Rather, he attributes their presence in the composition as inherent in the nature of cyclic-set materials (1977b, 34, n.21). In the aforementioned chapter in *The Listening Composer* Perle offers yet another interpretation of a specific passage in the same movement, based on his recent forays into dissonance, stating that "it is only since dissonant figuration has become an integral component of my compositional language that I have been able to offer what I think is a reasonable explanation [for a specific pitch, now interpreted as an octave-displaced suspension]" (1990a, 161-62). In the corresponding footnote, Perle comments that the previous interpretation provided in *Twelve-Tone Tonality* is quite different and less convincing (162, n.18). Moreover, Charles Porter reports that in his analysis of Perle's "Nocturne" in *Sonata A Quattro* Porter discovered the proportion 5:4 pervading the parameters of tempo, rhythm, duration, phrase, and form at multiple levels of structure (Porter 1995, 217). When Porter presented this finding to Perle, the composer denied consciously employing the proportion, other than in the opening measures' alternating tempi which express a 5:4 relation. Perle contended: "I never consciously work with numerical proportions except as they relate to metronome markings. Whatever other instances of the 5:4 proportion there are had to be intuitive" (219, n.5).

These claims of intuitive, unconscious awareness suggest that investigating earlier compositions for evidence of Perle's more recently developed concepts would be a fruitful endeavor. Such a task may lead the investigator to trace the development of these concepts along an evolutionary path through Perle's music, and to speculate on the degree of influence they may have had, although at a subconscious level, in the compositional process. To a certain extent the present study has begun such an investigation in its consideration of the implications of the synoptic key in Etude No.4, which was composed before Perle had formulated the concept.

Further questions arise pertaining to the contribution the recent additions to Perle's theory may make to its hierarchical structure. Perle states that if incidental sums and differences are observed in a twelve-tone tonal composition, they may imply deeper-level sums and differences resulting from background cyclic sets or arrays (1996, 237). If such background structures are detected, the analyst may wish to determine the extent of their participation and influence at the foreground. The analyst may investigate whether the relations between and among parameters established at the foreground intersect with those at the middleground and background, analogous to a Schenkerian model, or whether they exist as independent structural entities.

A related investigation would determine the extent to which Perle utilizes dissonant figurations. Will they remain as surface interpolations between array segments or will they gradually infiltrate the deeper layers of structure, forming gestures of longer range, thereby adding another dimension to the hierarchical structure of twelve-tone tonality? Together, these questions form part of a larger inquiry into the nature and direction of the theory's continuing evolution.

Perle (1993) has commented on the points of intersection that exist between his theory of twelve-tone tonality and the transformational theories of David Lewin, particularly with the concept of Klumpenhouwer networks (Lewin 1990). Klumpenhouwer networks represent pc sets in terms of the transpositional and inversional relationships among the component elements of the set. These networks are useful in establishing relations among pc sets that are neither transpositionally nor inversionally equivalent. Strongly isographic networks have isomorphic graphs; that is, their graphs express identical T_n and T_nI relationships.Two Klumpenhouwer networks whose T_n values are identical, but whose T_nI values have been transposed by the same amount in one network relative to the other, are said to be positively isographic. Two Klumpenhouwer networks whose T_n values are mod-12 complements, and whose T_nI values have been transposed by the same amount in one network relative to the other, are said to be negatively isographic.

Perle has said that these same isographic networks relationships exist in twelve-tone tonality. He defines a Klumpenhouwer network as "a chord analyzed in terms of its dyadic sums and differences" (1993, 300). In Perle's view, strongly isographic trichordal networks may be expressed as trichordal segments from the same cyclic set, with the T_n value representing the cyclic interval and the T_nI values representing the cyclic set's tonic sums. Positively and negatively isographic networks are analogous to cyclic set segments drawn from cyclic sets in the cognate relation (which are inversionally related and which share a tonic sum).

Perle also observes that the tetrachordal Klumpenhouwer networks Lewin discusses in his article (Lewin 1990, figure 17) are comparable to the sum tetrachords of twelve-tone tonality, which originate in aligned sum scales. He further notes that if the tetrachords being compared in the networks are not symmetrical, then they cannot be found as tetradic segments of a cyclic set; rather, they can be found as segments of derived sets (which combine two alternating cyclic intervals and four tonic sums). In addition, just as Lewin shows the various relations among compared sets' elements by reordering them in different "modes" of interpretation, Perle reorders the segments to belong to different cyclic sets.

Robert Morris (1998) has also observed the connections between Perle's cyclic sets and Lewin's Klumpenhouwer networks. Morris adapts Perle's ic5 cyclic set to serve as an example of a compositional space. Morris separates the inversionally related cycles into two vertically aligned rows, which he calls a *Perle space*. Morris draws triangles to form sets from different scs (of both cardinalities two and three), but which exhibit strong isographies as a result of the symmetrical nature of the cyclic set. The horizontal side preserves the transposition operation, while the vertical and diagonal sides preserve inversional operations. Morris further suggests the possibility of either flipping the triangles (by maintaining a common vertex or side) or moving among the isographic triangles to establish stable relationships among sets of differing scs.[5]

[5] Morris notes that Paul Lansky goes much further along these lines in his Ph.D. dissertation "Affine Music" (1973), which Morris views as a "comprehensive expansion of Perle's twelve-tone tonality using concepts and techniques from matrix algebra and affine geometry" (1998, 193, n.38).

Perle (1993) limits the connections he makes with Lewin's transformational networks to comparisons between Klumpenhouwer networks and segments from cyclic sets, sum scales, and derived sets. Future investigation might extend the transformational relationships within the context of twelve-tone tonality beyond array segments. For example, the dynamic nature of Lewin's transformational graphs suggests that they may be well suited to express systematic progressions among arrays, similar to the motion between arrays discussed in the analyses of etudes one and four in chapter four of the present study. In both etudes arrays transformed into another through various combinations of invariance and transposition.

Another possible application of transformational graphs to Perle's theory of twelve-tone tonality would be through the use of Lewin's "network of networks" (Lewin 1990). Such a network might be constructed to reflect the hierarchical structure of twelve-tone tonality, for example, in terms of arrays, keys, aggregate sums, and tonalities.

Outside the domain of twelve-tone tonality, some of the entities introduced in chapter three suggest several avenues for further exploration. In its contribution to the study of similarity relations the R_{SYM} relation focuses specifically on symmetrically variable icv entries. This invites the analyst to investigate the extent to which this feature manifests itself compositionally, a study perhaps similar in scope to Richard Cohn's investigation into the properties of transpositional composition and inversional symmetry in Bartók's music (Cohn 1987, 1988). Another path of inquiry would pursue the question of hierarchical degrees of similarity within the R_{SYM} relation, based on the degree to which the variable entries differ. Preliminary work in this direction was begun in chapter three of the present study, with the application of Isaacson's IcVSIM values to scs in the R_{SYM} relation and in Forte's R_n relations.

The ICS families evolved through imbrication of cyclic sets. Imbrication applied to other entities of twelve-tone tonality also may generate other types of associations between scs. Potential subjects for imbrication might include combinations of different interval cycles rather than of the complementary cycles of cyclic sets, or perhaps of combinations of cyclic sets as arrays. The assumption is that the resulting associations would be influenced by the specific nature of the combination of the imbricated subjects, such as whether the arrays imbricated were composed of cyclic sets of the same or different ic, or of the same or different partitions of ic cycles, and so forth. It is imaginable that a network of relations may be constructed through the imbrication of these interval-cycle entities. Moreover, each of the ICS families comprises a collection of scs in close association. The notion of shared characteristics grouping scs into families suggests intersection with the field of sc genera theory (Forte 1988, Parks 1998), inviting further investigation.

* * *

This study has taken a multifaceted approach in its presentation of Perle's theory of twelve-tone tonality, with the goal of making the theory more accessible and comprehensible to the theoretical community. The study has aimed to show the depth and potential of the theory, both within and outside its own context. Through the exploration of its fundamental structural entity, the cyclic set, the study has introduced new forms of associations and relations among scs, which may prove relevant and powerful in future theoretical and analytical discourse.

Appendix One

The Cyclic Sets of All Cyclic Intervals (mod 12)[1]

	CYCLIC INTERVAL-0																	
	first partition			second partition			third partition			fourth partition			fifth partition			sixth partition		
p0i0:	0	0	(0	1	e	(1	2	t	(2	3	9	(3	4	8	(4	5	7	(5
i1p1:	0	1	(0	1	0	(1	2	e	(2	3	t	(3	4	9	(4	5	8	(5
p2i2:	0	2	(0	1	1	(1	2	0	(2	3	e	(3	4	t	(4	5	9	(5
i3p3:	0	3	(0	1	2	(1	2	1	(2	3	0	(3	4	e	(4	5	t	(5
p4i4:	0	4	(0	1	3	(1	2	2	(2	3	1	(3	4	0	(4	5	e	(5
i5p5:	0	5	(0	1	4	(1	2	3	(2	3	2	(3	4	1	(4	5	0	(5
p6i6:	0	6	(0	1	5	(1	2	4	(2	3	3	(3	4	2	(4	5	1	(5
i7p7:	0	7	(0	1	6	(1	2	5	(2	3	4	(3	4	3	(4	5	2	(5
p8i8:	0	8	(0	1	7	(1	2	6	(2	3	5	(3	4	4	(4	5	3	(5
i9p9:	0	9	(0	1	8	(1	2	7	(2	3	6	(3	4	5	(4	5	4	(5
ptit:	0	t	(0	1	9	(1	2	8	(2	3	7	(3	4	6	(4	5	5	(5
iepe:	0	e	(0	1	t	(1	2	9	(2	3	8	(3	4	7	(4	5	6	(5
	seventh partition			eighth partition			ninth partition			tenth partition			eleventh partition			twelfth partition		
p0i0:	6	6	(6	7	5	(7	8	4	(8	9	3	(9	t	2	(t	e	1	(e
i1p1:	6	7	(6	7	6	(7	8	5	(8	9	4	(9	t	3	(t	e	2	(e
p2i2:	6	8	(6	7	7	(7	8	6	(8	9	5	(9	t	4	(t	e	3	(e
i3p3:	6	9	(6	7	8	(7	8	7	(8	9	6	(9	t	5	(t	e	4	(e
p4i4:	6	t	(6	7	9	(7	8	8	(8	9	7	(9	t	6	(t	e	5	(e
i5p5:	6	e	(6	7	t	(7	8	9	(8	9	8	(9	t	7	(t	e	6	(e
p6i6:	6	0	(6	7	e	(7	8	t	(8	9	9	(9	t	8	(t	e	7	(e
i7p7:	6	1	(6	7	0	(7	8	e	(8	9	t	(9	t	9	(t	e	8	(e
p8i8:	6	2	(6	7	1	(7	8	0	(8	9	e	(9	t	t	(t	e	9	(e
i9p9:	6	3	(6	7	2	(7	8	1	(8	9	0	(9	t	e	(t	e	t	(e
ptit:	6	4	(6	7	3	(7	8	2	(8	9	1	(9	t	0	(t	e	e	(e
iepe:	6	5	(6	7	4	(7	8	3	(8	9	2	(9	t	1	(t	e	0	(e

[1] Each single parenthesis indicates the completion of the full cycle.

