Minimal Hereditary Dominating Pair Graphs

by

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Abstract

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This thesis describes structural properties of *hereditary dominating pair (HDP)* and minimal HDP graphs. A *dominating pair (DP)* in a connected graph is a pair of vertices such that every path between them is dominating. A graph G is HDP if every connected induced subgraph of G has a DP. The class of HDP graphs includes all asteroidal triple-free (AT-free) graphs — already extensively studied — and some graphs containing asteroidal triples (ATs). A *minimal* HDP graph H contains an AT $\{x, y, z\}$, and satisfies the following: if $\mathcal{P}^c_{a,b}$ is the set of all induced paths between vertices a and b that avoid the neighborhood of a vertex c, then every vertex of H belongs to a path in $\mathcal{P}^z_{x,y} \cup \mathcal{P}^y_{x,z} \cup \mathcal{P}^x_{y,z}$. The position of DP vertices in minimal HDP graphs is determined, as well as some structural properties dictated by the position of DP vertices.

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Contents

Chapter 1

Introduction

This thesis describes some structural properties of hereditary dominating pair (HDP) graphs and minimal HDP graphs. To describe these classes of graphs some definitions need to be introduced. A set of vertices of a graph is *dominating* if every vertex outside the set is adjacent to some vertex in the set. A *dominating pair* (DP) in a connected graph is a pair of vertices such that every path between them is dominating. A graph is *HDP* if all of its induced subgraphs have dominating pairs. The class of HDP graphs contains asteroidal triple-free (AT-free) graphs as a subclass (Corneil, Olariu, and Stewart 1997), as well as some graphs with asteroidal triples (ATs), such as a C_6. An *asteroidal triple* is defined to be an independent set of vertices such that each pair of vertices is joined by a path that avoids the neighborhood of the third (Lekkerkerker and Boland 1962).

The motivation for describing the structure of these graphs and the overview of the thesis will be presented in the next two subsections.

1.1 Motivation

Progress in graph theory has resulted in the identification and study of many different graph families. An extensive survey of currently known results about various graph classes loosely associated with perfect graphs was given by Brandstadt, Le, and Spinard

(1999). When examining the structure of AT-free graphs, Corneil, Olariu, and Stewart (1997) noticed that these graphs exhibit various types of linear structure; for example, every connected AT-free graph has a dominating pair. Clearly, all subclasses of AT-free graphs, such as co-comparability, trapezoid, interval, and permutation graphs (definitions appear in the next chapter), satisfy this property too. However, even though different forms of linear structure of these subclasses of AT-free graphs have been noticed before, the common linear structure of these subclasses was not noticed until the structure of their superclass, AT-free graphs, was examined by Corneil, Olariu, and Stewart (1997). For this reason, it is interesting to study the structure of graph classes that contain AT-free graphs as a subclass, for example, HDP graphs, since they might have an interesting structure that will reveal many important common properties of its subclasses.

The thesis considers some structural properties of HDP graphs as well as structural properties of minimal HDP graphs. Minimal HDP graphs are defined differently from other minimal graph families, such as, for example, minimal imperfect graphs. While minimal imperfect graphs are those that become perfect by removing any single vertex, the definition of minimal HDP graphs is based on a completely different concept. Since AT-free graphs are HDP and the structure of AT-free graphs has already been studied (Corneil, Olariu, and Stewart 1997), this thesis considers only HDP graphs that have asteroidal triples, and defines *minimal HDP graphs* in the following way. If $\{x, y, z\}$ is an asteroidal triple of an HDP graph H, and if $\mathcal{P}^c_{a,b}$ is defined to be the set of all induced paths between vertices a and b in H that avoid the neighborhood of a vertex c in H, then H is *minimal* if all of its vertices belong to a path in $\mathcal{P}^z_{x,y} \cup \mathcal{P}^y_{x,z} \cup \mathcal{P}^x_{y,z}$. Henceforth $\mathcal{P}_{x,y}$, $\mathcal{P}_{x,z}$, and $\mathcal{P}_{y,z}$ will be used to denote $\mathcal{P}^z_{x,y}$, $\mathcal{P}^y_{x,z}$, and $\mathcal{P}^x_{y,z}$ respectively. From this definition it can be seen that minimal HDP graphs form a rich family, since the paths in $\mathcal{P}_{x,y} \cup \mathcal{P}_{x,z} \cup \mathcal{P}_{y,z}$ can be of any length, can share vertices, and can have different vertex adjacency patterns. Also, different minimal HDP graphs will have different patterns in terms of possible positions of their dominating pair vertices. In addition, the position of

DP vertices in a minimal HDP graph G will force some structural properties on G. Thus, the structure of minimal HDP graphs will be a complex and important building block for understanding the structure of HDP graphs. Similar to the definition of minimal HDP graphs, *minimal AT graphs* can be defined.

Corneil, Olariu, and Stewart (1997) explain AT-free graph properties that are quite deep and involved. The authors took over five years to understand some aspects of the structure of AT-free graphs, from their first technical report on this subject (1992), to completion of the paper (1997). Therefore, it is reasonable to expect that it will be very difficult and time consuming to describe the structure of HDP graphs as well. Even the first look at the structure of minimal HDP graphs in this thesis reveals a rich and intricate structure.

1.2 Overview of the Thesis

The thesis has the following structure.

Chapter 2 explains the background work relevant to the thesis. It first describes the currently known hierarchy of graph classes related to HDP graphs. Then it gives a review of known structural results about AT-free and HDP graphs.

Chapters 3, 4, and 5 contain original work on HDP graphs. Chapter 3 describes the general structural properties of HDP graphs and minimal HDP graphs. It introduces definitions and claims common to all HDP and all minimal HDP graphs. It also gives properties of minimal HDP graphs in terms of the number of their "non-path-disjoint" vertices. The main theorem of this chapter, Theorem 26, establishes that every asteroidal triple in an HDP graph must have at least two of its AT vertices path-disjoint. In this way, minimal HDP graphs are divided into two subclasses, (2,2,2) and (1,2,2) graphs. Definitions of these two subclasses are given in Chapter 3.

Chapters 4 and 5 describe structural properties of (2,2,2) and (1,2,2) graphs respectively. The structure of (2,2,2) graphs is described first because this subclass is more restricted than the (1,2,2) graph subclass, i.e., a (1,2,2) graph can contain a (2,2,2) graph as its induced subgraph, while the opposite is not true. (2,2,2) graphs are further characterized in terms of the "length of the sides of the graph" into two-long-sided, one-long-sided, and no-long-sided graphs, while (1,2,2) graphs are similarly characterized into long-sided, and no-long sided graphs. Structural characteristics specific to the lengths of the sides of (2,2,2) and (1,2,2) graphs are described in appropriate subsections of Chapters 4 and 5, respectively. Standard definitions of the path *"length"* and the path *"size"* are used throughout the thesis to denote the number of edges and the number of vertices respectively.

The main results of Chapter 4 are given in Claims 13, 18, 21, 25, 30, and Corollaries 7 and 11. Claim 13 establishes that in a two-long-sided (2,2,2) graph with an AT $\{x, y, z\}$ and long paths in $\mathcal{P}_{x,y}$ and $\mathcal{P}_{x,z}$, the length of all paths in $\mathcal{P}_{y,z}$ must be equal to 2. The other main results of this chapter describe the positions of DP vertices with respect to AT vertices in different types of (2,2,2) graphs. These results completely characterize the positions of DP vertices in all (2,2,2) graphs.

Similarly, the main results of Chapter 5 are given in Claims 35, 39, 41, 44. Claim 35 establishes that in a (1,2,2) graph with an AT $\{x, y, z\}$ and a non-path disjoint vertex x, the length of all paths in $\mathcal{P}_{y,z}$ must be equal to 2. The other main results of this chapter describe the positions of DP vertices with respect to AT vertices in different types of (1,2,2) graphs. These results completely characterize the positions of DP vertices in all (1,2,2) graphs.

Finally, Chapter 6 contains concluding remarks and directions for further research in this area. One possible direction for further research is considered in more detail in section 6.1. In this section, different conjectures about lifting the Spine Property from AT-free graphs to HDP graphs are considered, and counterexamples to those conjectures are given. Also, a possible description of the position of DP vertices in HDP graphs that differs from the Spine Property seen in AT-free graphs is discussed.

Chapter 2

Background

2.1 Hierarchy of Graph Classes

As mentioned in the previous chapter, various graph classes have intensively been studied in the past. The most extensive survey of currently known results about graph classes loosely related to perfect graphs was done by Brandstadt, Le, and Spinard (1999). This thesis describes the structural properties of HDP and minimal HDP graphs, so it is important to know the relationship between HDP graphs and other graph classes in order to get a better intuitive feel about HDP graphs, as well as to understand which properties might hold for HDP graphs. Definitions of classes in the neighborhood of HDP graphs as well as the most important results for understanding the relationships between these classes are described below. Note that only a quick overview of the classes in the neighborhood of HDP graphs is given. The only results about these graph classes that are presented are those that are necessary to understand the hierarchy of these graph classes; many other properties are omitted and can be found in (Brandstadt, Le, and Spinard 1999).

Perfect Graphs

The following basic definitions will be used to define perfect graphs.

Definition 1 *Let $G = (V, E)$ be a graph.*

$V' \subseteq V$ is an independent set *in G if for all $u, v \in V', uv \notin E$,*

$V' \subseteq V$ is a clique *in G if for all $u, v \in V', u \neq v, uv \in E$,*

$\chi(G) = min\{k$: there is a partition of V into k disjoint independent sets$\}$,

$\omega(G) = max\{|V'| : V' \subseteq V$ and V' is a clique in $G\}$.

$\chi(G)$ is called the *chromatic number* of G, since it represents the smallest number of colors needed to properly color the vertices of G. $\omega(G)$ is called the *clique number* of G, since it represents the size of the largest complete subgraph of G. Clearly, $\chi(G) \geq \omega(G)$ for all G.

The following definition was introduced by Berge in the early 1960s (Berge 1960; Berge 1961).

Definition 2 *A graph G is* perfect *if for all induced subgraphs H of G, $\chi(H) = \omega(H)$.*

Golumbic (1980) presented many results on various perfect graph classes. One of the most important results concerning perfect graphs is the Perfect Graph Theorem (PGT) by Lovàsz (1972):

Theorem 1 *(Lovàsz 1972) The complement of a perfect graph is perfect.*

Comparability Graphs

Comparability graphs were defined by Ghouila-Houri (1962), and Gilmore and Hoffman (1964). The following is a simple formulation of the definition of comparability graphs:

Definition 3 *An undirected graph which is transitively orientable is called a* comparability graph, *or a* transitively orientable *graph.*

Transitively orientable *means that each edge of $G = (V, E)$ can be assigned a one-way direction, so that the resulting oriented graph satisfies the following condition: for all $a, b, c \in V$, $ab \in \vec{E}$ and $bc \in \vec{E}$ imply $ac \in \vec{E}$.*

The following result described the relationship between comparability and perfect graphs:

Theorem 2 *(Berge 1967) Every comparability graph is perfect.*

Cocomparability Graphs

A *cocomparability graph* is the complement of a comparability graph. Since, by Theorem 2, comparability graphs are perfect and since, by Theorem 1, the complement of a perfect graph is perfect, it can be concluded that cocomparability graphs are also perfect.

The following result established the relationship between cocomparability and AT-free graphs:

Theorem 3 *(Golumbic, Monma, and Trotter 1984) Every cocomparability graph is AT-free.*

Diametral Path Graphs

The *diameter* of a graph G, denoted by $diam(G)$, is defined to be the maximum distance between any pair of vertices in G. Kratsch (1995) introduced the following definitions:

Definition 4 *A pair of vertices u, v of a graph G such that the distance between u and v equals the diameter of G is called a* diametral pair (u, v) *of G.*

A shortest path P in G between the vertices of a diametral pair (u, v) is called a diametral path *of G.*

If in addition to being a diametral path, P is also a dominating set of G, then P is called a dominating diametral path *(DDP).*

A graph G is called a diametral path graph *if every connected induced subgraph of G has a dominating diametral path.*

Kratsch (1995) established that the class of AT-free graphs is properly contained in the class of diametral path graphs. He also defined a *dominating pair graph* as a graph in which every connected induced subgraph has a dominating pair. This definition is the same as the definition of an HDP graph used in this thesis. Kratsch (1995) also established the following result:

Theorem 4 *(Kratsch 1995) Any HDP graph is a diametral path graph.*

Chordal Graphs

Chordal graphs, also called *triangulated graphs*, are defined as follows:

Definition 5 *A graph G is* chordal *if each cycle in G of length at least 4 has at least one chord.*

These graphs were shown by Berge (1967) to be perfect. Kratsch (1995) presented the following theorem.

Theorem 5 *(Kratsch 1995) A chordal graph G is a diametral path graph if and only if it does not contain the graphs AT-1 and AT-2, shown in Figure 2.1, as induced subgraphs.*

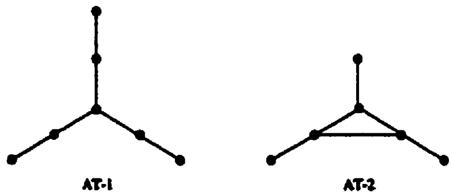


Figure 2.1: The minimal forbidden subgraphs for chordal diametral path graphs.

Trapezoid Graphs

The definition of trapezoid graphs was introduced by Corneil and Kamula (1987), and by Dagan, Golumbic, and Pinter (1988):

Definition 6 *G is a* trapezoid graph *if G is the intersection graph of a finite collection of trapezoids between two parallel lines.*

From the definition, interval graphs and permutation graphs described below are trapezoid graphs.

Permutation Graphs

The definition of permutation graphs was introduced by Even, Pnueli, and Lempel in (1971) and (1972):

Definition 7 *(1) Let $\mathcal{L}_1, \mathcal{L}_2$ be two parallel lines in the plane and label n points by $1, 2, 3, ..., n$ on $\mathcal{L}_1$ and also on $\mathcal{L}_2$. The straight line L_i connects i on $\mathcal{L}_1$ with i on $\mathcal{L}_2$.*

(2) Let $G_{\mathcal{L}} = (\{1, 2, ..., n\}, E_{\mathcal{L}})$ with $ij \in E_{\mathcal{L}}$ if L_i and L_j intersect each other.

(3) A graph G is called a permutation graph *if there is an intersection model $\mathcal{L}$ as described in condition (1) such that $G = G_{\mathcal{L}}$.*

The name *permutation graph* comes from the observation that the points on $\mathcal{L}_1$ and $\mathcal{L}_2$ can be seen as a permutation $\pi = \binom{1 \ldots n}{i_1 \ldots i_n}$, and $ij \in E_{\mathcal{L}}$ if and only if $(i-j)(\pi^{-1}(i) - \pi^{-1}(j)) < 0$, i.e., i and j form an inversion in π.

The following result explained the relationship between comparability, cocomparability, and permutation graphs.

Theorem 6 *(Pnueli, Lempel, and Even 1971) G is a permutation graph if and only if it is both comparability and cocomparability.*

Interval Graphs

Interval graphs are defined as follows:

Definition 8 *A graph G is an* interval graph *if the vertices of G can be put into a one-to-one correspondence with intervals on the real line, such that two vertices are adjacent in G if and only if the corresponding intervals have a nonempty intersection.*

Lekkerkerker and Boland (1962) characterized interval graphs as follows.

Theorem 7 *(Lekkerkerker and Boland 1962) A graph is an interval graph if and only if it is chordal and AT-free.*

Later, Gilmore and Hoffman (1964) strengthened (in light of the more recently known result that AT-free graphs strictly contain cocomparability graphs) this to:

Theorem 8 *(Gilmore and Hoffman 1964) A graph is an interval graph if and only if it is chordal and cocomparability.*

AT-free Graphs

The definition of AT-free graphs is given in Chapter 1. The structure of these graphs is explained in detail in section 2.2. Only a couple of major results with regards to the position of AT-free graphs in the graph class hierarchy are mentioned here.