CYCLIC INTERVAL-1

p0p1:	0	0	1	e	2	t	3	9	4	8	5	7	6	6	7	5	8	4	9	3	t	2	e	1	(0
i1i2:	0	1	1	0	2	e	3	t	4	9	5	8	6	7	7	6	8	5	9	4	t	3	e	2	(0
p2p3:	0	2	1	1	2	0	3	e	4	t	5	9	6	8	7	7	8	6	9	5	t	4	e	3	(0
i3i4:	0	3	1	2	2	1	3	0	4	e	5	t	6	9	7	8	8	7	9	6	t	5	e	4	(0
p4p5:	0	4	1	3	2	2	3	1	4	0	5	e	6	t	7	9	8	8	9	7	t	6	e	5	(0
i5i6:	0	5	1	4	2	3	3	2	4	1	5	0	6	e	7	t	8	9	9	8	t	7	e	6	(0
p6p7:	0	6	1	5	2	4	3	3	4	2	5	1	6	0	7	e	8	t	9	9	t	8	e	7	(0
i7i8:	0	7	1	6	2	5	3	4	4	3	5	2	6	1	7	0	8	e	9	t	t	9	e	8	(0
p8p9:	0	8	1	7	2	6	3	5	4	4	5	3	6	2	7	1	8	0	9	e	t	t	e	9	(0
i9it:	0	9	1	8	2	7	3	6	4	5	5	4	6	3	7	2	8	1	9	0	t	e	e	t	(0
ptpe:	0	t	1	9	2	8	3	7	4	6	5	5	6	4	7	3	8	2	9	1	t	0	e	e	(0
iei0:	0	e	1	t	2	9	3	8	4	7	5	6	6	5	7	4	8	3	9	2	t	1	e	0	(0

CYCLIC INTERVAL-e

p0pe:	0	0	e	e	t	t	9	9	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	(0
i1i0:	0	1	e	0	t	e	9	t	8	9	7	8	6	7	5	6	4	5	3	4	2	3	1	2	(0
p2p1:	0	2	e	1	t	0	9	e	8	t	7	9	6	8	5	7	4	6	3	5	2	4	1	3	(0
i3i2:	0	3	e	2	t	1	9	0	8	e	7	t	6	9	5	8	4	7	3	6	2	5	1	4	(0
p4p3:	0	4	e	3	t	2	9	1	8	0	7	e	6	t	5	9	4	8	3	7	2	6	1	5	(0
i5i4:	0	5	e	4	t	3	9	2	8	1	7	0	6	e	5	t	4	9	3	8	2	7	1	6	(0
p6p5:	0	6	e	5	t	4	9	3	8	2	7	1	6	0	5	e	4	t	3	9	2	8	1	7	(0
i7i6:	0	7	e	6	t	5	9	4	8	3	7	2	6	1	5	0	4	e	3	t	2	9	1	8	(0
p8p7:	0	8	e	7	t	6	9	5	8	4	7	3	6	2	5	1	4	0	3	e	2	t	1	9	(0
i9i8:	0	9	e	8	t	7	9	6	8	5	7	4	6	3	5	2	4	1	3	0	2	e	1	t	(0
ptp9:	0	t	e	9	t	8	9	7	8	6	7	5	6	4	5	3	4	2	3	1	2	0	1	e	(0
ieit:	0	e	e	t	t	9	9	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0	(0

CYCLIC INTERVAL-2																										
	first partition													second partition												
p0i2:	0	0	2	t	4	8	6	6	8	4	t	2	(0	1	e	3	9	5	7	7	5	9	3	e	1	(1
i1p3:	0	1	2	e	4	9	6	7	8	5	t	3	(0	1	0	3	t	5	8	7	6	9	4	e	2	(1
p2i4:	0	2	2	0	4	t	6	8	8	6	t	4	(0	1	1	3	e	5	9	7	7	9	5	e	3	(1
i3p5:	0	3	2	1	4	e	6	9	8	7	t	5	(0	1	2	3	0	5	t	7	8	9	6	e	4	(1
p4i6:	0	4	2	2	4	0	6	t	8	8	t	6	(0	1	3	3	1	5	e	7	9	9	7	e	5	(1
i5p7:	0	5	2	3	4	1	6	e	8	9	t	7	(0	1	4	3	2	5	0	7	t	9	8	e	6	(1
p6i8:	0	6	2	4	4	2	6	0	8	t	t	8	(0	1	5	3	3	5	1	7	e	9	9	e	7	(1
i7p9:	0	7	2	5	4	3	6	1	8	e	t	9	(0	1	6	3	4	5	2	7	0	9	t	e	8	(1
p8it:	0	8	2	6	4	4	6	2	8	0	t	t	(0	1	7	3	5	5	3	7	1	9	e	e	9	(1
i9pe:	0	9	2	7	4	5	6	3	8	1	t	e	(0	1	8	3	6	5	4	7	2	9	0	e	t	(1
pti0:	0	t	2	8	4	6	6	4	8	2	t	0	(0	1	9	3	7	5	5	7	3	9	1	e	e	(1
iep1:	0	e	2	9	4	7	6	5	8	3	t	1	(0	1	t	3	8	5	6	7	4	9	2	e	0	(1

CYCLIC INTERVAL-t																										
	first partition													second partition												
p0it:	0	0	t	2	8	4	6	6	4	8	2	t	(0	1	e	e	1	9	3	7	5	5	7	3	9	(1
i1pe:	0	1	t	3	8	5	6	7	4	9	2	e	(0	1	0	e	2	9	4	7	6	5	8	3	t	(1
p2i0:	0	2	t	4	8	6	6	8	4	t	2	0	(0	1	1	e	3	9	5	7	7	5	9	3	e	(1
i3p1:	0	3	t	5	8	7	6	9	4	e	2	1	(0	1	2	e	4	9	6	7	8	5	t	3	0	(1
p4i2:	0	4	t	6	8	8	6	t	4	0	2	2	(0	1	3	e	5	9	7	7	9	5	e	3	1	(1
i5p3:	0	5	t	7	8	9	6	e	4	1	2	3	(0	1	4	e	6	9	8	7	t	5	0	3	2	(1
p6i4:	0	6	t	8	8	t	6	0	4	2	2	4	(0	1	5	e	7	9	9	7	e	5	1	3	3	(1
i7p5:	0	7	t	9	8	e	6	1	4	3	2	5	(0	1	6	e	8	9	t	7	0	5	2	3	4	(1
p8i6:	0	8	t	t	8	0	6	2	4	4	2	6	(0	1	7	e	9	9	e	7	1	5	3	3	5	(1
i9p7:	0	9	t	e	8	1	6	3	4	5	2	7	(0	1	8	e	t	9	0	7	2	5	4	3	6	(1
pti8:	0	t	t	0	8	2	6	4	4	6	2	8	(0	1	9	e	e	9	1	7	3	5	5	3	7	(1
iep9:	0	e	t	1	8	3	6	5	4	7	2	9	(0	1	t	e	0	9	2	7	4	5	6	3	8	(1

CYCLIC INTERVAL-3																											
	first partition									second partition									third partition								
p0p3:	0	0	3	9	6	6	9	3	(0	1	e	4	8	7	5	t	2	(1	2	t	5	7	8	4	e	1	(2
i1i4:	0	1	3	t	6	7	9	4	(0	1	0	4	9	7	6	t	3	(1	2	e	5	8	8	5	e	2	(2
p2p5:	0	2	3	e	6	8	9	5	(0	1	1	4	t	7	7	t	4	(1	2	0	5	9	8	6	e	3	(2
i3i6:	0	3	3	0	6	9	9	6	(0	1	2	4	e	7	8	t	5	(1	2	1	5	t	8	7	e	4	(2
p4p7:	0	4	3	1	6	t	9	7	(0	1	3	4	0	7	9	t	6	(1	2	2	5	e	8	8	e	5	(2
i5i8:	0	5	3	2	6	e	9	8	(0	1	4	4	1	7	t	t	7	(1	2	3	5	0	8	9	e	6	(2
p6p9:	0	6	3	3	6	0	9	9	(0	1	5	4	2	7	e	t	8	(1	2	4	5	1	8	t	e	7	(2
i7it:	0	7	3	4	6	1	9	t	(0	1	6	4	3	7	0	t	9	(1	2	5	5	2	8	e	e	8	(2
p8pe:	0	8	3	5	6	2	9	e	(0	1	7	4	4	7	1	t	t	(1	2	6	5	3	8	0	e	9	(2
i9i0:	0	9	3	6	6	3	9	0	(0	1	8	4	5	7	2	t	e	(1	2	7	5	4	8	1	e	t	(2
ptp1:	0	t	3	7	6	4	9	1	(0	1	9	4	6	7	3	t	0	(1	2	8	5	5	8	2	e	e	(2
iei2:	0	e	3	8	6	5	9	2	(0	1	t	4	7	7	4	t	1	(1	2	9	5	6	8	3	e	0	(2

CYCLIC INTERVAL-9																											
	first partition									second partition									third partition								
p0p9:	0	0	9	3	6	6	3	9	(0	1	e	t	2	7	5	4	8	(1	2	t	e	1	8	4	5	7	(2
i1it:	0	1	9	4	6	7	3	t	(0	1	0	t	3	7	6	4	9	(1	2	e	e	2	8	5	5	8	(2
p2pe:	0	2	9	5	6	8	3	e	(0	1	1	t	4	7	7	4	t	(1	2	0	e	3	8	6	5	9	(2
i3i0:	0	3	9	6	6	9	3	0	(0	1	2	t	5	7	8	4	e	(1	2	1	e	4	8	7	5	t	(2
p4p1:	0	4	9	7	6	t	3	1	(0	1	3	t	6	7	9	4	0	(1	2	2	e	5	8	8	5	e	(2
i5i2:	0	5	9	8	6	e	3	2	(0	1	4	t	7	7	t	4	1	(1	2	3	e	6	8	9	5	0	(2
p6p3:	0	6	9	9	6	0	3	3	(0	1	5	t	8	7	e	4	2	(1	2	4	e	7	8	t	5	1	(2
i7i4:	0	7	9	t	6	1	3	4	(0	1	6	t	9	7	0	4	3	(1	2	5	e	8	8	e	5	2	(2
p8p5:	0	8	9	e	6	2	3	5	(0	1	7	t	t	7	1	4	4	(1	2	6	e	9	8	0	5	3	(2
i9i6:	0	9	9	0	6	3	3	6	(0	1	8	t	e	7	2	4	5	(1	2	7	e	t	8	1	5	4	(2
ptp7:	0	t	9	1	6	4	3	7	(0	1	9	t	0	7	3	4	6	(1	2	8	e	e	8	2	5	5	(2
iei8:	0	e	9	2	6	5	3	8	(0	1	t	t	1	7	4	4	7	(1	2	9	e	0	8	3	5	6	(2

CYCLIC INTERVAL-4

	first partition							second partition							third partition							fourth partition						
p0i4:	0	0	4	8	8	4	(0	1	e	5	7	9	3	(1	2	t	6	6	t	2	(2	3	9	7	5	e	1	(3
i1p5:	0	1	4	9	8	5	(0	1	0	5	8	9	4	(1	2	e	6	7	t	3	(2	3	t	7	6	e	2	(3
p2i6:	0	2	4	t	8	6	(0	1	1	5	9	9	5	(1	2	0	6	8	t	4	(2	3	e	7	7	e	3	(3
i3p7:	0	3	4	e	8	7	(0	1	2	5	t	9	6	(1	2	1	6	9	t	5	(2	3	0	7	8	e	4	(3
p4i8:	0	4	4	0	8	8	(0	1	3	5	e	9	7	(1	2	2	6	t	t	6	(2	3	1	7	9	e	5	(3
i5p9:	0	5	4	1	8	9	(0	1	4	5	0	9	8	(1	2	3	6	e	t	7	(2	3	2	7	t	e	6	(3
p6it:	0	6	4	2	8	t	(0	1	5	5	1	9	9	(1	2	4	6	0	t	8	(2	3	3	7	e	e	7	(3
i7pe:	0	7	4	3	8	e	(0	1	6	5	2	9	t	(1	2	5	6	1	t	9	(2	3	4	7	0	e	8	(3
p8i0:	0	8	4	4	8	0	(0	1	7	5	3	9	e	(1	2	6	6	2	t	t	(2	3	5	7	1	e	9	(3
i9p1:	0	9	4	5	8	1	(0	1	8	5	4	9	0	(1	2	7	6	3	t	e	(2	3	6	7	2	e	t	(3
pti2:	0	t	4	6	8	2	(0	1	9	5	5	9	1	(1	2	8	6	4	t	0	(2	3	7	7	3	e	e	(3
iep3:	0	e	4	7	8	3	(0	1	t	5	6	9	2	(1	2	9	6	5	t	1	(2	3	8	7	4	e	0	(3

CYCLIC INTERVAL-8

	first partition							second partition							third partition							fourth partition						
p0i8:	0	0	8	4	4	8	(0	1	e	9	3	5	7	(1	2	t	t	2	6	6	(2	3	9	e	1	7	5	(3
i1p9:	0	1	8	5	4	9	(0	1	0	9	4	5	8	(1	2	e	t	3	6	7	(2	3	t	e	2	7	6	(3
p2it:	0	2	8	6	4	t	(0	1	1	9	5	5	9	(1	2	0	t	4	6	8	(2	3	e	e	3	7	7	(3
i3pe:	0	3	8	7	4	e	(0	1	2	9	6	5	t	(1	2	1	t	5	6	9	(2	3	0	e	4	7	8	(3
p4i0:	0	4	8	8	4	0	(0	1	3	9	7	5	e	(1	2	2	t	6	6	t	(2	3	1	e	5	7	9	(3
i5p1:	0	5	8	9	4	1	(0	1	4	9	8	5	0	(1	2	3	t	7	6	e	(2	3	2	e	6	7	t	(3
p6i2:	0	6	8	t	4	2	(0	1	5	9	9	5	1	(1	2	4	t	8	6	0	(2	3	3	e	7	7	e	(3
i7p3:	0	7	8	e	4	3	(0	1	6	9	t	5	2	(1	2	5	t	9	6	1	(2	3	4	e	8	7	0	(3
p8i4:	0	8	8	0	4	4	(0	1	7	9	e	5	3	(1	2	6	t	t	6	2	(2	3	5	e	9	7	1	(3
i9p5:	0	9	8	1	4	5	(0	1	8	9	0	5	4	(1	2	7	t	e	6	3	(2	3	6	e	t	7	2	(3
pti6:	0	t	8	2	4	6	(0	1	9	9	1	5	5	(1	2	8	t	0	6	4	(2	3	7	e	e	7	3	(3
iep7:	0	e	8	3	4	7	(0	1	t	9	2	5	6	(1	2	9	t	1	6	5	(2	3	8	e	0	7	4	(3

CYCLIC INTERVAL-5

p0p5:	0	0	5	7	t	2	3	9	8	4	1	e	6	6	e	1	4	8	9	3	2	t	7	5	(0
i1i6:	0	1	5	8	t	3	3	t	8	5	1	0	6	7	e	2	4	9	9	4	2	e	7	6	(0
p2p7:	0	2	5	9	t	4	3	e	8	6	1	1	6	8	e	3	4	t	9	5	2	0	7	7	(0
i3i8:	0	3	5	t	t	5	3	0	8	7	1	2	6	9	e	4	4	e	9	6	2	1	7	8	(0
p4p9:	0	4	5	e	t	6	3	1	8	8	1	3	6	t	e	5	4	0	9	7	2	2	7	9	(0
i5it:	0	5	5	0	t	7	3	2	8	9	1	4	6	e	e	6	4	1	9	8	2	3	7	t	(0
p6pe:	0	6	5	1	t	8	3	3	8	t	1	5	6	0	e	7	4	2	9	9	2	4	7	e	(0
i7i0:	0	7	5	2	t	9	3	4	8	e	1	6	6	1	e	8	4	3	9	t	2	5	7	0	(0
p8p1:	0	8	5	3	t	t	3	5	8	0	1	7	6	2	e	9	4	4	9	e	2	6	7	1	(0
i9i2:	0	9	5	4	t	e	3	6	8	1	1	8	6	3	e	t	4	5	9	0	2	7	7	2	(0
ptp3:	0	t	5	5	t	0	3	7	8	2	1	9	6	4	e	e	4	6	9	1	2	8	7	3	(0
iei4:	0	e	5	6	t	1	3	8	8	3	1	t	6	5	e	0	4	7	9	2	2	9	7	4	(0

CYCLIC INTERVAL-7

p0p7:	0	0	7	5	2	t	9	3	4	8	e	1	6	6	1	e	8	4	3	9	t	2	5	7	(0
i1i8:	0	1	7	6	2	e	9	4	4	9	e	2	6	7	1	0	8	5	3	t	t	3	5	8	(0
p2p9:	0	2	7	7	2	0	9	5	4	t	e	3	6	8	1	1	8	6	3	e	t	4	5	9	(0
i3it:	0	3	7	8	2	1	9	6	4	e	e	4	6	9	1	2	8	7	3	0	t	5	5	t	(0
p4pe:	0	4	7	9	2	2	9	7	4	0	e	5	6	t	1	3	8	8	3	1	t	6	5	e	(0
i5i0:	0	5	7	t	2	3	9	8	4	1	e	6	6	e	1	4	8	9	3	2	t	7	5	0	(0
p6p1:	0	6	7	e	2	4	9	9	4	2	e	7	6	0	1	5	8	t	3	3	t	8	5	1	(0
i7i2:	0	7	7	0	2	5	9	t	4	3	e	8	6	1	1	6	8	e	3	4	t	9	5	2	(0
p8p3:	0	8	7	1	2	6	9	e	4	4	e	9	6	2	1	7	8	0	3	5	t	t	5	3	(0
i9i4:	0	9	7	2	2	7	9	0	4	5	e	t	6	3	1	8	8	1	3	6	t	e	5	4	(0
ptp5:	0	t	7	3	2	8	9	1	4	6	e	e	6	4	1	9	8	2	3	7	t	0	5	5	(0
iei6:	0	e	7	4	2	9	9	2	4	7	e	0	6	5	1	t	8	3	3	8	t	1	5	6	(0

CYCLIC INTERVAL-6															
	first partition					second partition					third partition				
p0i6:	*0*	0	*6*	6	*(0*	*1*	e	*7*	5	*(1*	*2*	t	*8*	4	*(2*
i1p7:	*0*	1	*6*	7	*(0*	*1*	0	*7*	6	*(1*	*2*	e	*8*	5	*(2*
p2i8:	*0*	2	*6*	8	*(0*	*1*	1	*7*	7	*(1*	*2*	0	*8*	6	*(2*
i3p9:	*0*	3	*6*	9	*(0*	*1*	2	*7*	8	*(1*	*2*	1	*8*	7	*(2*
p4it:	*0*	4	*6*	t	*(0*	*1*	3	*7*	9	*(1*	*2*	2	*8*	8	*(2*
i5pe:	*0*	5	*6*	e	*(0*	*1*	4	*7*	t	*(1*	*2*	3	*8*	9	*(2*
	fourth partition					fifth partition					sixth partition				
p0i6:	*3*	9	*9*	3	*(3*	*4*	8	*t*	2	*(4*	*5*	7	*e*	1	*(5*
i1p7:	*3*	t	*9*	4	*(3*	*4*	9	*t*	3	*(4*	*5*	8	*e*	2	*(5*
p2i8:	*3*	e	*9*	5	*(3*	*4*	t	*t*	4	*(4*	*5*	9	*e*	3	*(5*
i3p9:	*3*	0	*9*	6	*(3*	*4*	e	*t*	5	*(4*	*5*	t	*e*	4	*(5*
p4it:	*3*	1	*9*	7	*(3*	*4*	0	*t*	6	*(4*	*5*	e	*e*	5	*(5*
i5pe:	*3*	2	*9*	8	*(3*	*4*	1	*t*	7	*(4*	*5*	0	*e*	6	*(5*

Appendix Two

Set Class Membership in ICS Families[1]

(a) trichordal scs:						
	ICS-1	ICS-2	ICS-3	ICS-4	ICS-5	ICS-6
3-1	✓	✓				
3-2	✓	✓	✓			
3-3	✓		✓	✓		
3-4	✓			✓	✓	
3-5	✓				✓	✓
3-6		✓		✓		
3-7		✓	✓		✓	
3-8		✓		✓		✓
3-9		✓			✓	
3-10			✓			✓
3-11			✓	✓	✓	
3-12				✓		

(b) tetrachordal scs:						
	ICS-1	ICS-2	ICS-3	ICS-4	ICS-5	ICS-6
4-1	✓	✓				
4-3	✓		✓			
4-7	✓			✓		
4-8	✓				✓	
4-9	✓				✓	✓
4-10		✓	✓			
4-17			✓	✓		
4-20				✓	✓	
4-21		✓		✓		
4-23		✓			✓	
4-25		✓		✓		✓
4-26			✓		✓	
4-28			✓			✓

[1] The symbol ✓ in each row indicates the corresponding set's membership in the ICS-family given in the column heading.

(c) pentachordal scs:						
	ICS-1	ICS-2	ICS-3	ICS-4	ICS-5	ICS-6
5-1	✓	✓				
5-2		✓				
5-3	✓					
5-6	✓					
5-7	✓				✓	
5-10			✓			
5-16			✓			
5-20					✓	
5-21				✓		
5-23		✓				
5-25			✓			
5-27					✓	
5-32			✓			
5-33		✓		✓		
5-35		✓			✓	

(d) hexachordal scs:						
	ICS-1	ICS-2	ICS-3	ICS-4	ICS-5	ICS-6
6-1	✓	✓				
6-z4	✓					
6-z6	✓					
6-7	✓				✓	
6-8		✓				
6-z13			✓			
6-20				✓		
6-z23			✓			
6-z26					✓	
6-32		✓			✓	
6-35		✓		✓		
6-z38					✓	
6-z49			✓			
6-z50			✓			

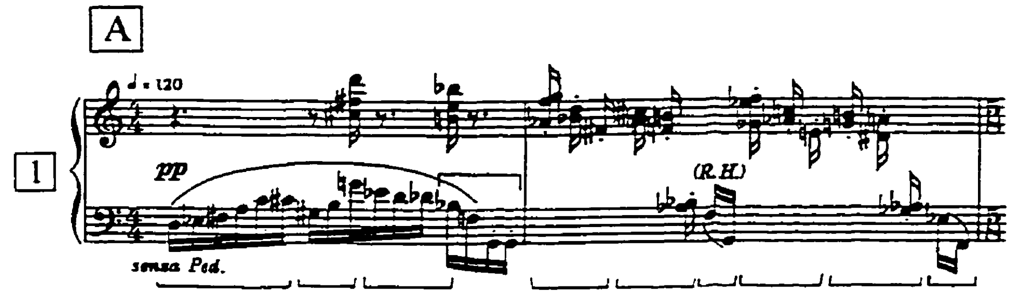
(e) heptachordal scs:						
	ICS-1	ICS-2	ICS-3	ICS-4	ICS-5	ICS-6
7-1	✓	✓				
7-2		✓				
7-5	✓					
7-7	✓				✓	
7-14					✓	
7-23		✓				
7-31			✓			
7-35		✓			✓	

(f) octachordal scs:						
	ICS-1	ICS-2	ICS-3	ICS-4	ICS-5	ICS-6
8-1	✓	✓				
8-6	✓				✓	
8-9	✓				✓	
8-10		✓				
8-23		✓			✓	
8-28			✓			

(g) nonachordal scs:						
	ICS-1	ICS-2	ICS-3	ICS-4	ICS-5	ICS-6
9-1	✓	✓				
9-2		✓				
9-5	✓				✓	
9-7		✓				
9-9		✓			✓	