Corneil, Olariu, and Stewart (1997) proved the following theorem which implies that AT-free graphs are HDP:

Theorem 9 *(Corneil, Olariu, and Stewart 1997) Every connected AT-free graph contains a dominating pair.*

In fact, they also showed (using the terminology of Kratsch (1995)):

Theorem 10 *(Corneil, Olariu, and Stewart 1997) Any AT-free graph is a diametral path graph.*

Clearly, AT-free graphs need not be perfect, since C_5 is AT-free.

HDP Graphs

The definition of HDP graphs is given in Chapter 1. The known structural results about these graphs are presented in section 2.3. Again, only major results that help in understanding the position of HDP graphs in the graph class hierarchy are mentioned here.

Kratsch (1995) proved the following result:

Theorem 11 *(Kratsch 1995) Any HDP graph is a diametral path graph.*

As mentioned above, all AT-free graphs are HDP graphs. In addition, some graphs that have asteroidal triples are also HDP, for example, C_6. There has not been much research on graphs that contain asteroidal triples. Thus, as mentioned in Chapter 1, graphs that contain asteroidal triples which are also HDP (AT∩HDP graphs) are the topic of this thesis.

The summary of the relationships between the described graph classes is represented in Figure 2.2.

2.2 Structure of AT-free Graphs

It has already been mentioned in Chapter 1 that the definition of an asteroidal triple was first introduced by Lekkerkerker and Boland (1962). Since 1989 Corneil, Olariu, and

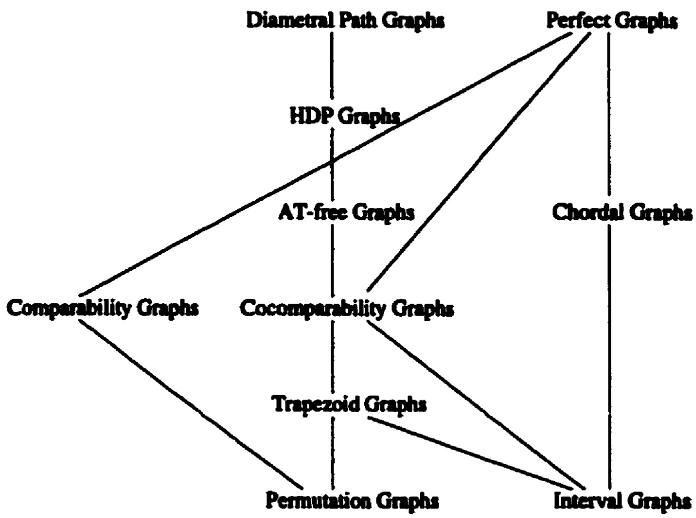


Figure 2.2: Hierarchy of the Graph Classes in the Neighborhood of HDP Graphs.

Stewart have been working intensively on AT-free graphs and proved many structural and algorithmic properties of these class of graphs [see for example their papers from (1994, 1995b, 1995a, 1997)]. Since this thesis deals with the structural properties of HDP graphs, only structural properties of AT-free graphs are presented in this section.

Theorem 9 by Corneil, Olariu, and Stewart (1997) mentioned above is implied by an even stronger result:

Theorem 12 *(Corneil, Olariu, and Stewart 1997) Let x be an arbitrary vertex of a connected AT-free graph G. Either (x, x) is a dominating pair, or else for a suitable choice of vertices y and z in $N'(x)$, (y, x), or (y, z) is a dominating pair. Here, $N'(x)$ denotes the set of all vertices adjacent to x in the complement $\bar{G}$ of G.*

They also proved the following result that Kratsch (1995) used in his work on diametral path graphs:

Theorem 13 *(Corneil, Olariu, and Stewart 1997) In every connected AT-free graph some dominating pair achieves the diameter.*

They gave two characterizations of AT-free graphs. The first one characterized AT-free graphs based on dominating pairs, while the second one characterized AT-free graphs in terms of minimal triangulations.

For the first characterization, they introduced the spine property as follows. A connected graph H with a dominating pair satisfies the *spine property* if for every nonadjacent dominating pair (α, β) in H there exists a neighbor α' of α such that (α', β) is a dominating pair of the connected component of $H \setminus \alpha$ containing β. Then they proved:

Theorem 14 *(Corneil, Olariu, and Stewart 1997) (The Spine theorem) A graph G is AT-free if and only if every connected induced subgraph H of G satisfies the spine property.*

This result enabled them to formulate a *spine* of G as follows. Let $G = (V, E)$ be a connected AT-free graph with (x, y) as an arbitrary nonadjacent dominating pair. Construct a sequence $x_0, x_1, ..., x_k$ of vertices of G and a sequence $G_0, G_1, ..., G_k$ of subgraphs of G defined in the following way:

(i) $G_0 = G$ and $x_0 = x$,

(ii) for all $i (0 \leq i \leq k-1)$, $x_i y \notin E$ and $x_k y \in E$,

(iii) for all $i (1 \leq i \leq k)$, let G_i denote the subgraph of G_{i-1} induced by the component of $G_{i-1} \setminus \{x_{i-1}\}$ containing y,

(iv) for all $i (1 \leq i \leq k)$, let x_i be a vertex in G_i adjacent to x_{i-1} and such that (x_i, y) is a dominating pair in G_i.

The Spine theorem guarantees the existence of the sequence $x_1, ..., x_k$. They referred to the sequence $x_1, ..., x_k$ as a spine of G. They also emphasized that the existence of a sequence of vertices and a sequence of subgraphs defined in (i) through (iv) above does not necessarily imply that the graph is AT-free.

Their second characterization of AT-free graphs in terms of minimal triangulations was as follows. For an arbitrary graph $G = (V, E)$, a *triangulation* $T(G)$ of G is a set of edges such that the graph $G' = (V, E \cup T(G))$ is chordal. A triangulation $T(G)$ is *minimal* when no proper subset of $T(G)$ is a triangulation of G. They quoted R. H. Möhring (1996) for the following result:

Theorem 15 *(R. H. Möhring 1996) If G is an AT-free graph, then for every minimal triangulation $T(G)$ of G, the graph $G' = (V, E \cup T(G))$ is an interval graph.*

Then they proved the converse of Theorem 15 which resulted in the following theorem:

Theorem 16 *(Corneil, Olariu, and Stewart 1997) A graph G is AT-free if and only if, for every minimal triangulation $T(G)$ of G, the graph $G' = (V, E \cup T(G))$ is an interval graph.*

They also gave some interesting results about augmenting AT-free graphs that confirm the linear structure of AT-free graphs, since the dominating pair can be stretched to a new dominating pair. They called a vertex v of an AT-free graph G *pokable* if the graph G' obtained from G by adding a degree 1 vertex adjacent to v is AT-free; otherwise v is called *unpokable*. A dominating pair (x, y) is referred to as *pokable* if both x and y are pokable. After the introduction of these definitions, they proved the following results:

Theorem 17 *(Corneil, Olariu, and Stewart 1997) Every connected AT-free graph contains a pokable dominating pair; furthermore, every connected AT-free graph which is not a clique contains a nonadjacent pokable dominating pair.*

Theorem 18 *(Corneil, Olariu, and Stewart 1997) (The Composition theorem) Given two AT-free graphs G_1 and G_2 and pokable dominating pairs (x_1, y_1) and (x_2, y_2) in G_1 and G_2 respectively, let G be the graph constructed from G_1 and G_2 by identifying vertices x_1 and x_2. Then G is an AT-free graph.*

Another interesting result by Corneil, Olariu, and Stewart (1997) deals with contracting AT-free graphs. First, they introduced the following definitions. For an AT-free graph $G = (V, E)$ with at least two vertices and a pokable dominating pair (x, y), define a binary relation R on G by writing for every pair u, v of vertices, $uRv \iff D(u, x) = D(v, x)$, where $D(a, b)$ denotes the set of vertices that intercept all a, b-paths. Clearly, R is an equivalence class; denote by $C_1, ..., C_k (k \geq 1)$ the equivalence classes of $G|R$. A class C_i is called *nontrivial* if $|C_i| \geq 2$. Then they proved:

Claim 1 *(Corneil, Olariu, and Stewart 1997) $G|R$ contains at least one nontrivial equivalence class.*

They called a nontrivial class C of $G|R$ *valid* if C induces a connected subgraph of G. Then they proved:

Claim 2 *(Corneil, Olariu, and Stewart 1997) $G|R$ contains at least one valid equivalence class.*

They also introduced the following definition. Let S be a set of vertices of G. The graph G' is said to arise from G by an *S-contraction* if G' contains all the vertices of $G \setminus S$ along with a new vertex s adjacent, in G', to all the vertices in $G \setminus S$ that were adjacent, in G, to some vertex in S. They proved:

Theorem 19 *(Corneil, Olariu, and Stewart 1997) Let C be an arbitrary valid equivalence class of $G|R$. The graph G' obtained from G by a C-contraction is AT-free.*

They also gave the following results about high diameter AT-free graphs:

Theorem 20 *(Corneil, Olariu, and Stewart 1997) Let G be a connected AT-free graph with diameter at least four. There exist nonempty, disjoint sets X and Y of vertices of G such that (x, y) is a dominating pair if and only if $x \in X$ and $y \in Y$.*

Corneil, Olariu, and Stewart (1997) also established some results about graphs that contain asteroidal triples. First, they defined the following. If $\pi = u_1, u_2, ..., u_k$ and $\pi_1 = v_1, v_2, ..., v_l$ are two paths, then the path $u_1, u_2, ..., u_i$ with $i \leq k$ is referred to as a *prefix* of π. A vertex w is a *cross* point of π and π_1 if $w = u_i = v_j$ and the four vertices $u_{i-1}, v_{j-1}, u_{i+1}$, and v_{j+1} are all defined and distinct. They denoted by G a graph containing an AT and chose an induced subgraph H of G with the least number of vertices such that some triple $\{x, y, z\}$ is an AT in H. They denoted by $\pi(x, y), \pi(x, z)$, and $\pi(y, z)$ paths in H demonstrating that $\{x, y, z\}$ is an AT. They also wrote $\pi(x, y)$: $x = u_1, u_2, ..., u_k = y$, $\pi(x, z) : x = v_1, v_2, ..., u_l = z$, and $\pi(y, z) : y = w_1, w_2, ..., w_t = z$. Then they showed the following:

Claim 3 *(Corneil, Olariu, and Stewart 1997) No pair of paths among $\pi(x, y)$, $\pi(x, z)$, and $\pi(y, z)$ has a cross point.*

Claim 4 *(Corneil, Olariu, and Stewart 1997) Let i be the largest subscript for which there exists a subscript j such that $u_i = v_j$ and $u_{i+1} \neq v_{j+1}$. Then $i = j$ and $u_t = v_t$ for all $1 \leq t \leq i$.*

Lemma 1 *(Corneil, Olariu, and Stewart 1997) There exist unique vertices x', y', z' in H such that*

(i) the unique path between x and x' is a prefix of both $\pi(x, y)$ and $\pi(x, z)$,

(ii) the unique path between y and y' is a prefix of both $\pi(y, x)$ and $\pi(y, z)$,

(iii) the unique path between z and z' is a prefix of both $\pi(z, x)$ and $\pi(z, y)$.

Claim 5 *(Corneil, Olariu, and Stewart 1997) The vertices x', y' and z' are either all distinct or else they coincide.*

Corollary 1 *(Corneil, Olariu, and Stewart 1997) Vertices x', y' and z' coincide if and only if H is isomorphic to the graph in Figure 2.3.*

Broersma, Kloks, Kratsch, and Muller (1999) presented an $O(n^4)$ algorithm to compute the maximum weight of an independent set in a given AT-free graph. To do this, they used the following definitions to describe some additional structural properties of AT-free graphs. In a graph $G = (V, E)$, they denoted by $C^x(y)$ the connected component of $G[V \setminus N[x]]$ containing the vertex y. Also, they denoted by $r(x)$ the number of

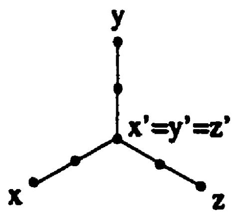


Figure 2.3: Illustration of Corollary 1

connected components of $G[V \setminus N[x]]$. For nonadjacent vertices x and y in G, they said that a vertex $z \in V \setminus \{x, y\}$ is *between* x and y if there is an x, z-path avoiding $N[y]$ and there is an y, z-path avoiding $N[x]$. For nonadjacent vertices x and y in G they said that the *interval* $I = I(x, y)$ is the set of all vertices that are between x and y. Thus, $I(x, y) = C^x(y) \cap C^y(x)$.

They described the following results about splitting intervals in AT-free graphs. Let $G = (V, E)$ be an AT-free graph, let $I = I(x, y)$ be an interval, let $s \in I$, and let $I_1 = I(x, s)$ and $I_2 = I(s, y)$.

Lemma 2 *(Broersma, Kloks, Kratsch, and Muller 1999) x and y are in different connected components of $G[V \setminus N[s]]$.*

Corollary 2 *(Broersma, Kloks, Kratsch, and Muller 1999) $I_1 \cap I_2 = \emptyset$.*

Lemma 3 *(Broersma, Kloks, Kratsch, and Muller 1999) $I_1 \subsetneq I$ and $I_2 \subsetneq I$.*

Theorem 21 *(Broersma, Kloks, Kratsch, and Muller 1999) There exist connected components $C_1^s, C_2^s, ..., C_t^s$ of $G[V \setminus N[s]]$ such that*

$$I - N[s] = I_1 \cup I_2 \cup \bigcup_{i=1}^{t} C_i^s.$$

Corollary 3 *(Broersma, Kloks, Kratsch, and Muller 1999) Every connected component of $G[I - (N[s] \cup I_1 \cup I_2)]$ is a connected component of $G[V \setminus N[s]]$.*

They also described the results about splitting connected components of AT-free graphs. As before, they let $G = (V, E)$ be an AT-free graph, let x be a vertex of G, let

C^x be a connected component of $G[V \setminus N[x]]$, let $y \in C^x$, and let $I = I(x,y)$. They studied the connected components of $G[C^x - N[y]]$ and gave the following results.

Theorem 22 *(Broersma, Kloks, Kratsch, and Muller 1999) Let D be a connected component of $G[C^x - N[y]]$. Then $N[D] \cap (N[x] \setminus N[y]) = \emptyset$ if and only if D is a connected component of $G(V \setminus N[y])$ with $D \subseteq C^x$.*

Corollary 4 *(Broersma, Kloks, Kratsch, and Muller 1999) Let B be a connected component of $G[C^x - N[y]]$. Then $N[B] \cap (N[x] \setminus N[y]) \neq \emptyset$ if and only if $B \subseteq C^y(x)$.*

Theorem 23 *(Broersma, Kloks, Kratsch, and Muller 1999) Let $B_1, ..., B_l$ be all the connected components of $G[C^x - N[y]]$, that are contained in $C^y(x)$. Then $I = \bigcup_{i=1}^{l} B_i$.*

2.3 Structure of HDP Graphs

Kratsch (1995) described some of the structural properties of diametral path graphs. As mentioned above, the class of diametral path graphs is a superclass of HDP graphs (see Theorem 4), so all results that apply to diametral path graphs apply to HDP graphs as well.

The definition of diametral path graphs was given above. Kratsch (1995) also defined a *dominating shortest path* of a graph G as a path between vertices x and y in G of length $d_G(x,y)$, where $d_G(x,y)$ is the distance between vertices x and y in G, i.e., it is the length of the shortest path between x and y. Then he proved the following theorem.

Theorem 24 *(Kratsch 1995) A graph G is a diametral path graph if and only if every connected induced subgraph H of G has a dominating shortest path.*

It has already been asserted in Theorem 10 that AT-free graphs are diametral path graphs. Moreover, Kratsch (1995) gave examples of many graphs that are diametral path graphs, but not AT-free, i.e., the class of diametral path graphs properly contains AT-free graphs as a subclass.

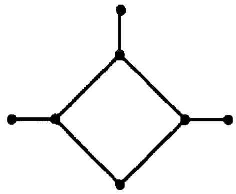


Figure 2.4: A diametral path graph that is not HDP.

Theorem 11 above explained the relationship between HDP graphs and diametral path graphs. Furthermore, diametral path graphs properly contain HDP graphs (see example in Figure 2.4). We also know that HDP graphs properly contain AT-free graphs (an example is graph C_6).

Theorem 5 above gave the relationship between chordal and diametral path graphs. Kratsch (1995) also characterized diametral path graphs in terms of *PATH-MCDS*, where PATH-MCDS is a minimum connected dominating set that induces a path in G.

Theorem 25 *(Kratsch 1995) Let $G = (V, E)$ be a connected diametral path graph with $diam(G) > 4$. Then G has a PATH-MCDS.*

Chapter 3

General Structure of HDP Graphs

This chapter and the following two chapters contain original work.

This chapter describes some structural properties of both HDP and minimal HDP graphs. The main result of the chapter is Theorem 26 which establishes that every AT of an AT∩HDP graph must have at least two of its vertices path-disjoint. Definition 9 introduces a notion of a path-disjoint vertex. Theorem 26 is important since it implies that only two types of minimal AT∩HDP graphs exist: those with all AT vertices being path-disjoint, and those with one non-path-disjoint AT vertex in at least one of their ATs. Consequently, these two types of minimal HDP graphs are introduced in Definitions 12 and 13.

The following lemma, i.e., Lemma 4, is used in the proof of Claim 6. Claim 6 is applied extensively in the proofs of claims throughout the thesis. Some examples if its use are in the proofs of Corollary 5, and Claims 8, 9, 11. The main result of this chapter, Theorem 26, follows from Claim 11.

The following notation will be used throughout this chapter. Let H be an HDP∩AT graph with an AT $\{x, y, z\}$. As stated previously, let $\mathcal{P}_{x,y}$ denote the set of all induced paths from x to y that avoid $N(z)$, $\mathcal{P}_{x,z}$ the set of all induced paths from x to z that avoid $N(y)$, and $\mathcal{P}_{y,z}$ the set of all induced paths from y to z that avoid $N(x)$ in H.

Let $P_{x,y}$, $P_{x,z}$, $P_{y,z}$ be arbitrary induced paths that establish the AT $\{x, y, z\}$ in H. The reader is reminded that standard definitions of the path length and the path size will be used to represent the number of edges and the number of vertices respectively. In the definitions and claims in this chapter which say "let H be defined as above," or "let H be defined as in the paragraph preceding Lemma 4," it is assumed that H is defined as in this paragraph. The same holds for $\mathcal{P}_{x,y}$, $\mathcal{P}_{x,z}$, and $\mathcal{P}_{y,z}$.

Lemma 4 *If H is an HDP graph, then there does not exist an AT $\{x, y, z\}$ in H with paths $P_{x,y}$, $P_{x,z}$, and $P_{y,z}$ establishing AT $\{x, y, z\}$, such that all three paths have a common vertex w.*

Proof: Assume to the contrary. That is, assume H is an HDP graph with an AT $\{x, y, z\}$, induced paths $P_{x,y}$, $P_{x,z}$, $P_{y,z}$ that establish an AT $\{x, y, z\}$, and a common vertex w of these three paths. Let H' be the graph induced on the union of paths $P_{w,x} \cup P_{w,y} \cup P_{w,z}$ of H, where $P_{w,x}$ is the path between w and x induced on $P_{x,y}$, $P_{w,y}$ is the path between w and y induced on $P_{x,y}$, and $P_{w,z}$ is the path between w and z induced on $P_{x,z}$. Since $\{x, y, z\}$ was an AT of H, the definition of H' implies that $\{x, y, z\}$ is also an AT of H'. The length of $P_{w,x}$ is at least 2 because $\{x, y, z\}$ is an AT of H'. Similarly, the length of $P_{w,y}$ and the length of $P_{w,z}$ are both at least 2.

Since H is an HDP, H' has a DP, say (α, β). Each of the DP vertices must belong to one of $P_{w,x}$, $P_{w,y}$, and $P_{w,z}$, by definition of H'. The DP vertices cannot both belong to $P_{w,x}$ because in that case a path between them induced on $P_{x,w}$ avoids $N(z)$, which would contradict the assumption that (α, β) is a DP. This is because z cannot be adjacent to any vertex of $P_{x,w}$, since otherwise the path $P_{x,y}$, whose subpath is the path $P_{x,w}$ by definition, would not avoid the neighborhood of z contradicting the fact that $\{x, y, z\}$ is an AT of H'. Similarly, both DP vertices cannot belong to $P_{w,y}$ and cannot belong to $P_{w,z}$. Therefore, they must belong to different paths in $\{P_{w,x} \cup P_{w,y} \cup P_{w,z}\}$. W.l.o.g. assume that $\alpha \in P_{w,y}$ and $\beta \in P_{w,z}$. Now, the path from α to β induced on $P_{\alpha,w} \cup P_{w,\beta}$, where $P_{\alpha,w}$ is the path between α and w induced on $P_{w,y}$ and $P_{w,\beta}$ is the path between β and w induced on $P_{w,x}$, avoids the neighborhood of x, again contradicting the fact that (α, β) is a DP of H'. $P_{\alpha,w} \cup P_{w,\beta}$ avoids $N(x)$ because it is actually a subpath of $P_{y,x}$. Thus, H' cannot have a DP. This contradicts the assumption that H is HDP. □

The path-disjoint and non-path-disjoint vertices mentioned before are defined as follows.

Definition 9 *Let H, $\mathcal{P}_{x,y}$, and $\mathcal{P}_{x,z}$ be defined as above. An AT vertex x is called* path-disjoint with respect to y, z *if for all paths $P \in \mathcal{P}_{x,y}$ and for all paths $Q \in \mathcal{P}_{x,z}$, $P \cap Q = \{x\}$.*

An AT vertex x is called non-path-disjoint *if there exist paths $P \in \mathcal{P}_{x,y}$ and $Q \in \mathcal{P}_{x,z}$ such that $P \cap Q \supseteq \{x, x'\}$, where $x \neq x'$.*

Note that sometimes "x is path disjoint" will be used to mean "x is path disjoint with respect to y, z."

Definition 9 motivates defining a significant neighbor of a non-path-disjoint vertex, as well as defining a 1-disjoint graph. These two terms are defined as follows.

Definition 10 *A neighbor x' of x in an HDP$\cap$AT graph H with an AT $\{x, y, z\}$ and a non-path-disjoint vertex x is called a* significant neighbor of x *if $x' \in P_{x,y} \cap P_{x,z}$, for some induced paths $P_{x,y} \in \mathcal{P}_{x,y}$ and $P_{x,z} \in \mathcal{P}_{x,z}$.*

Definition 11 *Consider an HDP$\cap$AT graph H with a non-path-disjoint vertex x and a significant neighbor x' of x, such that $H \setminus \{x'\}$ is disconnected into a connected component containing y and z and not containing x, and some other connected components. Such a graph will be called* 1-disjoint w.r.t. x.

Claim 6 *Let H be defined as in the paragraph preceding Lemma 4, let x be a non-path-disjoint vertex of H w.r.t. y, z, and let $P_{x,y}$, $P_{x,z}$, $P_{y,z}$ be induced paths establishing the AT. If any $x' \in P_{x,y} \cap P_{x,z}$, where $x' \neq x$, then xx' is an edge in H.*

Proof: Assume xx' is not an edge in H. Then, since $x \neq x'$, there must exist a path $P_{x,x'}$ between x and x' of length at least 2. W.l.o.g. let the path $P_{x,x'}$ be a subpath of $P_{x,y}$. Also, let $P_{x',y}$ and $P_{x',z}$ be subpaths of $P_{x,y}$ between x' and y, and of $P_{x,z}$ between x' and z respectively. The assumption that $x' \in P_{x,y} \cap P_{x,z}$ implies that the length of $P_{x',y}$ and the length of $P_{x',z}$ are both at least 2, since otherwise x' would be in the neighborhood of either y, or z, or both, contradicting the fact that $P_{x,y}$, $P_{x,z}$, and $P_{y,z}$ establish an AT $\{x, y, z\}$ in H.

Now look at the graph H' induced on $P_{x',x} \cup P_{x',y} \cup P_{x',z}$. Note that in H', $\{x, y, z\}$ is still an AT. But now, x' is a common vertex of the paths that establish an AT in H' and therefore, by Lemma 4, H' cannot have a DP. Since H' is an induced subgraph of H, this contradicts the assumption that H is an HDP graph. □

Corollary 5 *Let H be defined as in the paragraph preceding Lemma 4 with a non-path-disjoint vertex x w.r.t. y, z. Let $P_{x,y} \in \mathcal{P}_{x,y}$ and $P_{x,z} \in \mathcal{P}_{x,z}$ be such that $x' \in P_{x,y} \cap P_{x,z}$, and $x' \neq x$. $H \setminus \{x\}$ has only one connected component C.*

Proof: Assume to the contrary. Let $H \setminus \{x\}$ have two connected components, C containing y, z, and D not containing them. Consider any vertex $\tilde{x}$ of D that is adjacent to x in H. Now, $\{\tilde{x}, y, z\}$ is also an AT in H, but $\tilde{x}x'$ is not an edge, contradicting Claim 6. □

The following claim presents an interesting result that will be used in the proofs throughout this thesis, such as for example, in the proofs of Claims 8, 39, and Corollary 8.

Claim 7 *For any HDP graph G and any of its DP's (α, β), if $H \lhd G$, H is connected and $\alpha, \beta \in H$, then (α, β) is a DP of H.*

Proof: Assume that (α, β) is not a DP of H. Then, since H is connected, there exists a path P from α to β in H that misses a vertex $w \in H$. However, P also belongs to G and misses w in G, contradicting (α, β) being a DP of G. □

The following claim will be used in the proofs of Claims 11 and 37. The reader is reminded that the main result of this chapter is a corollary of Claim 11.

Claim 8 *Let H be defined as in the paragraph preceding Lemma 4 with induced paths $P_{x,y}$, $P_{x,z}$, $P_{y,z}$ establishing the AT, such that $x' \in P_{x,y} \cap P_{x,z}$, and $x' \neq x$. Let $H \setminus \{x'\}$ be disconnected, i.e. H is 1-disjoint w.r.t. x, and let C denote the connected component of $H \setminus \{x'\}$ that contains y and z. Then, for every DP (α, β) of H, one of α, β is in $H \setminus C$ and the other is in C. Furthermore, if H is minimal, then $H \setminus C = \{x, x'\}$.*

Proof: Let $\tilde{H}$ be the subgraph of H induced on $P_{x,y} \cup P_{x,z} \cup P_{y,z} \cup \{\alpha, \beta\}$. By Claim 7, (α, β) is a DP of $\tilde{H}$ as well.

Suppose both $\alpha, \beta \in C \cap \tilde{H}$. Since $\tilde{H}$ is an induced subgraph of H, any path L from α to β in $C \cap \tilde{H}$ is also in H. No vertex v of L is adjacent to x in H for the following reasons. Since $v \in L \in C \cap \tilde{H}$, $v \in P_{x,y} \cup P_{x,z} \setminus \{x, x'\}$. If $v \in \{y, z\}$, then $vx \notin E$ in H, since $\{x, y, z\}$ is an AT of H. If $v \in P_{x,y} \setminus \{x, x', y\}$, or $v \in P_{x,z} \setminus \{x, x', z\}$, then $vx \notin E$ in H, since $P_{x,y}$ and $P_{x,z}$ are induced paths. If $v \in P_{y,z} \setminus \{y, z\}$, then $xv \notin E$ in H, since all vertices on the path $P_{y,z}$ must miss x because paths $P_{x,y}$, $P_{x,z}$, $P_{y,z}$ establishing the AT $\{x, y, z\}$. Therefore, all vertices of L miss x contradicting (α, β) being a DP of H. Therefore, at least one of α, β must be in $H \setminus C$.

Now, suppose both $\alpha, \beta \in H \setminus C$. Then, any path from α to β in $H \setminus C$ misses both y and z contradicting (α, β) being a DP of H. Therefore, not both α and β are in $H \setminus C$.

It is proven that at least one DP vertex must be in $H \setminus C$, but not both of them are in $H \setminus C$. Thus, one of α, β is in $H \setminus C$ and the other is in C. An example is in Figure 3.1.

Now prove that if H is minimal, then $H \setminus C = \{x, x'\}$. Let H be a minimal HDP graph. Let $H \setminus C = \{x, x', v\}$, $x \neq v \neq x'$. $vx' \in E$, since otherwise it would contradict Claim 6. Now, either $xv \in E$, or $xv \notin E$. If $xv \in E$, then v belongs to a path in $\mathcal{P}_{x,y}$ that is not an induced path, which contradicts the definition of $\mathcal{P}_{x,y}$. If $xv \notin E$, then v

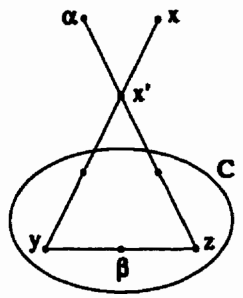


Figure 3.1: Illustration of Claim 8.

does not belong to any path in $\mathcal{P}_{x,y} \cup \mathcal{P}_{x,z} \cup \mathcal{P}_{y,z}$, which again contradicts the definition of a minimal HDP graph. □

Claim 9 *Let H be 1-disjoint w.r.t. x. If $v \notin C$, then v is either x' or it is adjacent to x'.*

Proof: Assume $v \neq x'$ and v is not adjacent to x'. Then, there exists a path Q in $H \setminus C$ of length at least 2 between x' and v. Let $P_{v,y}$ be the path from v to y induced on $Q \cup P_{x',y}$, and let $P_{v,z}$ be the path from v to z induced on $Q \cup P_{x',z}$. But now, $\{v, y, z\}$ is an AT and $\{x'\} \subseteq P_{v,y} \cap P_{v,z}$, and $vx' \notin E$ contradicting Claim 6. □

The following claim will be used in the proofs of Claims 35 and 37.

Claim 10 *Let H be 1-disjoint w.r.t. x, and let (α, β) be a DP of H. If $\alpha \notin C$, then $\beta \notin P_{x,y} \cup P_{x,z}$ for any path $P_{x,y}$ between x and y that avoids $N(z)$, and any path $P_{x,z}$ between x and z that avoids $N(y)$.*

Proof: W.l.o.g. if $\beta \in P_{x,y}$, then the path from α to β induced on $P_{x,y} \cup \{\alpha, x'\}$ does not hit z contradicting (α, β) being a DP of H. □

The claims above will be used to prove the following.

Claim 11 *If x is non-path-disjoint with respect to y, z, and y is non-path-disjoint with respect to x, z in a graph H with an AT $\{x, y, z\}$, i.e., $\exists x' \in P_{x,y} \cap P_{x,z}$, $x' \neq x$, and*

$\exists y' \in P_{x,y} \cap P_{y,z}$, $y' \neq y$, for some paths $P_{x,y} \in \mathcal{P}_{x,y}$, $P_{x,z} \in \mathcal{P}_{x,z}$, and $P_{y,z} \in \mathcal{P}_{y,z}$, then H is not an HDP graph.