Appendix Three

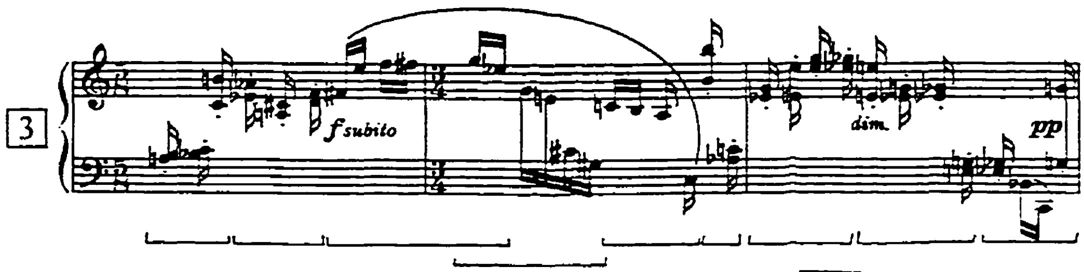
Annotated Score of *Etude No.1* from George Perle's *Six Etudes for Piano* (1973-76)



i1i2: 102 768 3t4 77 768 | 859 --
i3i0: 693 21e 21e t5 5t2 | 1et 57

i1i2: 586
p0p9: t27

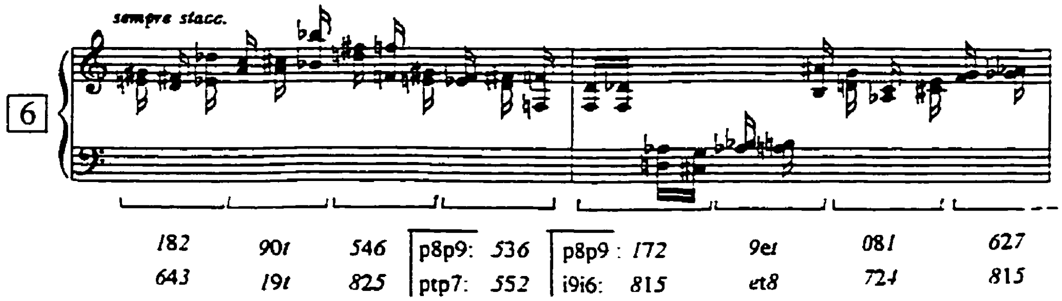
i9it: 364 637 --
p8p5: 805 e98 35

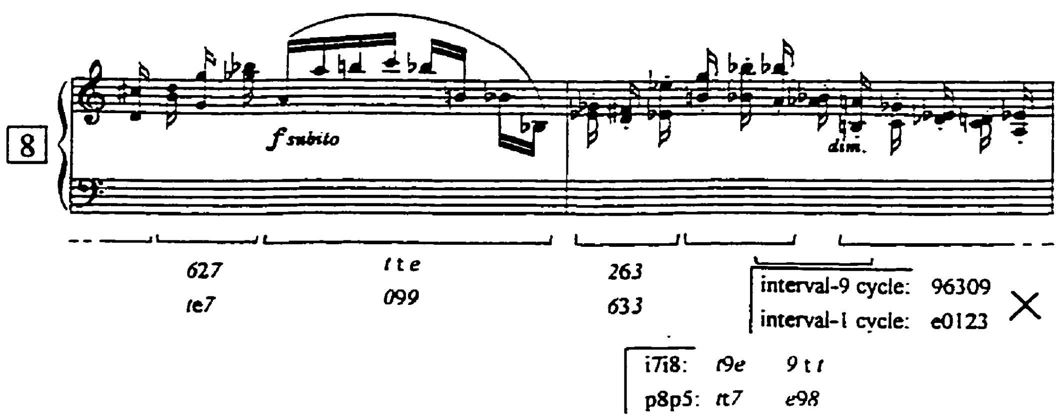


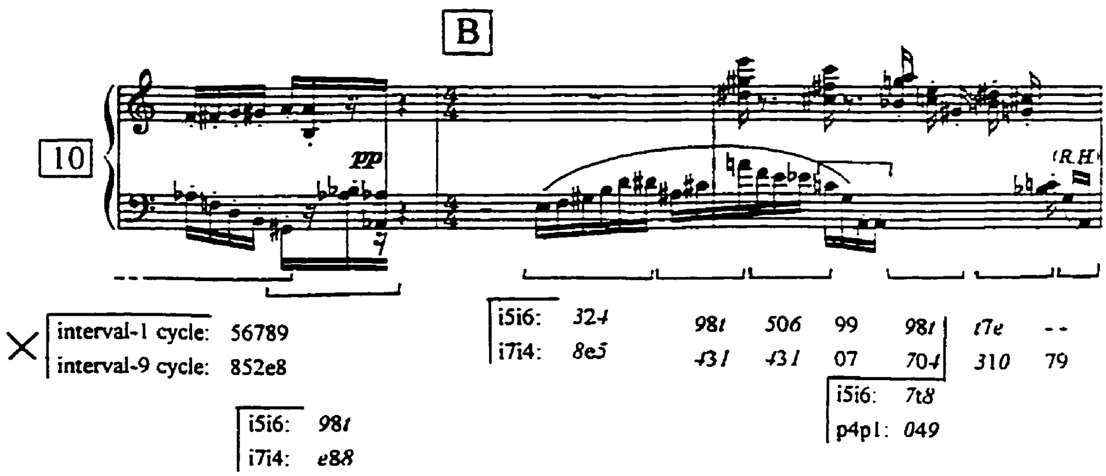
ptpe: t0e 192 647 738 --- ee
iei8: 0e9 835 653 471 0e9 08

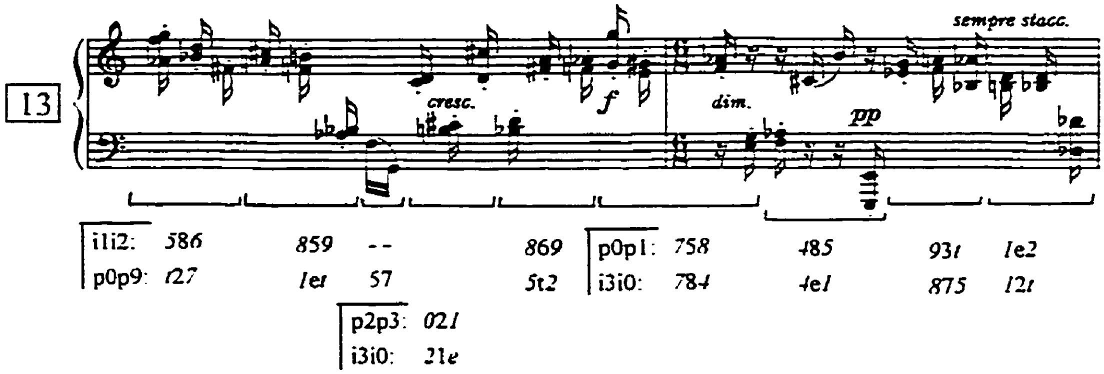
i9it: 364 637
ptp7: 734 t07

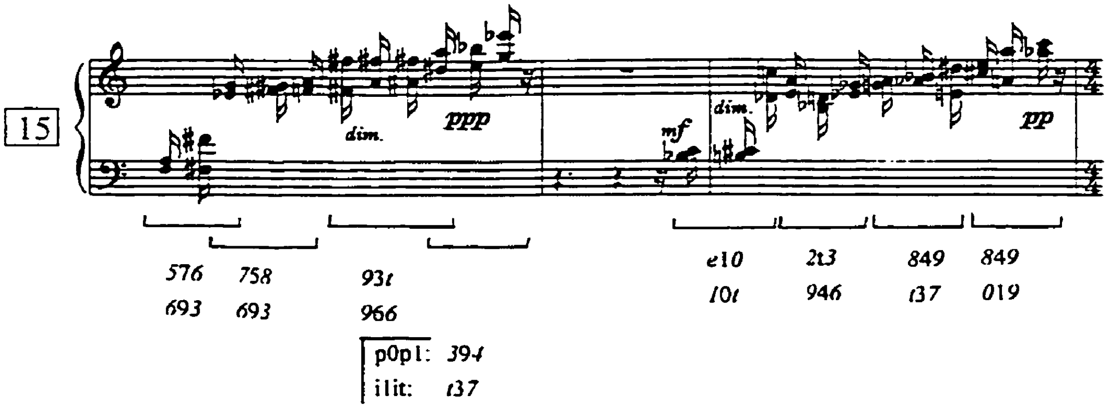
ptpe: 374
ptp7: 643

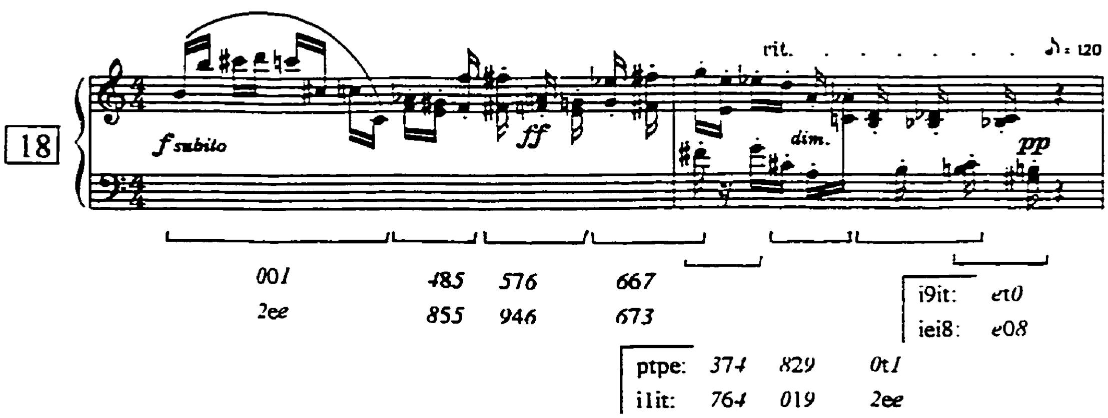

sempre stacc.
6
182 643
90t 19t
546 825
p8p9: 536
ptp7: 552
p8p9 : 172
i9i6: 815
9et et8
081 724
627 815

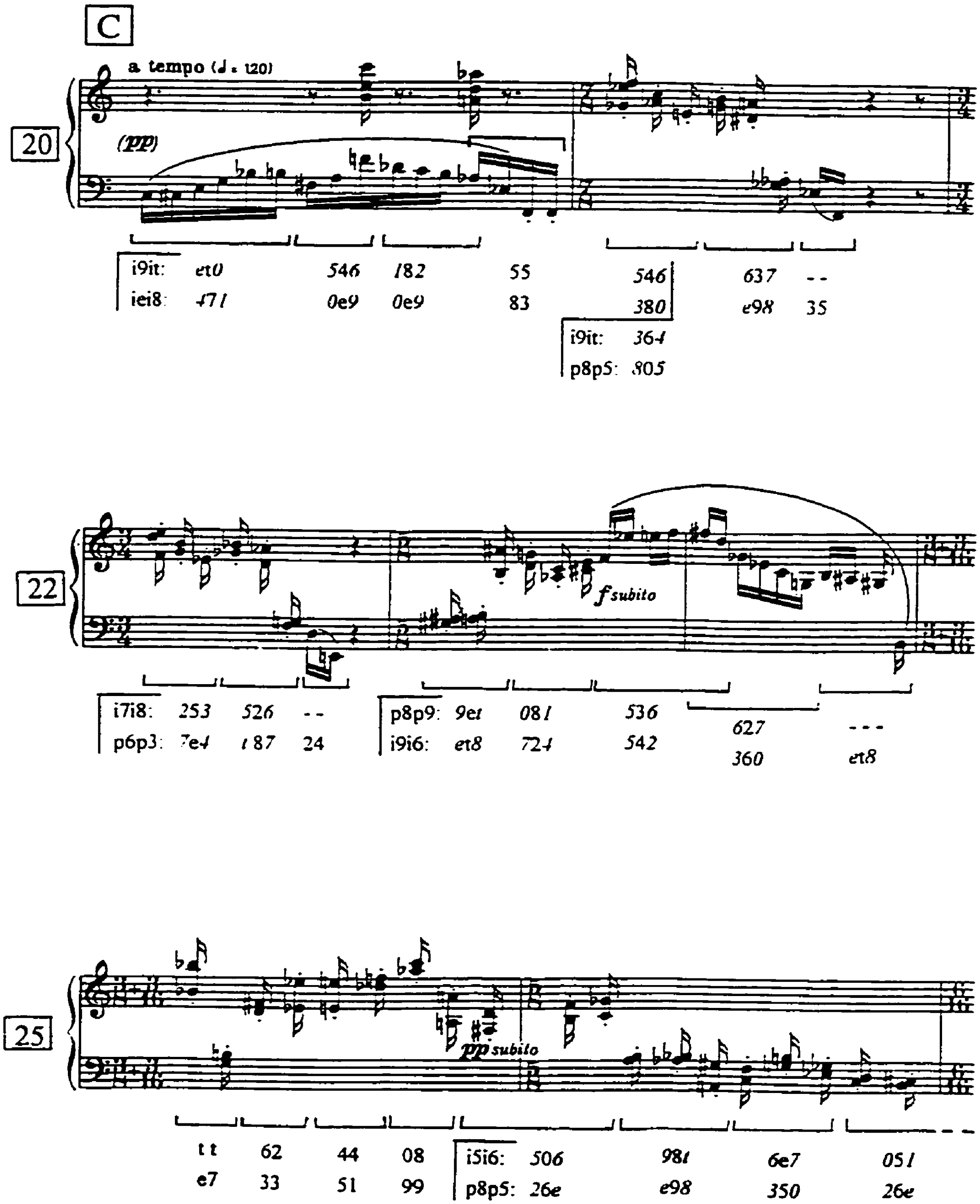

8
f subito
dim.
627 te7
tte 099
263 633
interval-9 cycle: 96309
interval-1 cycle: e0123
i7i8: t9e 9tt
p8p5: tt7 e98