Proof: Assume to the contrary. Thus, H is HDP. By Claim 6, xx' and yy' are edges in H. First note that $x' \neq y'$. This is because if $x' = y'$, then y intercepts path $P_{x,z}$ in H contradicting the fact that $\{x, y, z\}$ is an AT in H.

Let $\tilde{H}$ be the subgraph of H induced on $P_{x,y} \cup P_{x,z} \cup P_{y,z}$. Now, $\tilde{H}$ is a minimal HDP graph. Let (α, β) be a DP of $\tilde{H}$. By Claim 8, $\tilde{H} \setminus C_x = \{x, x'\}$ contains a DP vertex, w.l.o.g. say $\alpha \in \tilde{H} \setminus C_x$, where C_x is the connected component of $\tilde{H} \setminus \{x'\}$ that contains y and z. By the same reasoning, $\beta \in \tilde{H} \setminus C_y = \{y, y'\}$, where C_y is the connected component of $\tilde{H} \setminus \{y'\}$ that contains x and z. Consider the path joining α and β that is induced on $\{\alpha, \beta\} \cup P_{x',y'}$, where $P_{x',y'}$ is a subpath of $P_{x,y}$ between x' and y'. This path misses z, since $\{x, y, z\}$ is an AT, contradicting (α, β) being a DP of $\tilde{H}$. □

The following theorem is the main result of this chapter. It follows from the previous claim.

Theorem 26 *If H is an $AT \cap HDP$ graph, then for any AT in H at least two of its AT vertices are path-disjoint.*

Proof: Follows immediately from Claim 11. □

This theorem motivates the introduction of the Definitions 12, 13, 14, and 15. These definitions separate minimal HDP and minimal AT graphs into subclasses called (2,2,2), (1,2,2), (1,1,2), and (1,1,1) graphs, depending on the number of non-path-disjoint AT vertices. Note that (1,2,2) notation is used for minimal HDP graphs with exactly one non-path-disjoint AT vertex, and that the order in which path-disjoint and non-path-disjoint AT vertices appear does not matter; i.e., (2,1,2) and (2,2,1) notation need not to be used. The same holds for (1,1,2) notation.

Definition 12 *A minimal HDP graph is called a* (2,2,2) graph *if it has no non-path-disjoint AT vertices.*

Definition 13 *A minimal HDP graph is called a* (1,2,2) graph *if it has exactly one non-path-disjoint AT vertex.*

Definition 14 *A minimal AT graph is called a* (1,1,2) graph *if it has exactly two non-path-disjoint AT vertices.*

Definition 15 *A minimal AT graph is called a* (1,1,1) graph *if it has exactly three non-path-disjoint AT vertices.*

From Theorem 26, and Definitions 12 and 13 it can be concluded that the only AT graphs that are minimal HDP are (2,2,2) and (1,2,2) graphs. Therefore, the following two chapters will talk about (2,2,2) and (1,2,2) graphs. Before analyzing the structure of (2,2,2) and (1,2,2) graphs in detail, some additional definitions and claims about the general structure of HDP∩AT graphs will be introduced.

Recall that standard definitions of the path length and the path size are used throughout this thesis to denote the number of edges and the number of vertices respectively. The size of a path P is denoted by $|P|$.

Definition 16 *A path of size bigger than* 3 *is called a* long path.

For (1,2,2) graphs the definition of $\mathcal{R}_{x,y}$, and $\mathcal{R}_{x,z}$ is needed. Let $\mathcal{R}_{x,y} \subset \mathcal{P}_{x,y}$ be the set of all paths $P \in \mathcal{P}_{x,y}$ such that $\exists Q \in \mathcal{P}_{x,z}$ and $x' \in P \cap Q$ such that $x' \neq x$. Similarly, let $\mathcal{R}_{x,z} \subset \mathcal{P}_{x,z}$ be the set of all paths $P \in \mathcal{P}_{x,z}$ such that $\exists Q \in \mathcal{P}_{x,y}$ and $x' \in P \cap Q$ such that $x' \neq x$. Let $\mathcal{R}_{x',y}$ be the set of subpaths of paths in $\mathcal{R}_{x,y}$ between x' and y, and $\mathcal{R}_{x',z}$ be the set of subpaths of paths in $\mathcal{R}_{x,z}$ between x' and z.

Definition 17 *A (2,2,2) graph H will be called* two-long-sided *if there exists $P_{x,y} \in \mathcal{P}_{x,y}$ and there exists $P_{x,z} \in \mathcal{P}_{x,z}$ in H such that $|P_{x,y}| > 3$, and $|P_{x,z}| > 3$. A (1,2,2) graph H*

will be called two-long-sided *if there exists* $R_{x',y} \in \mathcal{R}_{x',y}$ *and there exists* $R_{x',z} \in \mathcal{R}_{x',z}$ *in* H *such that* $|R_{x',y}| > 3$, *and* $|R_{x',z}| > 3$.

Definition 18 *A (2,2,2) graph H will be called* one-long-sided *if there exists* $P_{x,y} \in \mathcal{P}_{x,y}$ *in* H *such that* $|P_{x,y}| > 3$, *and both* $\mathcal{P}_{x,z}$ *and* $\mathcal{P}_{y,z}$ *in H consist of* P_3*'s only. A (1,2,2) graph H will be called* one-long-sided *if there exists* $R_{x',y} \in \mathcal{R}_{x',y}$ *in H such that* $|R_{x',y}| > 3$, *and both* $\mathcal{R}_{x',z}$ *and* $\mathcal{P}_{y,z}$ *in H consist of* P_3*'s only.*

Definition 19 *A (2,2,2) graph H will be called* no-long-sided *if* $\mathcal{P}_{x,y}$, $\mathcal{P}_{x,z}$, *and* $\mathcal{P}_{y,z}$ *in H consist of* P_3*'s only. A (1,2,2) graph H will be called* no-long-sided *if* $\mathcal{R}_{x',y}$, $\mathcal{R}_{x',z}$, *and* $\mathcal{P}_{y,z}$ *in H consist of* P_3*'s only.*

Definition 20 *A (2,2,2) or a (1,2,2) graph is called* long-sided *if it is either one-long-sided, or two-long-sided.*

Note that any (2,2,2) and any (1,2,2) graph is either two-long-sided, or one-long-sided, or no-long-sided. The structure of these three types of (2,2,2) graphs will be considered separately in sections 4.1, 4.2, and 4.3. The structure of these three types of (1,2,2) graphs will be considered in sections 5.1, and 5.2.

The structure of two-long-sided (2,2,2) graphs will differ depending on the lengths of paths in $\mathcal{P}_{x,y}$ and $\mathcal{P}_{x,z}$. A specific structure will result if in addition to having long paths in $\mathcal{P}_{x,y}$ and $\mathcal{P}_{x,z}$, a two-long-sided (2,2,2) graph H also has a path of length 2 in $\mathcal{P}_{x,y}$, or in $\mathcal{P}_{x,z}$. Similarly, a specific structure will result if a two-long-sided (2,2,2) graph has more than one long path in $\mathcal{P}_{x,y}$, or more than one long path in $\mathcal{P}_{x,z}$. The position of DP vertices in these two sub-families of two-long-sided (2,2,2) graphs will be examined in subsections 4.1.1 and 4.1.2. For that purpose, the paths of length 2 will be called ***short paths***, and the set of paths that contains a long path, i.e., $\mathcal{P}_{x,y}$ or $\mathcal{P}_{x,z}$, will be called a ***long side***. Similarly, the structure of one-long-sided (2,2,2) graphs with the long side containing a short path will be described in subsection 4.2.1, and the structure of one-long-sided (2,2,2) graphs with the long side containing at least two long paths will be described in subsection 4.2.2. Equivalent results hold for (1,2,2) graphs and will be explained in subsections 5.1.1, 5.1.2.

The structure of all these subfamilies of graphs will be examined in order to determine the position of their DP's in relation to position of their AT vertices.

The following three lemmas present results about the position of DP vertices in HDP graphs. They will be used in the proofs of Claims 13, 18, 30, 39, and Lemma 8. Note that sometimes the terminology will be abused by letting $\mathcal{P}_{x,y}$ denote both the set of paths between x and y that miss z, and the union of the vertices on these paths.

Lemma 5 *Let (α, β) be a DP of an AT$\cap$HDP graph H. α and β cannot both belong to $\mathcal{P}_{x,y}$, cannot both belong to $\mathcal{P}_{x,z}$, and cannot both belong to $\mathcal{P}_{y,z}$.*

Proof: W.l.o.g. assume that α and β both belong to $\mathcal{P}_{x,y}$. Since the subgraph induced on the vertices in $\mathcal{P}_{x,y}$ is connected, there is a path between them that misses z, contradicting (α, β) being a DP. □

Lemma 6 *Let (α, β) be a DP of a graph H.*

(1) If H is a two-long-sided (2,2,2) graph, then α and β cannot belong to the union of the internal vertices of $P_{x,y}$ and $P_{x,z}$, where $P_{x,y}$ and $P_{x,z}$ are long paths in $\mathcal{P}_{x,y}$ and $\mathcal{P}_{x,z}$ respectively.

(2) If H is a (1,2,2) graph, then α and β cannot belong to the union of the vertices of $R_{x',y}$ and $R_{x',z}$, where $R_{x',y}$ and $R_{x',z}$ are paths in $\mathcal{R}_{x',y}$ and $\mathcal{R}_{x',z}$ respectively.

Proof: Assume to the contrary.

(1) W.l.o.g. let α be an internal vertex of $P_{x,y}$ and let β be an internal vertex of $P_{x,z}$. Note that α cannot be adjacent to both x and y, and that β cannot be adjacent to both x and z, since $|P_{x,y}| > 3$ and $|P_{x,z}| > 3$. If $\alpha y \in E$ and $\beta z \in E$, then the path from α to β induced on $\alpha y \cup P_{y,z} \cup \beta z$ does not hit x contradicting (α, β) being a DP of H, where

$P_{y,z}$ is any path in $\mathcal{P}_{y,z}$. If one of these two edges αy and βz does not exist, i.e. w.l.o.g. if $\alpha y \notin E$, then the path from α to β induced on $P_{\alpha,x} \cup P_{x,\beta}$ does not hit y contradicting (α, β) being a DP, where $P_{\alpha,x}$ is the subpath of $P_{x,y}$ between α and x, and $P_{x,\beta}$ is the subpath of $P_{x,z}$ between x and β. Therefore, α and β do not both belong to the union of the internal vertices of $P_{x,y}$ and $P_{x,z}$.

(2) W.l.o.g. let $\alpha \in R_{x',y}$ and $\beta \in R_{x',z}$. Clearly, either $\beta = x'$, or $\beta \in R_{x',z} \setminus \{x'\}$. If $\beta = x'$ (and $\alpha \in R_{x',y}$), then both α, β belong to $\mathcal{P}_{x,y}$ contradicting Lemma 5. If $\beta \in R_{x',z} \setminus \{x'\}$, then we have the following cases:

(i) if $\alpha = x'$, then both α, β belong to $\mathcal{P}_{x,z}$ contradicting Lemma 5.

(ii) if $\alpha \in R_{x',y} \setminus \{x'\}$, then the path between α and β induced on $P_{\alpha,y} \cup P_{y,z} \cup P_{z,\beta}$ does not hit x, where $P_{\alpha,y}$ is the path between α and y induced on $R_{x',y}$, $P_{z,\beta}$ is the path between z and β induced on $R_{x',z}$, and $P_{y,z}$ is any path in $\mathcal{P}_{y,z}$; note that no vertex on $P_{\alpha,y}$ and no vertex of $P_{z,\beta}$ is adjacent to x, since $R_{x',y}$ and $R_{x',z}$ are induced paths. □

Lemma 7 *(a) Consider a two-long-sided (2,2,2) graph H with long paths $P_{x,y} \in \mathcal{P}_{x,y}$ and $P_{x,z} \in \mathcal{P}_{x,z}$. It is not the case that one DP vertex of H is an internal vertex of $P_{x,y}$ and the other one is equal to z. By symmetry, it is not the case that one DP vertex of H is an internal vertex of $P_{x,z}$ and the other one is equal to y.*

(b) Consider a (1,2,2) graph H with paths $R_{x',y} \in \mathcal{R}_{x',y}$ and $R_{x',z} \in \mathcal{R}_{x',z}$. It is not the case that one DP vertex of H is an internal vertex of $R_{x',y}$ and the other one is equal to z. By symmetry, it is not the case that one DP vertex of H is an internal vertex of $R_{x',z}$ and the other one is equal to y.

Proof: (a) Let (α, β) be a DP of a two-long-sided (2,2,2) graph H. W.l.o.g. assume that $\alpha = z$ and $\beta \in P_{x,y} \setminus \{x, y\}$. Since $|P_{x,y}| > 3$, β cannot be adjacent to both x and y. If $\beta x \notin E$, then the path from α to β induced on $P_{y,z} \cup P_{y,\beta}$ does not hit x contradicting (α, β) being a DP, where $P_{y,\beta}$ is the subpath of $P_{x,y}$ between y and β. If $\beta y \notin E$, then the path from α to β induced on $P_{x,z} \cup P_{x,\beta}$ does not hit y contradicting (α, β) being a DP, where $P_{x,\beta}$ is the subpath of $P_{x,y}$ between x and β.

(b) Corollary of Lemma 6 (2). □

Chapter 4

(2,2,2) Graphs

The claims in this section will describe some structure of (2,2,2) graphs. The goal is to describe enough structure of (2,2,2) graphs, so that the positions of all DP vertices in these graphs can always be determined. This goal is achieved and presented in Claims 18, 21, 25, 30, and Corollaries 7 and 11. These are the main results of this chapter. Each of them describes the positions of DP vertices in a specific type of (2,2,2) graphs, and together they describe the positions of DP vertices all types of (2,2,2) graphs. In addition, an interesting structural result appears in Claim 13.

The following notation will be used in this chapter. Let H be a (2,2,2) graph with an AT $\{x, y, z\}$. As before, let $\mathcal{P}_{x,y}$ be the set of induced paths between x and y that avoid $N(z)$, let $\mathcal{P}_{x,z}$ be the set of induced paths between x and z that avoid $N(y)$, and let $\mathcal{P}_{y,z}$ be the set of induced paths between y and z that avoid $N(x)$. In the claims in this chapter which say "let H be defined as in the paragraph preceding Claim 12," it is assumed that H is defined as in this paragraph. The same holds for $\mathcal{P}_{x,y}$, $\mathcal{P}_{x,z}$, and $\mathcal{P}_{y,z}$.

The following claim describes the positions of DP vertices in a restricted subfamily of (2,2,2) graphs.

Claim 12 *If $\forall P_{x,y} \in \mathcal{P}_{x,y}$, $\forall P_{x,z} \in \mathcal{P}_{x,z}$, and $\forall P_{y,z} \in \mathcal{P}_{y,z}$ there are no edges between internal vertices of $P_{x,y}$, $P_{x,z}$, and $P_{y,z}$, then a DP vertex of any DP of H must be in*

$\{x, y, z\}$.

Proof: Assume to the contrary. Thus, $\forall P_{x,y} \in \mathcal{P}_{x,y}$, $\forall P_{x,z} \in \mathcal{P}_{x,z}$, and $\forall P_{y,z} \in \mathcal{P}_{y,z}$ there are no edges between internal vertices of $P_{x,y}$, $P_{x,z}$, and $P_{y,z}$, and no DP vertex is in $\{x, y, z\}$. Denote by (α, β) any DP of H. W.l.o.g. assume $\alpha \in P_{x,y} \setminus \{x, y\}$, $\beta \in P_{x,z} \setminus \{x, z\}$ for some $P_{x,y} \in \mathcal{P}_{x,y}$ and some $P_{x,z} \in \mathcal{P}_{x,z}$. Denote by $P_{\alpha,x}$ the subpath of $P_{x,y}$ between α and x, and by $P_{\beta,x}$ the subpath of $P_{x,z}$ between β and x. But now, the path between α and β induced on $P_{\alpha,x} \cup P_{\beta,x}$ does not hit any internal vertex of any $P_{y,z} \in \mathcal{P}_{y,z}$ contradicting (α, β) being a DP. □

The following claim describes an interesting structural property of (2,2,2) graphs.

Claim 13 *If* $\exists P_{x,y} \in \mathcal{P}_{x,y}$ *and* $\exists P_{x,z} \in \mathcal{P}_{x,z}$ *in* H *such that* $|P_{x,y}| > 3$, $|P_{x,z}| > 3$, *then* $\forall P \in \mathcal{P}_{y,z}$, $|P| = 3$.

Proof: Assume to the contrary. Thus, $\exists P_{y,z} \in \mathcal{P}_{y,z}$ such that $|P_{y,z}| > 3$. So, $|P_{x,y}| > 3$, $|P_{x,z}| > 3$, $|P_{y,z}| > 3$. Let $\tilde{H}$ be the subgraph of H induced on $P_{x,y} \cup P_{x,z} \cup P_{y,z}$. Let (α, β) be any DP of $\tilde{H}$. Where could α and β be positioned?

By Lemma 6 (1), α and β do not belong to the union of the internal vertices of $P_{x,y}$, $P_{x,z}$, and $P_{y,z}$. Therefore, one of α, β must be in $\{x, y, z\}$. W.l.o.g. let $\alpha = x$. Then by Lemma 5, $\beta \notin P_{x,y} \cup P_{x,z}$. Therefore, $\beta \in P_{y,z} \setminus \{y, z\}$. β cannot be adjacent to both y and z, since $|P_{y,z}| > 3$. W.l.o.g. assume that $\beta z \notin E$. Now the path between α and β induced on $P_{y,\beta} \cup P_{x,y}$ does not hit z contradicting (α, β) being a DP, where $P_{y,\beta}$ is the subpath of $P_{y,z}$ between y and β.

Thus, $\tilde{H}$ does not have a DP contradicting H being HDP. □

Additional structural properties of (2,2,2) graphs are described in the following claim.

Claim 14 *In a (2,2,2) graph* H, *let* $P_{x,y} \in \mathcal{P}_{x,y}$, $P_{x,z} \in \mathcal{P}_{x,z}$, $P_{y,z} \in \mathcal{P}_{y,z}$, *and let* $u \in P_{x,y} \setminus \{x, y\}$, $v \in P_{x,z} \setminus \{x, z\}$ *be such that the length of the path between them induced* *on $P_{x,y} \cup P_{x,z}$ that includes x is at least 4. Let $P_{x,v}$ be the subpath of $P_{x,z}$ between x and v, and let $P_{x,u}$ be the subpath of $P_{x,y}$ between x and u. If $uv \in E$, then either*

(i) u is universal to $P_{x,v}$, or

(ii) v is universal to $P_{x,u}$, or

(iii) if $xv \in E$, and v is not adjacent to $w \in P_{u,x}$, and $wu \in E$, then either w is adjacent to a vertex in $P_{y,z} \setminus \{y, z\}$, or every vertex in $P_{y,z} \setminus \{y, z\}$ is adjacent either to u or to v.

Note that by symmetry, the same holds if $u \in P_{x,y} \setminus \{x, y\}, v \in P_{y,z} \setminus \{y, z\}$, or if $u \in P_{x,z} \setminus \{x, z\}, v \in P_{y,z} \setminus \{y, z\}$, and the conditions above are satisfied.

Proof: Assume to the contrary. Thus, $uv \in E$ and neither u is universal to $P_{x,v}$, nor v is universal to $P_{x,u}$ (negation of the condition (iii) above will be added in part (2)(a)(i) of this proof). Consider the subgraph $\tilde{H}$ of H induced on $P_{x,y} \cup P_{x,z} \cup P_{y,z}$. Since the length of the u, v-path induced on $P_{x,y} \cup P_{x,z}$ is at least 4, there are two cases to consider:

(1) $xu, xv \notin E$. Consider $\hat{H} = \tilde{H} \setminus \{P_{y,z} \setminus \{y, z\}\}$. $\hat{H}$ is (1,1,2), in particular, vertices y and z of the AT $\{x, y, z\}$ in $\hat{H}$ are non-path-disjoint, contradicting Theorem 26.

(2) Exactly one of xu, xv is an edge. W.l.o.g. let $xv \in E$. Clearly, u is not universal to $P_{x,v}$, since $xu \notin E$. Also, by assumption, v is not universal to $P_{x,u}$. Let $w \in P_{x,u}$ be a vertex not adjacent to v. Here, there are the following cases to consider:

(a) $wu \in E$, or $wx \in E$. Consider each of these two cases separately.

(i) Assume $wu \in E$, w is not adjacent to any vertex in $P_{y,z} \setminus \{y, z\}$, and there exists a vertex $p \in P_{y,z} \setminus \{y, z\}$ that is not adjacent to u and is not adjacent to v. Clearly, $wx \notin E$ since the length of the path between u and v induced on $P_{x,y} \cup P_{x,z}$ that includes x is at least 4 and $xv \in E$. Let $P_{x,w}$ be the subpath of $P_{x,y}$ between x and w, let $w' = P_{x,w} \cap N(w)$, let $x' = P_{x,y} \cap N(x)$, and let $P_{x',w'}$ be the subpath of $P_{x,y}$ between x' and w'. Now, $\tilde{H} \setminus P_{x',w'}$ is (1,1,2), in particular, vertices w and x of the AT $\{w, p, x\}$ are non-path-disjoint, contradicting Theorem 26; note that w is not adjacent to any vertex of $P_{v,z}$, where $P_{v,z}$ is the subpath of $P_{x,z}$ between v and z, since otherwise, i.e., if

$wq \in E, q \in P_{v,z}$, then w and q satisfy the condition of case (1) above (replace v by q, and u by w).

(ii) Now assume $wx \in E$. Denote by $\hat{H}$ the graph obtained by removing from $\tilde{H}$ all vertices on $P_{y,z} \setminus \{y, z\}$. Now, $\hat{H}$ is a (1,1,2) graph with an AT $\{w, y, z\}$, in particular, the non-path-disjoint vertices are y, and z, contradicting Theorem 26; note that if $wq \in E$ for some $q \in P_{v,z}$, then H is not a (2,2,2) graph, since the path $Q \in \mathcal{P}_{x,z}$ induced on $\{xw\} \cup \{wq\} \cup P_{q,z}$, where $P_{q,z}$ is the subpath of $P_{x,z}$ between q and z, has common vertices $\{x, w\}$ with $P_{x,y}$, i.e., x is not a path-disjoint vertex, contradicting the assumption that H is (2,2,2).

(b) $wu \notin E$, and $wx \notin E$. Now, the graph $\tilde{H} \setminus \{P_{y,z} \setminus \{y, z\}\}$ is (1,1,2), in particular, vertices y and z of the AT $\{w, y, z\}$ are non-path-disjoint contradicting Theorem 26; again, $wq \notin E$, for all $q \in P_{v,z}$, as explained in part (2)(a)(ii) above. □

It has been mentioned before that any (2,2,2) graph is either two-long-sided, or one-long-sided, or no-long-sided. The structure of these three types of (2,2,2) graphs will be considered separately in the next three sections.

4.1 Two-Long-Sided Graphs

In this section, assume that the graph is two-long-sided, i.e. that there exists $P_{x,y} \in \mathcal{P}_{x,y}$ and there exists $P_{x,z} \in \mathcal{P}_{x,z}$ in a (2,2,2) graph H such that $|P_{x,y}| > 3, |P_{x,z}| > 3$. By Claim 13, $\mathcal{P}_{y,z}$ consists of P_3's only. Denote by M the set of internal vertices of all paths in $\mathcal{P}_{y,z}$. These assumptions about the graph H hold in all claims in this section.

The standard definition of *distance* between two vertices u and v, denoted by $d(u, v)$, which defines it as the minimum length of a u, v-path, is used in this thesis. Recall that the length of a path is the number of its edges.

The following claim describes the structural property of two-long-sided graphs that will be used in proofs of various claims in this thesis, such as for example, in the proofs of Claims 16, 17, 20, and Corollary 9 in this chapter.

Claim 15 *The vertex of distance i from x on $P_{x,y}$, for $i \geq 2$, cannot be adjacent to the vertex of distance j from x on $P_{x,z}$, for $j \geq 2$.*

Proof: Assume to the contrary. Denote by u the vertex of distance i from x on $P_{x,y}$, for $i \geq 2$, and by v the vertex of distance j from x on $P_{x,z}$, for $j \geq 2$, with $uv \in E$. That is, $d(u,x) = i \geq 2$, $d(v,x) = j \geq 2$, and $uv \in E$. Note that $u \neq y$, because otherwise u could not be adjacent to any vertex on $P_{x,z}$, since y is an AT vertex. Similarly, $v \neq z$. Consider the subgraph $\tilde{H}$ of H induced on $P_{x,y} \cup P_{x,z}$. Since $uv \in E$, $\tilde{H}$ is a (1,1,2) graph, namely the AT vertices y and z of the AT $\{x,y,z\}$ of $\tilde{H}$ are non-path-disjoint, contradicting Theorem 26. □

Note that edges from the neighbor of x on a long path $P_{x,y}$ to a non-neighbor of x on a long path $P_{x,z}$ can occur in minimal HDP graphs. However, such an edge would imply the graph is (1,2,2) and is considered in the next chapter. Therefore, the following corollary.

Corollary 6 *In a two-long-sided (2,2,2) graph H with long paths $P_{x,y} \in \mathcal{P}_{x,y}$ and $P_{x,z} \in \mathcal{P}_{x,z}$, the vertex of distance i from x on $P_{x,y}$, for $i \geq 1$, cannot be adjacent to the vertex of distance j from x on $P_{x,z}$, for $j \geq 2$.*

Proof: Follows directly from Claim 15 and the paragraph preceding Corollary 6. □

The following lemma describes the positions of DP vertices in a specific type of an induced subgraph of a two-long-sided (2,2,2) graph. One of the main results of this chapter, namely Corollary 7, follows directly from this lemma. The lemma will also be used to prove Claims 16, 17, and 19. Claim 17 will further be used in the proofs of Claims 42 and 46.

Lemma 8 *Let H be a two-long-sided (2,2,2) graph with long paths $P_{x,y} \in \mathcal{P}_{x,y}$ and $P_{x,z} \in \mathcal{P}_{x,z}$, let $P_{y,z}$ be any path in $\mathcal{P}_{y,z}$ of H, and let (α,β) be a DP of the subgraph $\tilde{H}$ of H induced on $P_{x,y} \cup P_{x,z} \cup P_{y,z}$. Then one of α,β is in $N[x] = \{x\} \cup N(x)$ in $\tilde{H}$ and the other one is the internal vertex of $P_{y,z}$.*

Proof: Note that $\tilde{H}$ is a two-long-sided (2,2,2) graph.

By Lemma 5, α and β cannot both belong to $P_{x,y}$, cannot both belong to $P_{x,z}$, and cannot both belong to $P_{y,z}$.

By Lemma 6 (1), α and β cannot belong to the union of the internal vertices of $P_{x,y}$ and $P_{x,z}$.

Let $x' = P_{x,y} \cap N(x)$. It is not the case that one of α,β is an internal vertex of $P_{y,z}$ and the other one belongs to $P_{x,y} \setminus \{x, x', y\}$ for the following reason. Assume to the contrary. Thus, w.l.o.g. assume that $\alpha \in P_{y,z} \setminus \{y, z\}$ and $\beta \in P_{x,y} \setminus \{x, x', y\}$. Then the path from α to β induced on $\{\alpha y\} \cup P_{y,\beta}$ does not hit x contradicting (α,β) being a DP, where $P_{y,\beta}$ is the subpath of $P_{x,y}$ between β and y. Similarly, it is not the case that one of α,β is in $P_{y,z} \setminus \{y, z\}$ and the other one belongs to $P_{x,z} \setminus \{x, x'', z\}$, where $x'' = P_{x,z} \cap N(x)$.

By Lemma 7 (a), it is not the case that one of α,β is equal to z and the other one belongs to $P_{x,y} \setminus \{x, y\}$. Similarly, it is not the case that one of α and β is equal to y and the other one belongs to $P_{x,z} \setminus \{x, z\}$.

Therefore, the only possible position for (α,β) is that one of them is in $N[x]$ and the other one is in $P_{y,z} \setminus \{y, z\}$. □

Corollary 7 *Let H be a 2-long-sided (2,2,2) graph with no short paths in the long sides. One DP vertex of H must be in $N[x]$ and the other one must be an internal vertex of a path in $\mathcal{P}_{y,z}$.*

Proof: Follows directly from Lemma 8. □

The following two claims describe some structural properties of two-long-sided (2,2,2) graphs.

Claim 16 *Let x_1' and x_2' be the neighbors of x on long paths $P_{x,y}$ and $P_{x,z}$ in H respectively, and let x_1'' and x_2'' be of distance 2 from x on $P_{x,y}$ and $P_{x,z}$ respectively. Let v be a vertex in M. Then $\{x_1'x_2', x_1''v, x_2'v\} \cap E \neq \emptyset$. By symmetry, $\{x_1'x_2', x_2''v, x_1'v\} \cap E \neq \emptyset$ as well.*

Proof: Assume to the contrary. Thus, $x_1'x_2', x_1''v, x_2'v \notin E$. Note that $x_1''x_2' \notin E$ since H is a (2,2,2) graph. Denote by $P_{y,z} \in \mathcal{P}_{y,z}$ the path that contains v. Consider the subgraph $\tilde{H}$ of H induced on $P_{x,y} \cup P_{x,z} \cup P_{y,z}$. Denote by (α, β) a DP of $\tilde{H}$. From Lemma 8, in $\tilde{H}$ one of $\{\alpha, \beta\}$ is in $N[x]$ and the other one is in M. W.l.o.g. let $\alpha \in M$ and $\beta \in N[x]$.

However, β cannot belong to $\{x, x_2'\}$ together with $\alpha = v$ for the following reason. Assume that $\alpha = v$ and $\beta \in \{x, x_2'\}$. Then the path from α to β induced on $P_{x,z} \cup \{vz\}$ does not hit x_1'', contradicting (α, β) being a DP, since by Claim 15, no vertex of $P_{x,z}$ that is of distance 2 or more from x is adjacent to any vertex on $P_{x,y}$ that is of distance 2 or more from x, i.e., x_1'' is not adjacent to any non-neighbor of x on $P_{x,z}$; also, by assumption, $x_1''v \notin E$. Similarly, α cannot be equal to v together with β being equal to x_1', since otherwise the path between them induced on $\{P_{x,y} \setminus \{x\}\} \cup \{yv\}$ would not hit x_2'; this is true because x_2' is not adjacent to any non-neighbor of x on $P_{x,y}$, since H is (2,2,2); also, by assumption, $x_2'x_1', x_2'v \notin E$.

Thus, $\tilde{H}$ does not have a DP contradicting H being HDP. □

Claim 17 *All vertices of distance i from x on $P_{x,y}$, for $i \geq 3$, if they exist, must be adjacent to all vertices in M. By symmetry, the same holds for the vertices of distance i from x on $P_{x,z}$.*

Proof: Assume to the contrary. Thus, there exists a vertex of distance i from x on $P_{x,y}$, call it u, where $i \geq 3$, that is not adjacent to a vertex $v \in M$. Note that u is not adjacent to x', where x' is the neighbor of x on $P_{x,z}$, and $P_{x,z}$ is a long path in $\mathcal{P}_{x,z}$, since H is (2,2,2). Let $P_{y,z} \in \mathcal{P}_{y,z}$ be the path that contains v. Consider the subgraph $\tilde{H}$ of H induced on $P_{x,y} \cup P_{x,z} \cup P_{y,z}$. Denote by (α, β) a DP of $\tilde{H}$. From Lemma 8, one of $\{\alpha, \beta\}$ is in $N[x]$ and the other one is in M.