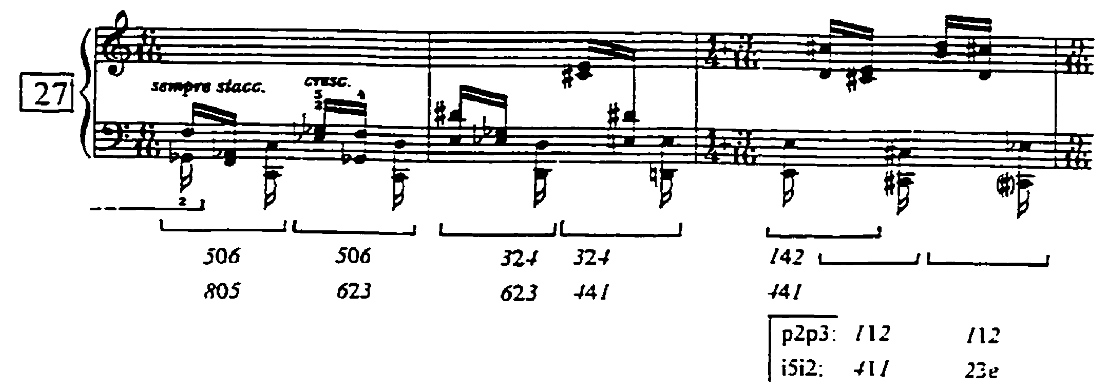

B
10
pp
(R.H.
interval-1 cycle: 56789
interval-9 cycle: 852e8
i5i6: 98t
i7i4: e88
i5i6: 324
i7i4: 8e5
98t 431
506 431
99 07
98t 704
i5i6: 7t8
p4p1: 049
t7e 310
-- 79

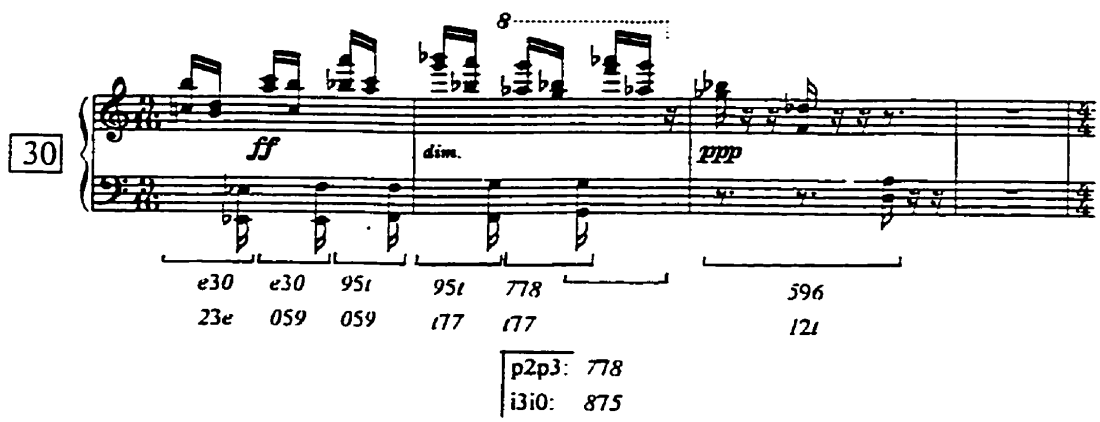

13
sempre stacc.
cresc.
f
dim.
pp
i1i2: 586 859 -- 869
p0p9: t27 1et 57 5t2
p0p1: 758 485 93t 1e2
i3i0: 784 4e1 875 12t
p2p3: 021
i3i0: 21e

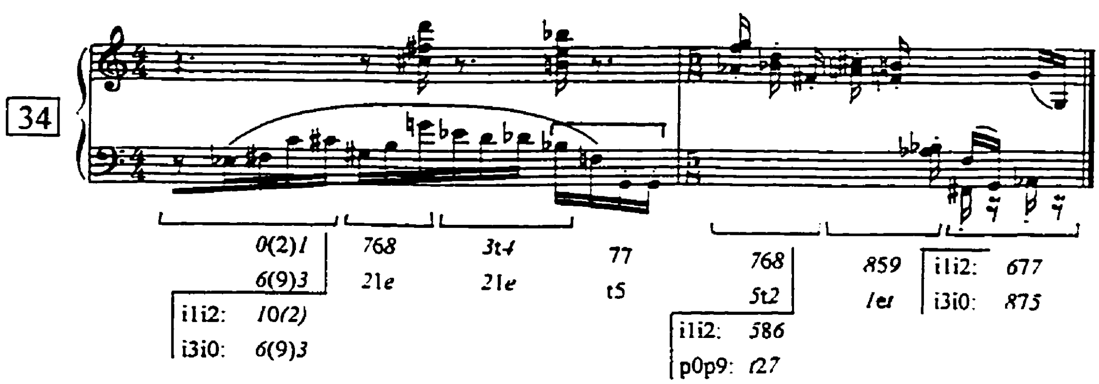

15
dim.
ppp
mf
dim.
pp
576 758 93t
693 693 966
p0p1: 394
i1it: t37
e10 2t3 849 849
10t 946 t37 019


18
f subito
ff
rit.
dim.
pp
= 120
001 485 576 667
2ee 855 946 673
i9it: et0
iei8: e08
ptpe: 374 829 0t1
i1it: 764 019 2ee


C
a tempo (♩. = 120)
20
(pp)
i9it: et0 546 182 55 546 637 - -
iei8: 471 0e9 0e9 83 380 e98 35
i9it: 364
p8p5: 805
22
f subito
i7i8: 253 526 - -
p6p3: 7e4 t87 24
p8p9: 9et 081 536 627 - - -
i9i6: et8 724 542 360 et8
25
pp subito
tt 62 44 08
e7 33 51 99
i5i6: 506 98t 6e7 051
p8p5: 26e e98 350 26e


27
sempre stacc.
cresc.
506 506 324 324 142
805 623 623 441 441
p2p3: 112 112
i5i2: 411 23e

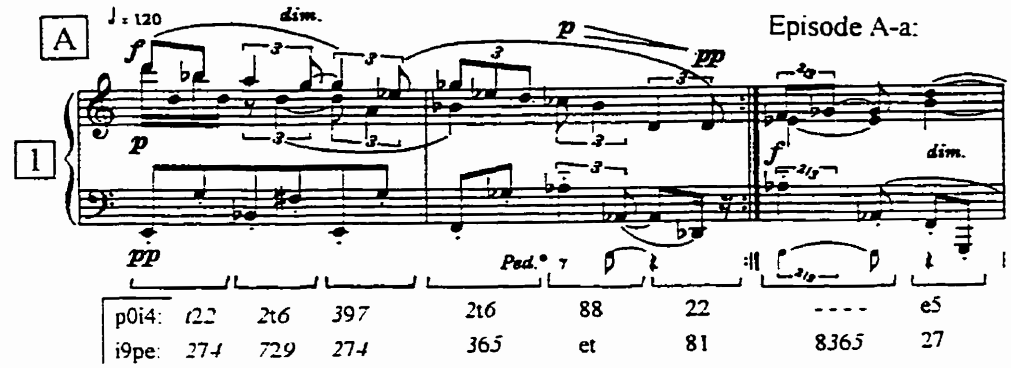

30
ff
dim.
ppp
e30 e30 95t 95t 778 596
23e 059 059 t77 t77 12t
p2p3: 778
i3i0: 875

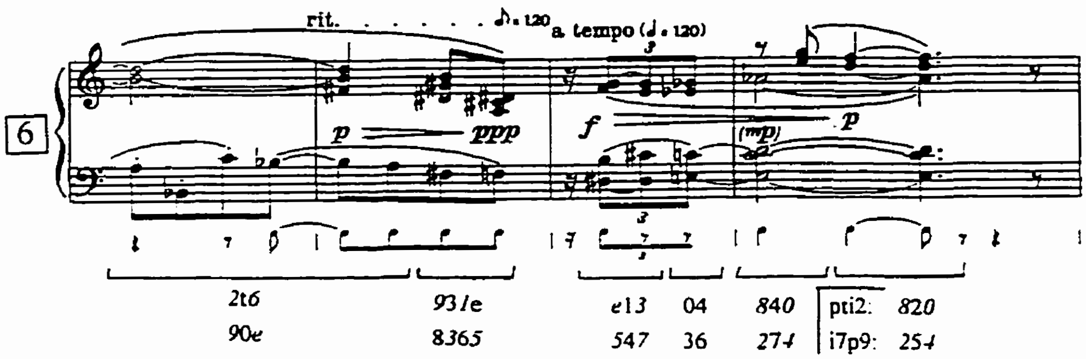

34
0(2)1 768 3t4 77 768 859
6(9)3 21e 21e t5 5t2 1et
i1i2: 10(2)
i3i0: 6(9)3
i1i2: 586
p0p9: t27
i1i2: 677
i3i0: 875

Appendix Four

Annotated Score of *Etude No.4* from George Perle's *Six Etudes for Piano* (1973-76)

Subject A-a:





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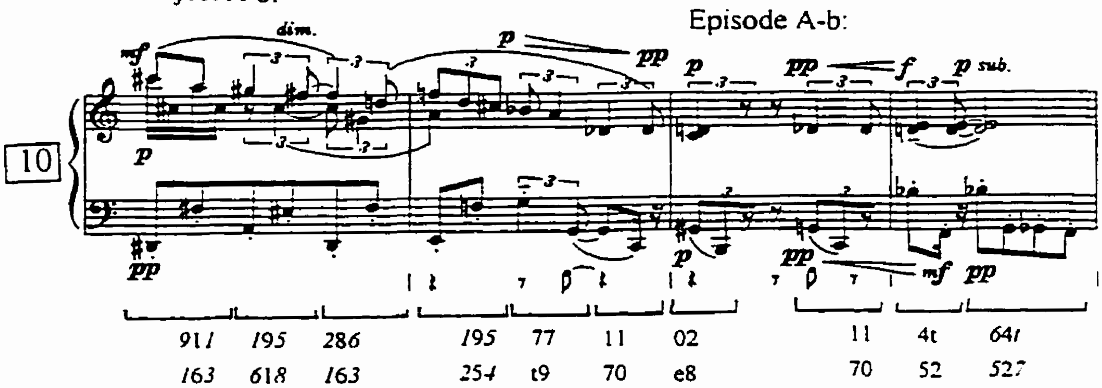
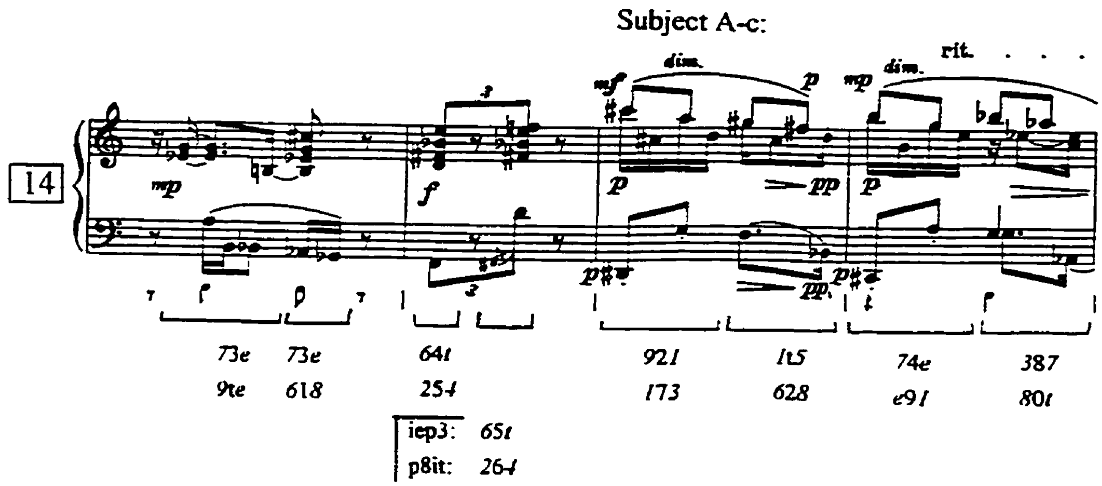
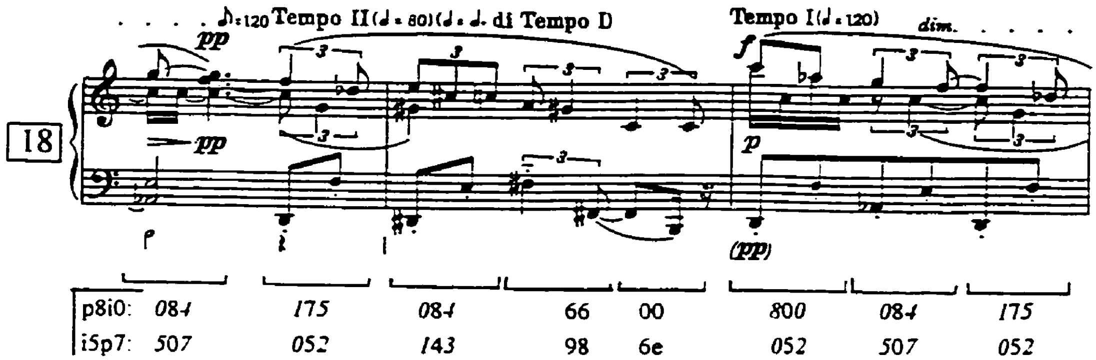
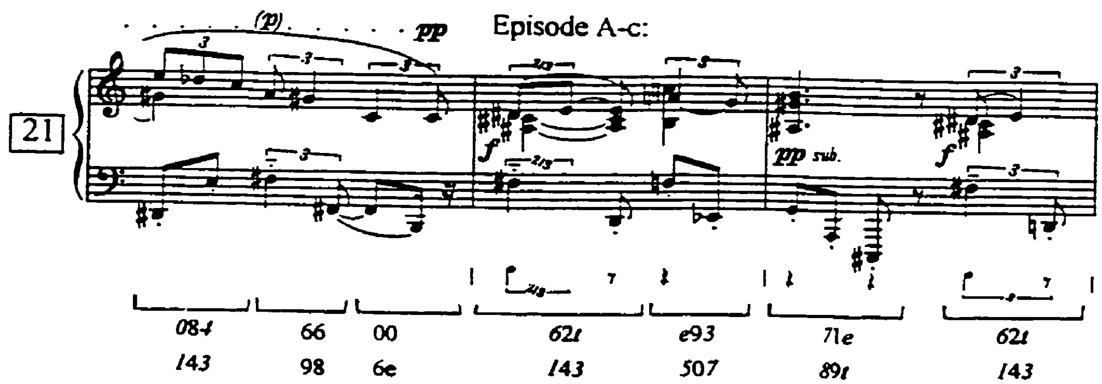
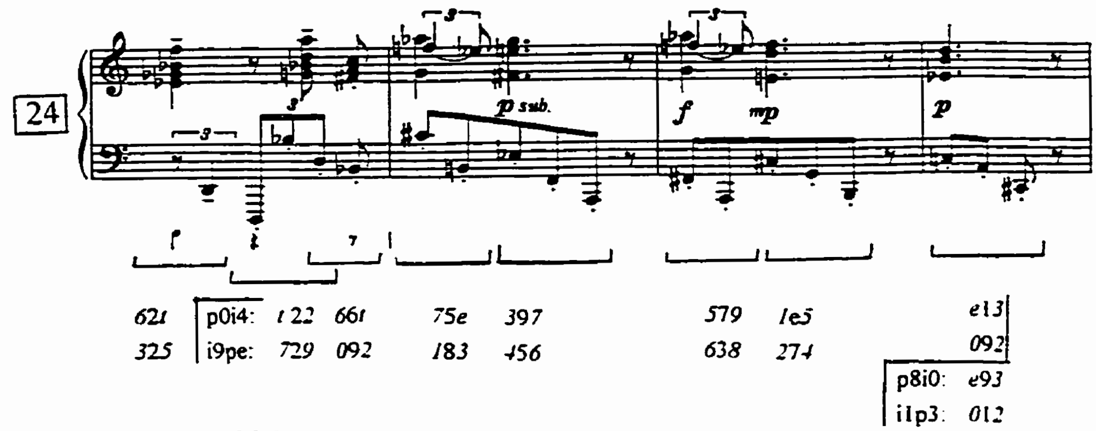



Subject A-c:
rit.
dim.
14
73e 73e 641 921 1t5 74e 387
9te 618 254 173 628 e91 801
iep3: 651
p8it: 264


Tempo II
Tempo I
dim.
18
p8i0: 084 175 084 66 00 800 084 175
i5p7: 507 052 143 98 6e 052 507 052


Episode A-c:
21
pp sub.
084 66 00 621 e93 71e 621
143 98 6e 143 507 89t 143



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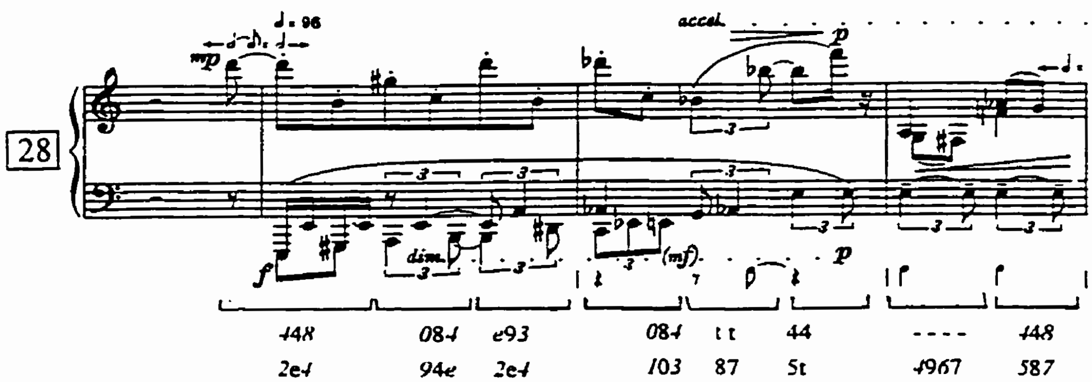
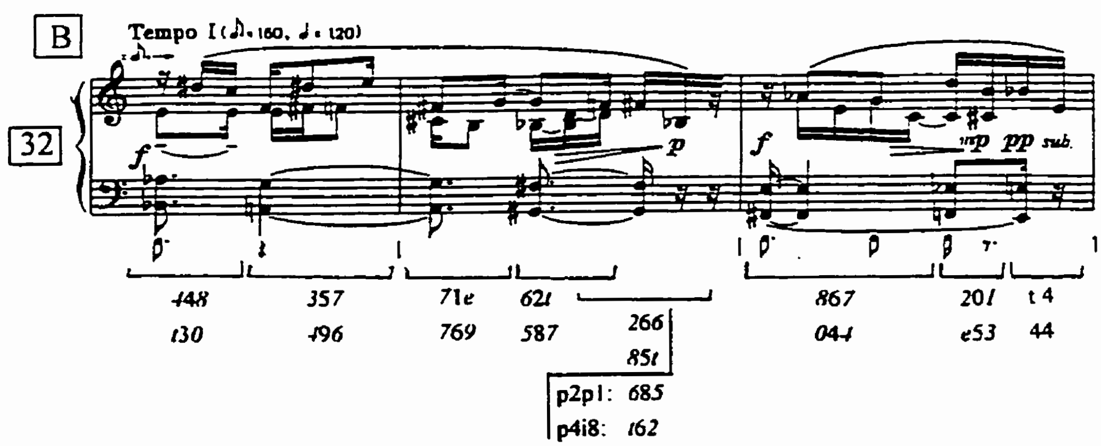
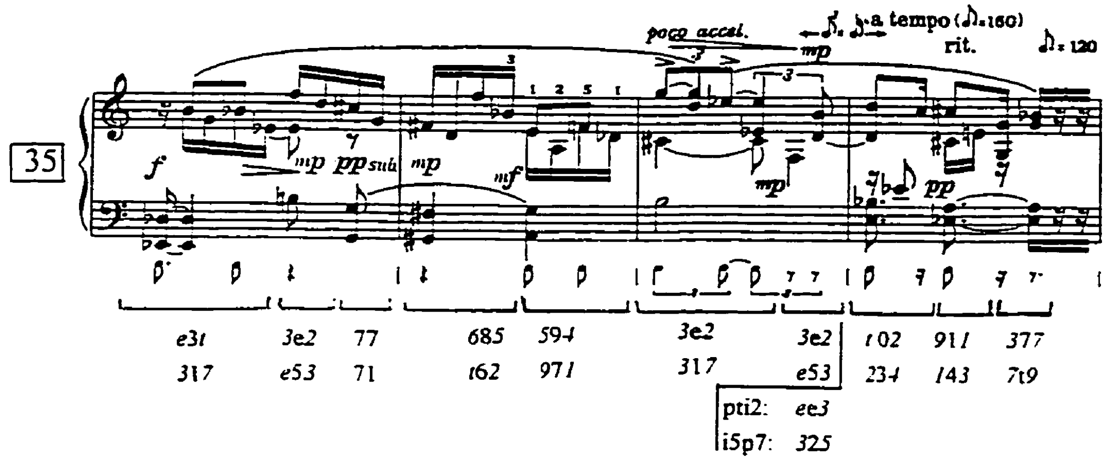
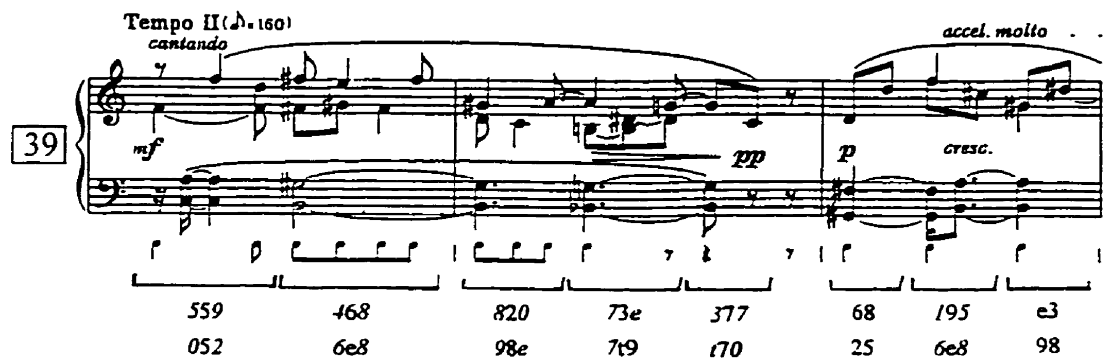