As in the proof of Claim 16, it is not the case that one of α and β is equal to v and the other one belongs to $\{x, x'\}$, since otherwise the path between them induced on $\{vz\} \cup P_{x,z}$ would miss u; note that by Claim 15, u cannot be adjacent to any non-neighbor of x on $P_{x,z}$, and in addition, by our assumption, u is not adjacent to v; note again that $ux' \notin E$, since H is (2,2,2).

Similarly, it is not the case that one of α and β is equal to v and the other one belongs to $P_{x,y} \cap N(x)$ for the following reason. Assume to the contrary. Thus, w.l.o.g. assume that $\alpha = v$ and $\beta \in P_{x,y} \cap N(x)$. Since $\beta x \in E$, the path from α to β induced on $\{vz\} \cup P_{x,z} \cup \{x\beta\}$ misses u; note that by our assumptions and Claim 15, u is not adjacent to any vertex on $P_{x,z}$ and is not adjacent to v.

Thus, $\tilde{H}$ does not have a DP contradicting H being HDP. □

4.1.1 A Long Side Has a Short Path

Let H be a two-long-sided (2,2,2) graph with a long side having a short path. W.l.o.g. let $\mathcal{P}_{x,y}$ have a short path. Denote by $P_{x,y}$ a long path in $\mathcal{P}_{x,y}$, by $P_{x,z}$ a long path in $\mathcal{P}_{x,z}$, and by P a path of length 2 in $\mathcal{P}_{x,y}$ in H. Denote by W the set of middle vertices of all P_3's in $\mathcal{P}_{x,y}$. As before, denote by M the set of middle vertices of all paths in $\mathcal{P}_{y,z}$. Let m be the middle vertex of $P_{y,z}$, where $P_{y,z}$ is an arbitrary path in $\mathcal{P}_{y,z}$.

The following claim represents one of the main results of this chapter. It describes the positions of DP vertices in two-long-sided (2,2,2) graphs with a long side having a short path. The three corollaries following this claim describe some of the structural properties that are dictated by specific positions of DP vertices in these graphs.

Claim 18 *Consider a two-long-sided (2,2,2) graph H with a short path in a long side as described in the first paragraph of this subsection. One DP vertex of H is in $N[x] = \{x\} \cup N(x)$ and the other one is in M, or one DP vertex is in $N[z] = \{z\} \cup N(z)$ and the other one is in W. Each of these two types of DPs can occur. Note that the symmetric result would hold if there was a P_3 in $\mathcal{P}_{x,z}$.*

Proof: Let (α, β) be a DP of H. Consider where α and β could be positioned in H.

By Lemma 5, α and β cannot both belong to $\mathcal{P}_{x,y}$, cannot both belong to $\mathcal{P}_{x,z}$, and cannot both belong to $\mathcal{P}_{y,z}$.

By Lemma 6 (1), α and β cannot belong to the union of the internal vertices of $P_{x,y}$ and $P_{x,z}$, where $P_{x,y}$ and $P_{x,z}$ are long paths in $\mathcal{P}_{x,y}$ and $\mathcal{P}_{x,z}$ respectively.

By Lemma 7 (a), it is not the case that one of α and β is equal to z and the other one belongs to $P_{x,y} \setminus \{x, y\}$, and also is not the case that one of them is equal to y and the other one belongs to $P_{x,z} \setminus \{x, z\}$.

If one of α and β is equal to m and the other one belongs to $P_{x,y} \setminus \{x, y, x'\}$, or if one of them is equal to m and the other one belongs to $P_{x,z} \setminus \{x, z, x''\}$, where $x' = N(x) \cap P_{x,y}$, $x'' = N(x) \cap P_{x,z}$, and m is any vertex in M, then the proof is as follows. W.l.o.g. assume that $\alpha \in P_{x,y} \setminus \{x, y, x'\}$ and $\beta = m$. Since $\alpha x \notin E$, the path between α and β induced on $P_{\alpha,y} \cup \{y\beta\}$ does not hit x contradicting (α, β) being a DP, where $P_{\alpha,y}$ is the subpath of $P_{x,y}$ between α and y.

Thus, the only two options for $\{\alpha, \beta\}$ are that one is in $N[x]$ and the other one is in M, or that one is in $N[z]$ and the other one is in W.

Examples showing that each of these two types of DPs can occur are given in Figure 4.1. DP vertices are shaded in these examples. □

Consider each of the possible positions for $\{\alpha, \beta\}$ from the previous claim separately.

The following notation will be used in the following three corollaries. Consider a two-long-sided (2,2,2) graph H with a short path in a long side as described in the first

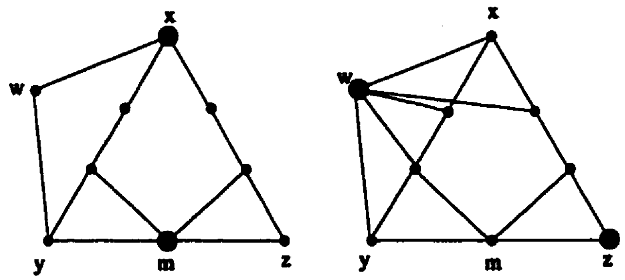


Figure 4.1:

paragraph of this subsection. Let m be a vertex in M, let w be a vertex in W, let $x' = P_{x,y} \cap N(x)$, $y' = P_{x,y} \cap N(y)$, $z' = P_{x,z} \cap N(z)$, $x'' = P_{x,z} \cap N(x)$, where $P_{x,y}$ and $P_{x,z}$ are long paths in $\mathcal{P}_{x,y}$ and $\mathcal{P}_{x,z}$ respectively, let $P_{y,z}$ be the path containing m, and let P be a P_3 in $\mathcal{P}_{x,y}$ whose mid-vertex is w. Let $P_{x',y'}$ be the subpath of $P_{x,y}$ between x' and y', let $\tilde{H}$ be the subgraph of H induced on $P_{x,y} \cup P_{x,z} \cup P_{y,z} \cup P$, and let $P_{x',y}$ be the subpath of $P_{x,y}$ between x' and y. Denote by U the set of midpoints of all P_3's in $\mathcal{P}_{x,y} \cup \mathcal{P}_{x,z}$; note that $W \subseteq U$. Denote by z'' the second neighbor of z on $P_{x,z}$.

Corollary 8 *If one of the DP vertices of H is equal to x and the other one is m, then all non-neighbors of x and y on the long paths in $\mathcal{P}_{x,y}$, if they exist, and all non-neighbors of x and z on the long paths in $\mathcal{P}_{x,z}$, if they exist, must either be universal to U, or adjacent to m, or both.*

Proof: Assume to the contrary. W.l.o.g. let v be a non-neighbor of x and y on the long path $P_{x,y} \in \mathcal{P}_{x,y}$ that is not adjacent to w and is not adjacent to m, where (x, m) is a DP of H. Consider the subgraph $\tilde{H}$ of H induced on $P \cup P_{x,y} \cup P_{x,z} \cup P_{y,z}$. By Claim 7, it is known that, since (x, m) is a DP of H, and since $x, m \in \tilde{H}$, (x, m) is also a DP of $\tilde{H}$. However, the path from x to m in $\tilde{H}$ induced on $P \cup \{ym\}$ does not hit v contradicting (x, m) being a DP of $\tilde{H}$. □

Corollary 9 *Let (w, z) be a DP of H. Then H has the following properties:*

(a) Let v be an internal vertex of $P_{x,y}$ different from x'. Then $vw \in E$.

(b) Let v be an internal vertex of $P_{x,y}$ different from y', or an internal vertex of $P_{x,z}$ different from z'. If $vw \notin E$, then v is universal to M.

(c) If $y'w, y'm \notin E$, then m is not adjacent to a non-neighbor of y' on $P_{x,y}$.

Proof: (a) Let $v \in P_{x,y} \setminus \{x, x', y\}$ be such that $vw \notin E$. Note that $vx'' \notin E$ since H is a (2,2,2) graph. Then the path from w to z induced on $P_{x,z} \cup \{xw\}$ misses v, contradicting (w, z) being a DP of H (note that by Claim 15 the only vertex on $P_{x,z}$ that v can be adjacent to is x'').

(b) W.l.o.g. let v be an internal vertex of $P_{x,y}$ different from y' such that $vw, vm \notin E$. Then the path from z to w induced on $P_{y,z} \cup \{yw\}$ misses v, contradicting (w, z) being a DP of H.

(c) If $y'w, y'm \notin E$ and m is adjacent to a non-neighbor q of y' on $P_{x,y}$, then the path from z to w induced on $\{zm\} \cup \{mq\} \cup P_{q,x} \cup \{xw\}$ does not hit y', contradicting (w, z) being a DP of H, where $P_{q,x}$ is the subpath of $P_{x,y}$ between q and x. □

Corollary 10 *If (w, m) is a DP of H, then every internal vertex of a long path $P_{x,z} \in \mathcal{P}_{x,z}$ and every internal vertex of a long path $P_{x,y} \in \mathcal{P}_{x,y}$ different from y' must be adjacent either to m, or to w, or to both. In addition, if $wm \in E$, then y' must also be adjacent to m, or to w, or to both.*

Proof: Assume to the contrary. Let v be an internal vertex of $P_{x,y}$ different from y', or an internal vertex of $P_{x,z}$. Since by assumption $vm, vw \notin E$, the path between m and w induced on $\{ym\} \cup \{yw\}$ does not hit v contradicting (m, w) being a DP. Similarly, if $wm \in E$ and y' is not adjacent to m and not adjacent to w, then the path from m to w induced on $\{mw\}$ does not hit y'. □

4.1.2 A Long Side Has at Least Two Long Paths

The following notation will be used in the claims in this subsection. Let H be a two-long-sided (2,2,2) graph that has at least two long paths on one of its long sides. W.l.o.g.

let $\mathcal{P}_{x,y}$ have at least two long paths. Denote by P and Q two long paths in $\mathcal{P}_{x,y}$ of H. Denote by $P_{x,z}$ a long path in $\mathcal{P}_{x,z}$. Remember that, by Claim 13, $\mathcal{P}_{y,z}$ consists of P_3's only. Let $P_{y,z}$ be any path in $\mathcal{P}_{y,z}$. Let $x_p = N(x) \cap P$, $x_q = N(x) \cap Q$, $x_z = N(x) \cap P_{x,z}$, $y_p = N(y) \cap P$, $y_q = N(y) \cap Q$, and let m be the middle vertex of $P_{y,z}$. Let $N_l(x)$ be the neighborhood of x on the long paths.

The following claim describes the positions of DP vertices in a subfamily of two-long-sided (2,2,2) graphs with a long side having at least two long paths.

Claim 19 *If there are no P_3's in $\mathcal{P}_{x,y}$ and $\mathcal{P}_{x,z}$, then one DP vertex of H is in $\{x\} \cup N_l(x)$ and the other one is in M.*

Proof: Corollary of Lemma 8. □

As mentioned before, let $D(u,v)$ denote the set of vertices that intercept all u,v-paths in a connected graph.

Claim 20 *$\forall P, Q \in \mathcal{P}_{x,y}$ that are long paths, and $\forall v_1 \in P \setminus \{x, x_p\}$ and $\forall v_2 \in Q \setminus \{x, x_q\}$, $v_1 \in D(x, v_2)$, or $v_2 \in D(x, v_1)$.*

Proof: Assume to the contrary. Let $P, Q \in \mathcal{P}_{x,y}$ be long paths, let $P_{x,z}$ be any long path in $\mathcal{P}_{x,z}$, and let x_z be the neighbor of x on $P_{x,z}$. Let $v_1 \in P \setminus \{x, x_p\}$, $v_2 \in Q \setminus \{x, x_q\}$ be such that $v_1 \notin D(x, v_2)$ and $v_2 \notin D(x, v_1)$. Note that $v_1 x_z \notin E$ and $v_2 x_z \notin E$ since H is a (2,2,2) graph. Let $P_{y,z}$ be any path in $\mathcal{P}_{y,z}$. Let $\check{H}$ be the subgraph of H induced on $P \cup Q \cup P_{x,z} \cup P_{y,z}$. Let m be the mid-vertex of $P_{y,z}$.

$\check{H} \setminus \{m\}$ does not have a DP any more, since $\{v_1, v_2, z\}$ is its AT that contradicts Claim 6. In particular, $\{v_1, v_2, z\}$ is an AT of $\check{H}$ for the following reasons:

(i) the path from v_1 to v_2 induced on $P_{v_1,y} \cup P_{v_2,y}$ misses z, where $P_{v_1,y}$ is the path between v_1 and y induced on P, and similarly, $P_{v_2,y}$ is the path between v_2 and y induced on Q; this is true by the definition of $\mathcal{P}_{x,y}$ and $\mathcal{P}_{x,z}$;

(ii) the path between z and v_1 induced on $P_{x,z} \cup P_{x,v_1}$ misses v_2, where P_{x,v_1} is the path between v_1 and x induced on P; this is true because $v_1 x_z \notin E$, since H is (2,2,2), and also, by Claim 15, v_1 is not adjacent to any non-neighbor of x on $P_{x,z}$;

(iii) similarly, the path between z and v_2 induced on $P_{x,z} \cup P_{x,v_2}$ misses v_1, where P_{x,v_2} is the path between v_2 and x induced on Q.

Clearly, in $\check{H}$, $xz \notin E$, contradicting Claim 6. □

4.2 One-Long-Sided Graphs

Let H be a one-long-sided (2,2,2) graph with an AT $\{x, y, z\}$. W.l.o.g. let $\mathcal{P}_{x,y}$ be the long side of H. Denote by M_1 the set of mid-points of all paths in $\mathcal{P}_{x,z}$, by M_2 the set of mid-points of all paths in $\mathcal{P}_{y,z}$. For now assume that the long side does not contain any short paths; the case when the long side has a short path will be discussed in subsection 4.2.1.

Note that for any internal vertex v of a long path $P_{x,y} \in \mathcal{P}_{x,y}$, edges between v and any vertex in M_1, and between v and any vertex in M_2 are allowed in these types of graphs. For example, the fact that $vm_1 \in E$ in H, for some $m_1 \in M_1$, might seem to contradict the fact that H is (2,2,2), since the path L between x and z induced on $P_{x,v} \cup \{vm_1\} \cup \{m_1 z\}$, where $P_{x,v}$ is the path between x and v induced on $P_{x,y}$, seems to be in $\mathcal{P}_{x,z}$, and it shares vertex v with $P_{x,y} \in \mathcal{P}_{x,y}$ making the graph H (1,2,2). However, this is not so, since, by definition, all paths in $\mathcal{P}_{x,z}$ are induced paths, while L is not induced, and therefore not in $\mathcal{P}_{x,z}$; in particular, $xm_1, vm_1 \in E$. Therefore, even if v is a non-neighbor of x and a non-neighbor of y on a long path $P_{x,y} \in \mathcal{P}_{x,y}$ in a one-long-sided (2,2,2) graph H described here, this does not contradict H being (2,2,2).

The following claim is one of the main results of this section. It describes the positions of DP vertices in one-long-sided (2,2,2) graphs. The three claims following it establish some structural properties that are dictated by specific positions of DP vertices in these

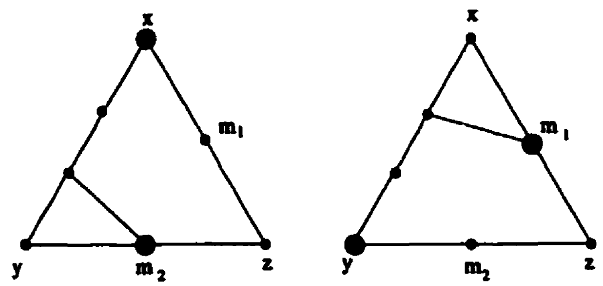


Figure 4.2:

graphs.

Claim 21 *One DP vertex of a one-long-sided (2,2,2) graph is in $N[x]$ and the other one is in M_2, or one of its DP vertices is in $N[y]$ and the other one is in M_1. Each of these two types of DPs can occur.*

Proof: Follows the proof of Claim 18.

Examples showing that each of these two types of DPs can occur are given in Figure 4.2. DP vertices are shaded in these examples. □

Claim 22 *If $\{\alpha,\beta\} = \{x, m_2\}$ for some vertex $m_2 \in M_2$, then for all $m_1 \in M_1$, every vertex in $P_{x,y} \setminus \{x, y, x'\}$, where $P_{x,y}$ is a long path in $\mathcal{P}_{x,y}$ and x' is the neighbor of x on $P_{x,y}$, is adjacent to m_1, or to m_2, or to both.*

Proof: Assume to the contrary. Let $v \in P_{x,y} \setminus \{x, y, x'\}$ be non-adjacent to some $m_1 \in M_1$, and non-adjacent to m_2. Then the path from α to β induced on $\{xm_1\} \cup \{m_1z\} \cup \{zm_2\}$ would not hit v contradicting (α,β) being a DP. □

Claim 23 *If $\{\alpha,\beta\} = \{x', m_2\}$ for some vertex $m_2 \in M_2$ and the neighbor x' of x on a long path $P_{x,y} \in \mathcal{P}_{x,y}$, then every vertex in M_1 is adjacent to a vertex in $\{m_2\} \cup P_{x,y} \setminus \{x, y\}$. By symmetry, if $\{\alpha,\beta\} = \{y', m_1\}$ for some vertex $m_1 \in M_1$ and the neighbor y' of y on a long path $P_{x,y} \in \mathcal{P}_{x,y}$, then every vertex in M_2 is adjacent to a vertex in $\{m_1\} \cup P_{x,y} \setminus \{x, y\}$.*

Proof: Assume to the contrary. Let $\{\alpha,\beta\} = \{x', m_2\}$ and a vertex $m_1 \in M_1$ is not adjacent to any vertex in $\{m_2\} \cup P_{x,y} \setminus \{x,y\}$. Then the path from α to β induced on $\{ym_2\} \cup P_{x,y} \setminus \{x\}$ does not hit m_1 contradicting (α,β) being a DP. □

Claim 24 *If $\{\alpha,\beta\} = \{m_1, m_2\}$ for some vertices $m_1 \in M_1$ and $m_2 \in M_2$, then every internal vertex of a long path $P_{x,y} \in \mathcal{P}_{x,y}$ is adjacent to m_1, or to m_2.*

Proof: This is true because otherwise the path from α to β induced on $\{m_1 z\} \cup \{m_2 z\}$ would not hit an internal vertex of $P_{x,y}$ that is not adjacent to $\{m_1, m_2\}$. □

4.2.1 The Long Side Has a Short Path

The following assumptions and notation hold for the claims in this subsection. Let H be a one-long-sided (2,2,2) graph with an AT $\{x,y,z\}$, the long side $\mathcal{P}_{x,y}$, a long path $P_{x,y} \in \mathcal{P}_{x,y}$, and a short path P in $\mathcal{P}_{x,y}$. As before, denote by M_1 the set of mid-points of all paths in $\mathcal{P}_{x,z}$, and by M_2 the set of mid-points of all paths in $\mathcal{P}_{y,z}$. Also, denote by W the set of mid-points of all paths in $\mathcal{P}_{x,y}$. Let x' be the neighbor of x on $P_{x,y}$. Let y' be the neighbor of y on $P_{x,y}$. Let m_1 be an arbitrary vertex in M_1, m_2 an arbitrary vertex in M_2, and w an arbitrary vertex in W. Let $P_{x,z}$ be any path in $\mathcal{P}_{x,z}$.

The following claim is one of the main results of this section. It describes the positions of DP vertices in one-long-sided (2,2,2) graphs in which the long side contains a short path. The three claims following it establish some structural properties that are dictated by specific positions of DP vertices in these graphs.

Claim 25 *DP vertices of a one-long-sided (2,2,2) graph with a short path in a long side satisfy the following. Either:*

(a) one DP vertex is in $N[x]$ and the other one is in M_2, or

(b) one DP vertex is in $N[y]$ and the other one is in M_1, or

(c) one DP vertex is in $N[z]$ and the other one is in W.

Each of these three types of DPs can occur.

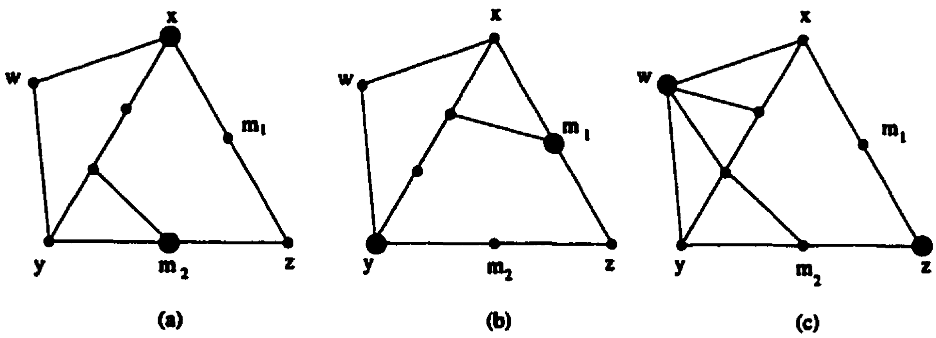


Figure 4.3:

Proof: Follows the proof of Claim 18.

Examples showing that each of these three types of DPs can occur are given in Figure 4.3. DP vertices are shaded in these examples. □

Claim 26 *Let (z, w) be a DP of H.*

(a) If $x'w \notin E$, then x' must be universal to M_2. By symmetry, if $y'w \notin E$, then y' must be universal to M_1.

(b) Let v be an internal vertex of $P_{x,y}$ different from x' and y'. If $vw \notin E$, then v is universal to $M_1 \cup M_2$. Also, if $x'w, x'y', y'w \notin E$, then x' is universal to $M_1 \cup M_2$, and similarly, if $y'w, x'y', x'w \notin E$, then y' is universal to $M_1 \cup M_2$.

Proof: (a) Assume to the contrary. Let $x'w \notin E$, and let $x'm_2 \notin E$, for some $m_2 \in M_2$. Let $m_2 \in P_{y,z}$. Then the path from z to w induced on $P_{y,z} \cup \{yw\}$ misses x' contradicting (z, w) being a DP.

(b) Assume to the contrary. Let $vw \notin E$ and, w.l.o.g., v is not adjacent to some vertex $m_1 \in M_1$. Then the path between z and w induced on $\{zm_1\} \cup \{m_1x\} \cup \{xw\}$ misses v.

Now, prove the second part of the claim. If $x'w, x'y', y'w \notin E$ and w.l.o.g. x' is not adjacent to some $m_1' \in M_1$, since y' is universal to M_1 (by part (a)), the path between z and w induced on $\{zm_1'\} \cup \{m_1'y'\} \cup \{y'y\} \cup \{yw\}$ misses x'. □

Claim 27 *If (m_2, w) is a DP of H, then:*

(a) every vertex in $M_1 \cup P_{x,y} \setminus \{y'\}$ must be adjacent to m_2, or to w, or to both;

(b) it is possible for vertices in $\{y'\} \cup M_2 \setminus \{m_2\}$ to be not adjacent to both m_2 and w.

Proof: (a) Assume to the contrary. Let $v \in M_1 \cup P_{x,y} \setminus \{y'\}$ be non-adjacent to m_2 and w. Then the path from m_2 to w induced on $\{m_2 y\} \cup \{yw\}$ does not hit v.

(b) To see that it is possible for y' not to be adjacent to m_2 and w consider the following. If y' is adjacent to all vertices in M_1, and if there are no edges between m_2 and internal vertices in $\mathcal{P}_{x,y}$ (note that internal vertices of $\mathcal{P}_{x,y} \setminus \{w\}$ can be adjacent to w), then every path from m_2 to w contains at least one of the vertices in $M_1 \cup \{y\}$, and therefore hits y'.

To see that it is possible for a vertex in $M_2 \setminus \{m_2\}$ not to be adjacent to m_2 and w consider the following. If for all internal vertices u of all paths in $\mathcal{P}_{x,y} \cup \mathcal{P}_{x,z}$, $m_2 u \notin E$, then any path from m_2 to w must include at least one of vertices y and z, and therefore every vertex $v \in M_2 \setminus \{m_2\}$ is hit by any path from m_2 to w. □

Claim 28 *Not all internal vertices of a path $P_{x,y}$ in H, such that $|P_{x,y}| > 4$, can be non-adjacent to both vertices in $\{m_1, m_2\}$, for any $m_1 \in M_1$ and any $m_2 \in M_2$.*

Proof: Assume to the contrary. Thus, all internal vertices of a path $P_{x,y}$ in H, $|P_{x,y}| > 4$, are non-adjacent to both vertices in $\{m_1, m_2\}$, for some $m_1 \in M_1$ and some $m_2 \in M_2$. Then the subgraph $\tilde{H}$ of H induced on $P_{x,y} \cup P_{x,z} \cup P_{y,z}$, where $m_1 \in P_{x,z} \in \mathcal{P}_{x,z}$ and $m_2 \in P_{y,z} \in \mathcal{P}_{y,z}$, has a chordless cycle of length at least 7, and therefore does not have a DP. □

4.2.2 The Long Side Has at Least Two Long Paths

Let H be a one-long-sided (2,2,2) graph with an AT $\{x, y, z\}$, the long side $\mathcal{P}_{x,y}$, and at least two long paths in $\mathcal{P}_{x,y}$. Let P and Q be the long paths in $\mathcal{P}_{x,y}$. As before, denote by M_1 the set of mid-points of all paths in $\mathcal{P}_{x,z}$, and by M_2 the set of mid-points of all paths in $\mathcal{P}_{y,z}$. Let x_p be the neighbor of x on P, let x_q be the neighbor of x on Q.

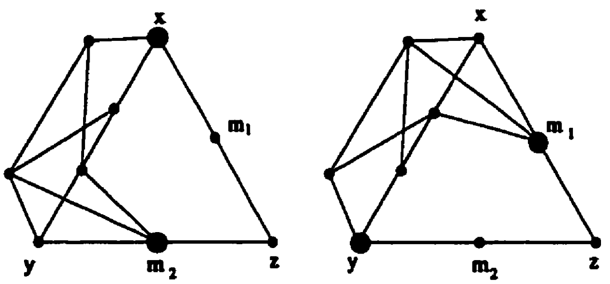


Figure 4.4:

The following corollary specifies the positions of DP vertices in one-long-sided (2,2,2) graphs in which the long side has at least two long paths.

Corollary 11 *If there are no short paths in $\mathcal{P}_{x,y}$, one DP vertex of H is in $N[x]$ and the other one is in M_2, or one DP vertex of H is in $N[y]$ and the other one is in M_1. Each of these two types of DPs can occur.*

Proof: Follows directly from Claim 21.

Examples showing that each of these three types of DPs can occur are given in Figure 4.4. DP vertices are shaded in these examples. □

The following claim describes an interesting structural property of these types of (2,2,2) graphs.

Claim 29 *$\forall P, Q \in \mathcal{P}_{x,y}$ that are long paths, and $\forall v_1 \in P \setminus \{x, x_p\}$ and $\forall v_2 \in Q \setminus \{x, x_q\}$, if $v_1 m_1, v_2 m_1 \notin E$ for some $m_1 \in M_1$, then $v_1 \in D(x, v_2)$, or $v_2 \in D(x, v_1)$.*

Proof: Follows the proof of Claim 20. □

4.3 No-Long-Sided Graphs

Let H be a no-long-sided (2,2,2) graph with an AT $\{x, y, z\}$. That is, all paths in $\mathcal{P}_{x,y} \cup \mathcal{P}_{x,z} \cup \mathcal{P}_{y,z}$ are P_3s. Denote by M_1 the set of mid-points of all paths in $\mathcal{P}_{x,y}$, by M_2

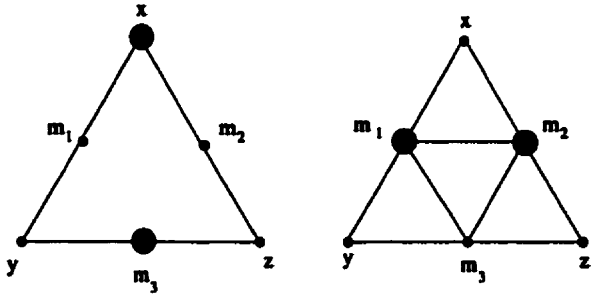


Figure 4.5:

the set of mid-points of all paths in $\mathcal{P}_{x,z}$, and by M_3 the set of mid-points of all paths in $\mathcal{P}_{y,z}$.

The following claim is one of the main results of this section. It describes the positions of DP vertices in no-long-sided (2,2,2) graphs. The two claims following it establish some structural properties that are dictated by specific positions of DP vertices in these graphs.

Claim 30 *A DP of a no-long-sided (2,2,2) graph consists of an AT vertex $v \in \{x, y, z\}$ and a mid-point of a path that avoids the neighborhood of v, or of a vertex in M_i and a vertex in M_j, where $i \neq j$, and $i, j \in \{1,2,3\}$. Each of these types of DPs can occur.*

Proof: Denote by (α, β) a DP of a no-long-sided (2,2,2) graph H with an AT $\{x, y, z\}$. By Lemma 5, α and β cannot both be in $\mathcal{P}_{x,y}$, cannot both be in $\mathcal{P}_{x,z}$, and cannot both be in $\mathcal{P}_{y,z}$. Therefore, since H is a no-long-sided (2,2,2) graph, the only two options for the position of α and β are an AT vertex $v \in \{x, y, z\}$ and a mid-point of a path that avoids the neighborhood of v, or a vertex in M_i and a vertex in M_j, where $i \neq j$, and $i, j \in \{1,2,3\}$.