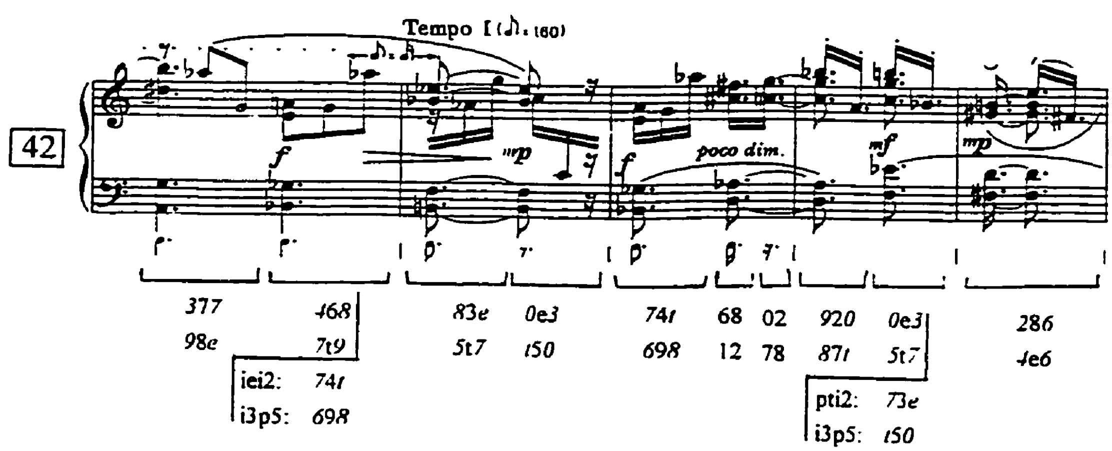
Phrase group B-d:



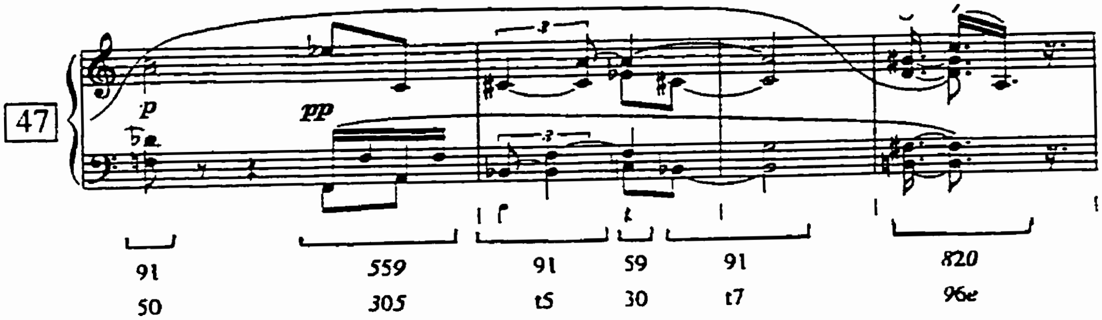


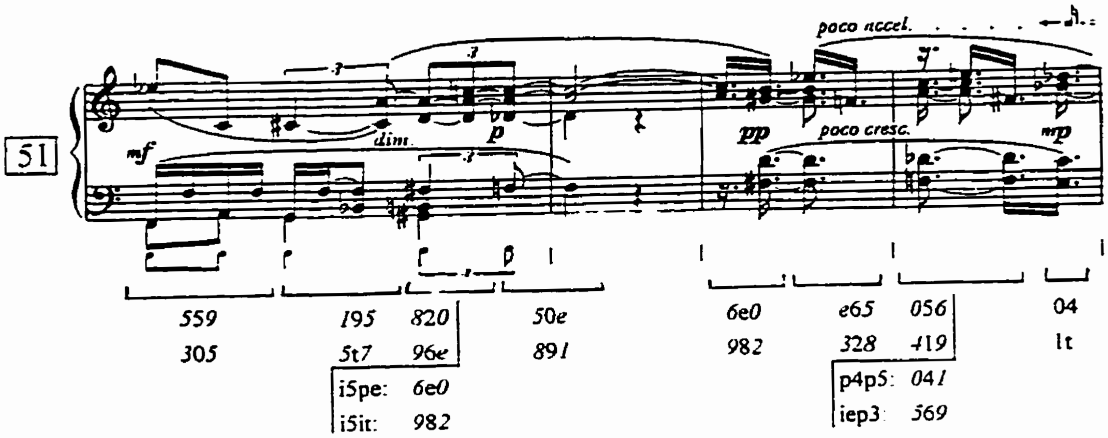
Phrase group B-e:





Phrase group B-f:





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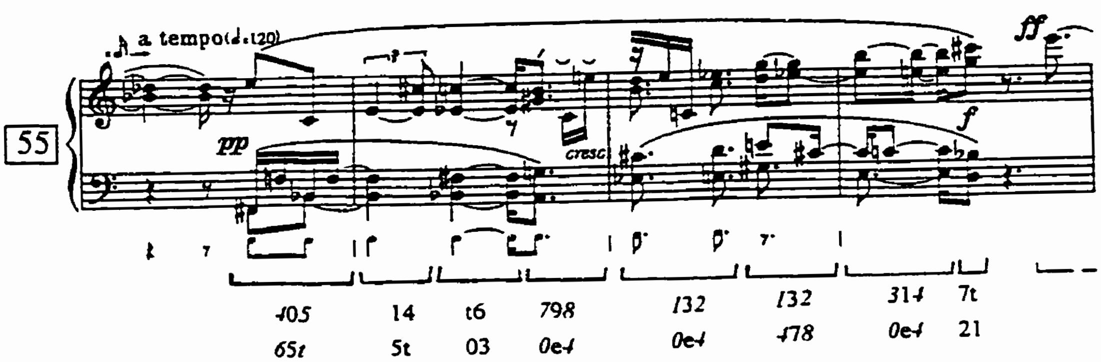
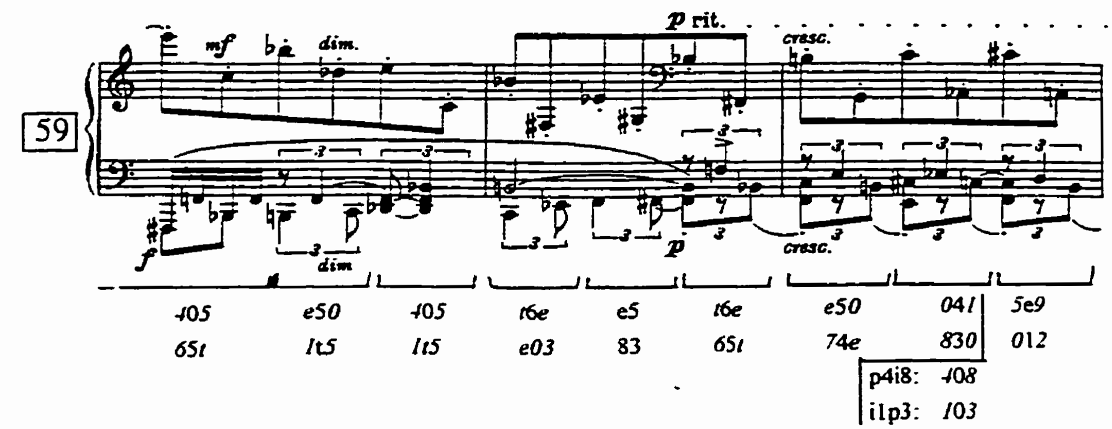
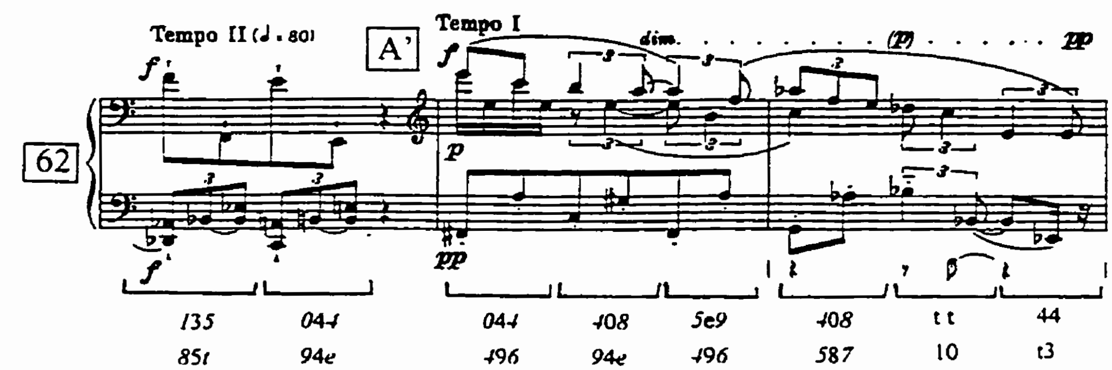
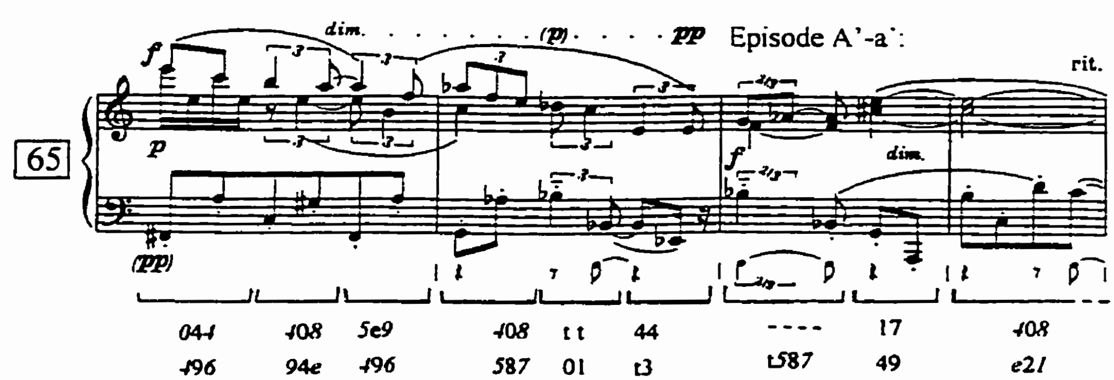


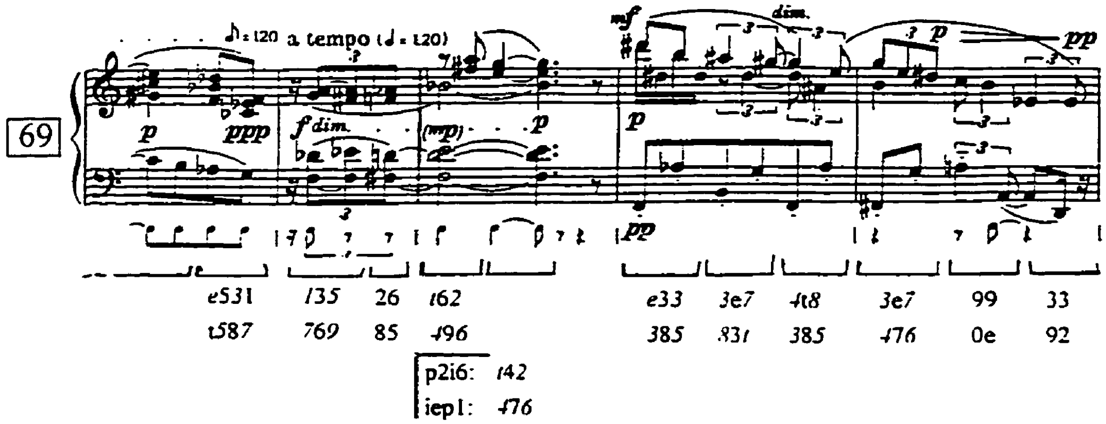

59
p rit.
mf
dim.
cresc.
f
dim
p
cresc.
405 65t
e50 1t5
405 1t5
t6e e03
e5 83
t6e 65t
e50 74e
041 830
5e9 012
p4i8: 408
i1p3: 103

Subject A'-a':

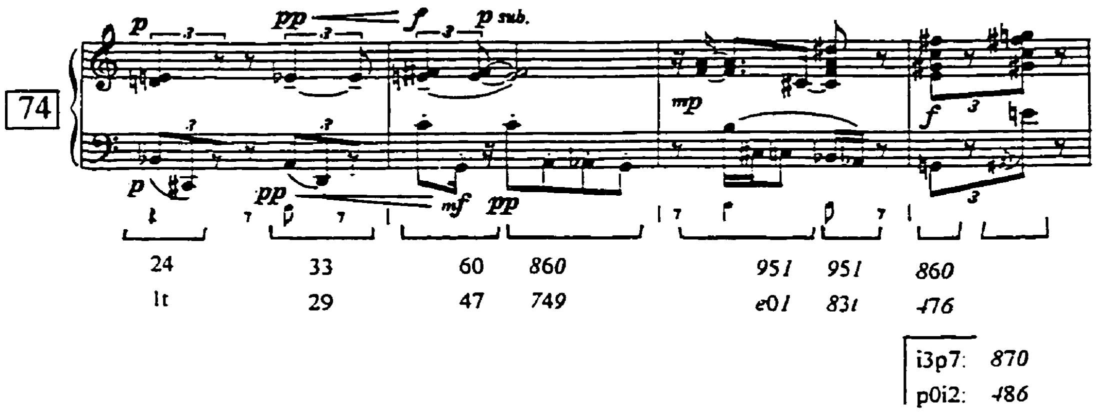

62
Tempo II (♩. 80)
A'
Tempo I
f
dim.
(p)
pp
p
135 85t
044 94e
044 496
408 94e
5e9 496
408 587
t t 10
44 t3

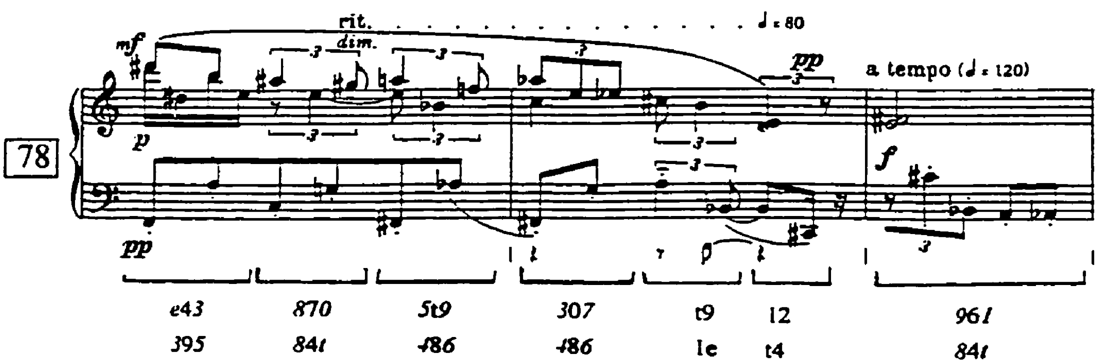

65
f
dim.
(p)
pp
Episode A'-a':
rit.
p
(pp)
dim.
044 496
408 94e
5e9 496
408 587
t t 01
44 t3
- - - - t587
17 49
408 e21

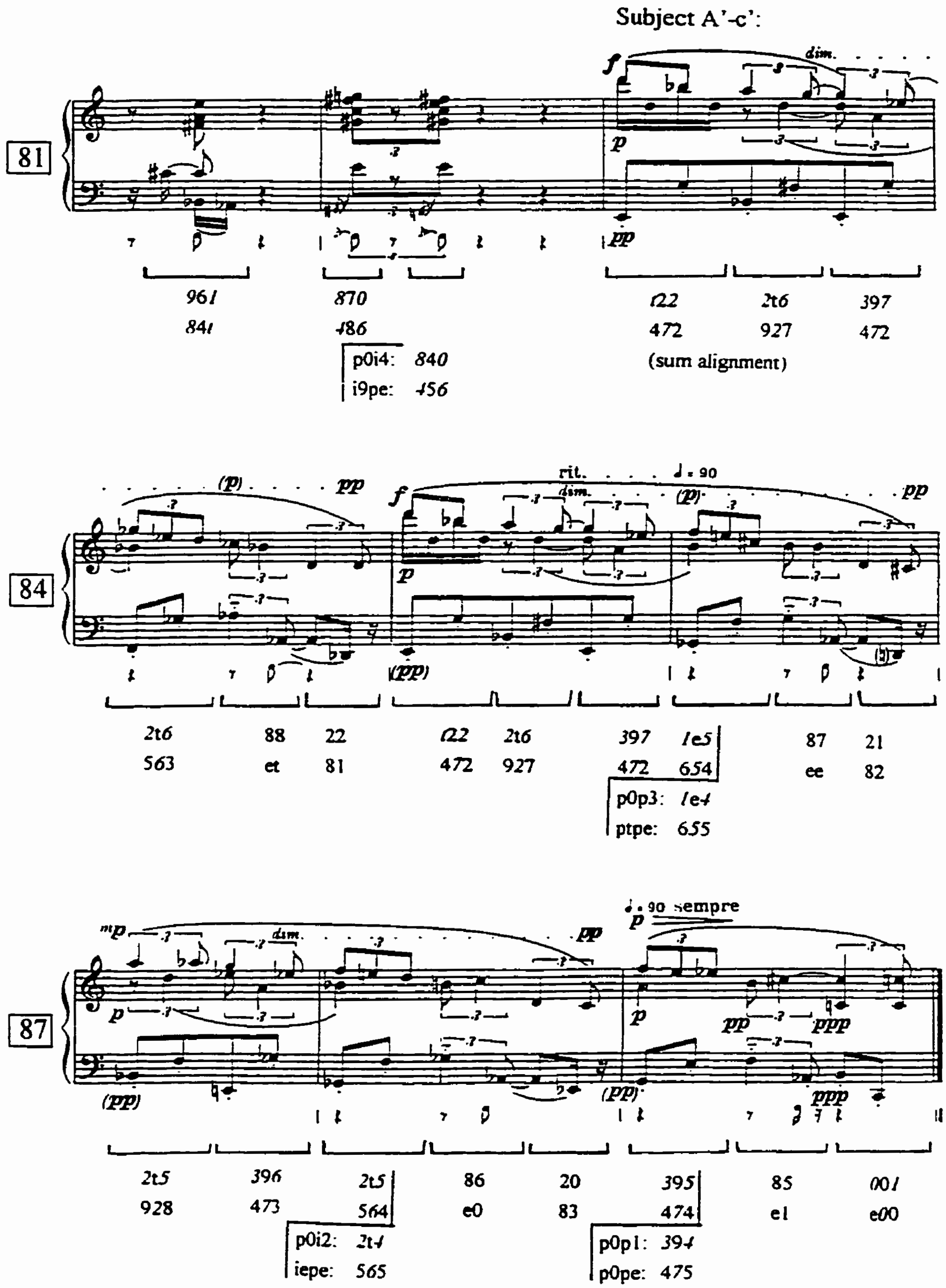
Subject A'-b':



Episode A'-b':






Subject A'-c':
81
961
84ı
870
486
p0i4: 840
i9pe: 456
ı22
472
2t6
927
397
472
(sum alignment)
84
rit.
dim.
2t6
563
88
et
22
81
ı22
472
2t6
927
397
472
ıe5
654
p0p3: ıe4
ptpe: 655
87
ee
21
82
87
dim.
90 sempre
2t5
928
396
473
2t5
564
p0i2: 2t4
iepe: 565
86
e0
20
83
395
474
p0p1: 394
p0pe: 475
85
e1
001
e00

Appendix Five

Glossary of Terms in Perle's Theory of Twelve-Tone Tonality

Aggregate sum. The sum of the four tonic sums in an array.

Array. A construct formed from the alignment of two cyclic sets. An array takes its name from its component cyclic sets.

Axis dyad. The central vertical dyad in an axis-dyad chord.

Axis-dyad chord. A collection of six pcs formed by pairing trichordal segments from each of the cyclic sets in an array.

Axis note. Any single pc in a cyclic set, flanked on either side by neighbour notes. Axis notes and neighbour notes occur in alternation in a cyclic set.

Axis of symmetry. The central point or line about which pc collections are symmetrically positioned. Symmetrically related pc collections have the same sum as the axis of symmetry.

Cognate relation. The relation between inversionally related cyclic sets that share a tonic sum.

Cyclic chord. A construct formed by two neighbour dyads without the intervening axis dyad. The horizontal dyads contain the cyclic intervals of the cyclic sets.

Cyclic interval. The generating interval of a cyclic set.

Cyclic set. The series of pcs formed by alternating members of inversionally related interval cycles.

Cyclic set sum. The sum of the two tonic sums in a cyclic set.

Difference alignment. An arrangement of cyclic sets such that the component P-cycles and I-cycles are vertically aligned. The vertical dyads yield a consistent pattern of alternating secondary differences.

Interval cycle. An ordered series of pcs based upon a single recurrent interval, which is measured by the number of semitones it spans. The interval cycle is completed by the return of the initial pc.

Interval system. A pair of integers denoting the cyclic intervals of an array.

Key. The symmetrical relationship between an array's cyclic sets in a sum alignment. The key is calculated as the sum of oppositely aligned tonic sums in the array, or as the sum of the secondary sums of the axis dyad and each of its neighbour dyads in an axis-dyad chord.

Mode. The intervallic relationship between an array's cyclic sets in a difference alignment. The mode is calculated as the difference between the corresponding tonic sums of the cyclic sets, or as the difference between the secondary differences of the axis dyad and each of its neighbour dyads in an axis-dyad chord.

Modulation through reinterpretation. Modulation in which various members of an axis-dyad chord are reordered, resulting in a reinterpretation of the axis-dyad chord as a segment of a different array.

Modulation through substitution. Modulation in which either the axis dyad or the neighbour dyads of an axis-dyad chord are replaced with others of either the same difference or sum.

Neighbour dyads. The vertical dyads flanking the axis dyad in an axis-dyad chord.

Neighbour note. A pc to the immediate right or left of an axis note. Axis notes and neighbour notes occur in alternation in a cyclic set.

Secondary differences. The vertical dyadic intervals resulting from a pair of cyclic sets in a difference alignment.

Secondary sums. The vertical dyadic sums resulting from a pair of cyclic sets in a sum alignment.

Semi-equivalence. The relation between pairs of cyclic sets that have the same tonic sums but with opposite p/i designations. Sets so related have the same ordering of pcs, but which originate in opposite P and I cycles. Semi-equivalent cyclic sets comprise complementary cyclic intervals.

Semi-inversion. The inversional relation between two cyclic sets of complementary intervals or two arrays of complementary interval systems, but whose tonic sum p/i designations do not match.

Semi-transposition. The transpositional relation between two cyclic sets of the same cyclic interval or two arrays of the same interval system, but whose tonic sum p/i designations do not match. The corresponding tonic sums both differ by an *odd* integer, indicating that the component elements are not transposed by the same interval.

Sum alignment. An arrangement of cyclic sets such that the P-cycle of one cyclic set is aligned with the I-cycle of the other, and vice versa. The vertical dyads yield a consistent pattern of alternating secondary sums.

Sum tetrachord. A segment of an array consisting of an axis dyad and only one of its neighbour dyads. The horizontal dyads indicate two of the four tonic sums of the array.

Synoptic array. A collection of arrays that share structural similarities in their interval systems.

Synoptic key, master array of. A collection of arrays whose cyclic intervals show the same sum. There are seven different master arrays of the synoptic key, from 0 to 6.

Synoptic mode, master array of. A collection of arrays whose cyclic intervals show the same difference. There are seven different master arrays in the synoptic mode, from 0 to 6.

Tonality. The symmetrical relation among arrays sharing the same axis of symmetry. There are three categories of tonality, based on the transpositional relationship between the arrays' symmetrical axes and determined by the arrays' aggregate sums. Tonality 0 constitutes all arrays whose aggregate sums are 0, 4, or 8 and whose axes consist of two repeated even integers. Tonality 1 constitutes all arrays whose aggregate sums are represented by an odd integer and whose axes consist of an odd and an even integer that differ by 1. Tonality 2 constitutes all arrays whose aggregate sums are 2, 6, or t and whose axes consist of two repeated odd integers.

Tonic axis-dyad chord. An axis-dyad chord whose axis dyad pcs are present simultaneously in a trichordal segment of one of the cyclic sets, and whose axis dyad sum duplicates a tonic sum. If the cyclic interval is the same for each of the cyclic sets of the array, the pcs of the neighbour dyads are present simultaneously in a tetradic segment of the other cyclic set.

Tonic sums. The sums formed by an axis note with each of its adjacent neighbour notes. The tonic sums are used to name the cyclic set, with each sum preceded by the lower-case letter **p** or **i**.

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