Examples showing that each of these types of DPs can occur are given in Figure 4.5. DP vertices are shaded in these examples. □

Claim 31 *Let (α, β) be a DP of a no-long-sided (2,2,2) graph H with an AT $\{x, y, z\}$. If $\alpha \in M_i$, $\beta \in M_j$, where $i \neq j$, and $i, j \in \{1,2,3\}$, then every vertex in M_k, for $k \notin \{i, j\}$ and $k \in \{1,2,3\}$, must either be adjacent to α, or to β.*

Proof: Assume to the contrary. W.l.o.g. let $\alpha \in M_1, \beta \in M_2$, and let v be a vertex in M_3 that is not adjacent to α and is not adjacent to β. Then the path from α to β induced on $\{\alpha x\} \cup \{x\beta\}$ does not hit v contradicting (α, β) being a DP of H. □

Claim 32 *Let (α, β) be a DP of a no-long-sided (2,2,2) graph H with an AT $\{x, y, z\}$. If $\{\alpha, \beta\} \in \{x, m_3\}$ for $m_3 \in M_3$, then m_3 cannot be adjacent to any vertex $p \in M_1 \cup M_2$ unless every vertex of M_3 is adjacent to either m_3 or p. By symmetry, the same holds for $\{\alpha, \beta\} \in \{y, m_2\}$ for $m_2 \in M_2$, and for $\{\alpha, \beta\} \in \{z, m_1\}$ for $m_1 \in M_1$.*

Proof: W.l.o.g. let $\alpha = x$ and $\beta = m_3$, $m_3 \in M_3$. Assume to the contrary. Thus, let $m_3 p \in E$, $p \in M_1 \cup M_2$, and a vertex $m_3' \in M_3$ is neither adjacent to m_3, nor to p. Then the path from α to β induced on $\{\alpha p\} \cup \{p\beta\}$ does not hit m_3' contradicting (α, β) being a DP. □

Chapter 5

(1,2,2) Graphs

The claims in this section will describe some structure of (1,2,2) graphs. As in the previous chapter, the goal is to describe enough structure of (1,2,2) graphs, so that the positions of all DP vertices in these graphs can always be determined. This goal is achieved and presented in Claims 39, 41, and 44. These are the main results of this chapter. Each of them describes the positions of DP vertices in a specific type of (1,2,2) graphs, and together they describe the positions of DP vertices all types of (1,2,2) graphs. In addition, an interesting structural result appears in Claim 35.

In this chapter we will assume the following. Let $\mathcal{P}_{x,y}$, $\mathcal{P}_{x,z}$ and $\mathcal{P}_{y,z}$ be defined as before in a (1,2,2) graph H with AT $\{x, y, z\}$. Let x be a non-path disjoint AT vertex of H with $x' \in P_{x,y} \cap P_{x,z}$, $x \neq x'$, for some $P_{x,y} \in \mathcal{P}_{x,y}$, and some $P_{x,z} \in \mathcal{P}_{x,z}$. Let $P_{x',y}$ be the subpath of $P_{x,y}$ between x' and y, $P_{x',z}$ the subpath of $P_{x,z}$ between x' and z, and let $P_{y,z}$ be any path in $\mathcal{P}_{y,z}$. In the claims in this chapter which say "let H be defined as in the paragraph preceding Claim 33," it is assumed that H is defined as in this paragraph. The same holds for $P_{x,y}$, $P_{x,z}$, and $P_{y,z}$.

The following two results present some structural properties of (1,2,2) graphs.

Claim 33 *For any $a \in P_{x',y} \setminus \{x'\}$ and any $b \in P_{x',z} \setminus \{x'\}$, $ab \notin E$.*

Proof: Assume to the contrary. Thus, $ab \in E$. Take a subgraph $\hat{H}$ of H induced on

$P_{x,y} \cup P_{x,z}$. Note that $\hat{H}$ is also an AT graph because it has the path $P_{x,y}$ that avoids the neighborhood of z, $P_{x,z}$ that avoids the neighborhood of y, and the path from y to z induced on $P_{y,a} \cup \{ab\} \cup P_{b,z}$, where $P_{y,a}$ is the subpath of $P_{x,y}$ between y and a, and $P_{b,z}$ is the subpath of $P_{x,z}$ between b and z, that avoids the neighborhood of x. (Note that no vertex in $P_{x',y} \cup P_{x',z} \setminus \{x'\}$ is adjacent to x since all paths in $\mathcal{P}_{x,y} \cup \mathcal{P}_{x,z}$ are induced.) But now, vertices x, y, and z in $\hat{H}$ are all non-path-disjoint contradicting Theorem 26. Therefore, there does not exist an edge from a to b in H.□

Corollary 12 *For any $a \in P_{x',y} \setminus \{x', y\}$ and any $c \in P_{y,z} \setminus \{y, z\}$, if $ac \in E$, then $cy \in E$.*

Proof: Assume to the contrary. Thus, $ac \in E$, and $cy \notin E$. Since $cy \notin E$, there must exist a subpath of $P_{y,z}$ from c to y of length bigger than one. Let $P'_{y,z}$ be the path from y to z induced on $P_{ya} \cup \{ac\} \cup P_{cz}$, where $P_{y,a}$ is the subpath of $P_{x,y}$ from y to a, and P_{cz} is the subpath of $P_{y,z}$ from c to z. Since $cy \notin E$, $P_{x,y}$ and $P'_{y,z}$ have a common vertex different from y. Consider graph $\check{H}$ induced on $P_{x,y}$, $P'_{y,z}$, and $P_{x,z}$. This graph is AT since $P_{x,y}$ avoids the neighborhood of z, $P'_{y,z}$ avoids the neighborhood of x, and $P_{x,z}$ avoids the neighborhood of y. Note that since c is not adjacent to y in H, y is not path-disjoint with respect to x, z in $\check{H}$. Now, $\check{H}$ has two non-path-disjoint vertices, x and y, and thus, by Theorem 26, is not an HDP graph which contradicts H being HDP.

Note that in the same way it can be proven that for any $b \in P_{x',z} \setminus \{x', z\}$, if $bc \in E$, then $cz \in E$. □

The following claim describes one of the possible positions of DP vertices in (1,2,2) graphs.

Claim 34 *Let H be defined as in the paragraph preceding Claim 33. Also, let $P_{x,y}$, $P_{x,z}$, and $P_{y,z}$ be defined as in the paragraph preceding Claim 33, and let (α, β) be a DP of H. If $\alpha \in H \setminus C$, where C is the connected component of $H \setminus \{x'\}$ that contains y and z, and if $\beta \in P_{y,z}$, then β is adjacent to both y and z.*

Proof: Assume to the contrary. Thus, assume that β is not adjacent to z. Then, the path from β to α that consists of the subpath of $P_{y,z}$ between β and y, and the y,α-path induced on $P_{x,y} \cup \{\alpha\}$ does not hit z contradicting (α,β) being a DP. □

The following claim establishes an interesting structural property of (1,2,2) graphs.

Claim 35 *Let H, $P_{x,y}$, and $P_{x,z}$ be defined as in the paragraph preceding Claim 33. All paths in $\mathcal{P}_{y,z}$ are P_3's.*

Proof: Assume to the contrary. Let $P'_{y,z}$ be a path in $\mathcal{P}_{y,z}$ that is of length bigger than 2. Let $\tilde{H}$ be the subgraph of H induced on $P_{x,y} \cup P_{x,z} \cup P'_{y,z}$. Clearly, $\tilde{H}$ is 1-disjoint w.r.t. x. Let (α,β) be a DP of $\tilde{H}$. One of α,β must be in $\{x,x'\}$, since otherwise the path between them induced on $V(\tilde{H}) \setminus \{x,x'\}$ would miss x. (Note that no vertices in $V(\tilde{H}) \setminus \{x,x'\}$ are adjacent to x since $P_{x,y}$ and $P_{x,z}$ are induced, and $P'_{y,z} \in \mathcal{P}_{y,z}$.) By Claim 10, $\beta \in P'_{y,z} \setminus \{y,z\}$ in $\tilde{H}$. Since $P'_{y,z}$ is not a P_3, β is not adjacent to at least one of y,z. W.l.o.g. let $\beta y \notin E$. But now, the path induced on $P_{\beta,z} \cup P_{x,z}$, where $P_{\beta,z}$ is the subpath of $P'_{y,z}$ between β and z, is an α,β-path missing y contradicting (α,β) being a DP of $\tilde{H}$. □

Remember that $\mathcal{R}_{x,y}$ is defined in Chapter 3 to be the set of all induced paths between x and y that avoid $N(z)$ such that $\forall P \in \mathcal{R}_{x,y}\ \exists Q \in \mathcal{P}_{x,z}$ such that $x' \in P \cap Q$, $x' \neq x$. Also, $\mathcal{R}_{x',y}$ is defined to be the set of subpaths of paths in $\mathcal{R}_{x,y}$ between x' and y. Similar definitions hold for $\mathcal{R}_{x,z}$ and $\mathcal{R}_{x',z}$. The following claim gives a structural property of (1,2,2) graphs.

Claim 36 *Let H be defined as in the paragraph preceding Claim 33. Let y' be the neighbor of y on a path in $\mathcal{R}_{x',y}$, and let z' be the neighbor of z on a path in $\mathcal{R}_{x',z}$. If $deg(y') = deg(z') = 2$ and $\tilde{\mathcal{R}}_{x,y} \neq \emptyset \neq \tilde{\mathcal{R}}_{x,z}$, then H is not HDP.*

Proof: Assume to the contrary. Let $P_1 \in \tilde{\mathcal{R}}_{x,y}$ and let $Q_1 \in \tilde{\mathcal{R}}_{x,z}$. Let $P_2 \in \mathcal{R}_{x,y}$, $Q_2 \in \mathcal{R}_{x,z}$, and let $y' = P_2 \cap N(y)$ and $z' = Q_2 \cap N(z)$ be of degree 2. Let R be any path in $\mathcal{P}_{y,z}$. Let $\tilde{H}$ be the subgraphs of H induced on $P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup R$. However, $\tilde{H} \setminus \{\{v \in P_2 | v \notin \{y, y', x\}\} \cup \{v \in Q_2 | v \notin \{z, z', x\}\}\}$ has AT $\{x, y', z'\}$ that contradicts Theorem 26, in particular, both y' and z' are non-path-disjoint. □

Chapter 4 differentiated between two-long-sided and one-long-sided graphs. When dealing with (1,2,2) graphs it is not necessary to have this distinction; instead, *long sided* graphs, i.e., (1,2,2) graphs for which at least one path in $\mathcal{R}_{x',y} \cup \mathcal{R}_{x',z}$ is of length bigger than 2, are studied. In the study of such graphs, some simple structure of these graphs that is forced by a particular placement of the DP vertices will be emphasized. Clearly, every path between DP vertices must dominate the graph. This results in some edges that will not be explicitly mentioned.

5.1 Long-Sided Graphs

As before, let x be non-path-disjoint w.r.t. y, z in an HDP∩AT graph H with AT $\{x, y, z\}$, and let x' be a significant neighbor of x. Let $\mathcal{P}_{y,z}$ be the set of all induced paths between y and z in H that avoid $N(x)$. Let M be the set of midpoints of all paths in $\mathcal{P}_{y,z}$. Let $D(u, v)$ denote the set of vertices that intercept all u, v-paths.

Claims 37 and 38 describe some structure of long-sided (1,2,2) graphs.

Claim 37 *Every non-neighbor of x' in a long path in $\mathcal{R}_{x',y}$ must be universal to M. By symmetry, the same holds for $\mathcal{R}_{x',z}$ as well.*

Proof: Assume to the contrary. Thus, there exists a non-neighbor v of x' on a long path $P_{x',y} \in \mathcal{R}_{x',y}$ that is not adjacent to a vertex $m \in M$. Let $\tilde{H}$ be the subgraph of H induced on $P_{x',y} \cup \{x\} \cup P_{y,z} \cup P_{x',z}$, where $P_{x',z}$ is any path in $\mathcal{R}_{x',z}$, and $m \in P_{y,z}$. Let (α, β) be a DP of $\tilde{H}$. Since $\tilde{H}$ is 1-disjoint w.r.t. x, by Claim 8, one DP vertex of $\tilde{H}$ is in $\tilde{H} \setminus C$ and the other one is in C, where C is the connected component of $\tilde{H} \setminus \{x'\}$ containing y, z. W.l.o.g. assume $\alpha \in \tilde{H} \setminus C = \{x, x'\}$ (note that since $\tilde{H}$ is minimal, by

Claim 8, $\tilde{H} \setminus C = \{x, x'\}$). Then, by Claims 10 and 35, β is the midpoint of $P_{y,z}$ in $\tilde{H}$, i.e., $\beta = m$. Since $v \in P_{x',y} \setminus \{x', y\}$, by Claim 33 and the assumption that $vx' \notin E$, v is not adjacent to any vertex in $P_{x',z}$. Now, the path from α to β induced on $\{\beta z\} \cup P_{x',z} \cup \{x\}$ does not hit v contradicting (α, β) being a DP of $\tilde{H}$. □

As before, let x be non-path-disjoint w.r.t. y, z with a significant neighbor x' in an HDP∩AT graph H with AT $\{x, y, z\}$. Let $\mathcal{R}_{x,y}$ be defined as in Chapter 3, and let $\tilde{\mathcal{R}}_{x,y}$ be the set of all induced paths between x and y that avoid $N(z)$ and do not share vertices with paths in $\mathcal{R}_{x,y}$ other than x and y. Similarly, let $\mathcal{R}_{x,z}$ be defined as in Chapter 3, and let $\tilde{\mathcal{R}}_{x,z}$ be the set of all induced paths between x and z that avoid $N(y)$ and do not share vertices with paths in $\mathcal{R}_{x,z}$ other than x and z. Let $\mathcal{P}_{y,z}$ be defined as before to be the set of all induced paths between y and z that avoid the neighborhood of x.

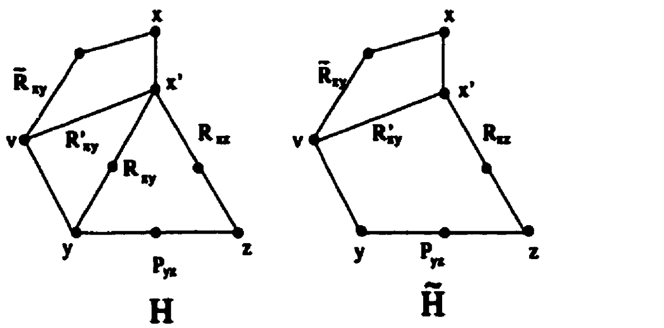
Claim 38 *Let H be defined as in the paragraph preceding this claim. If $\tilde{\mathcal{R}}_{x,y} \neq \emptyset$, or $\tilde{\mathcal{R}}_{x,z} \neq \emptyset$, then H is not 1-disjoint w.r.t. x.*

Proof: Let $\tilde{\mathcal{R}}_{x,y} \neq \emptyset$. Let $P \in \tilde{\mathcal{R}}_{x,y}$. By definition of $\tilde{\mathcal{R}}_{x,y}$, $x' \notin P$, and therefore, $P \in H \setminus \{x'\}$. Thus, the connected component of $H \setminus \{x'\}$ that contains y and z also contains x, i.e. H is not 1-disjoint w.r.t. x. □

Denote by M the set of mid-points of all paths in $\mathcal{P}_{y,z}$, by W_1 the set of mid-points of all short paths in $\tilde{\mathcal{R}}_{x,y}$, and by W_2 the set of mid-points of all short paths in $\tilde{\mathcal{R}}_{x,z}$. Note that W_1 and W_2 might be empty.

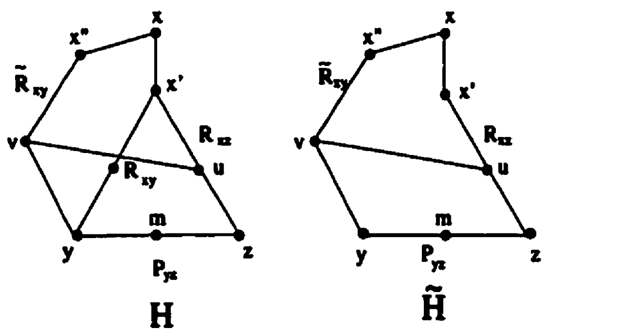
The following claim is one of the main results of this chapter. It describes the positions of DP vertices in long-sided (1,2,2) graphs.

Claim 39 *One DP vertex of a long-sided (1,2,2) graph H is in $N[x]$ and the other one is in M, or one DP vertex of H is in $N[z]$ and the other one is in W_1, or one DP vertex of H is in $N[y]$ and the other one is in W_2. Each of these three types of DPs can occur.*



In H, paths $\tilde{R}_{xy}$ and R'_{xy} share vertex v different from x and y contradicting the definition of $\tilde{R}_{xy}$.

Figure 5.1:



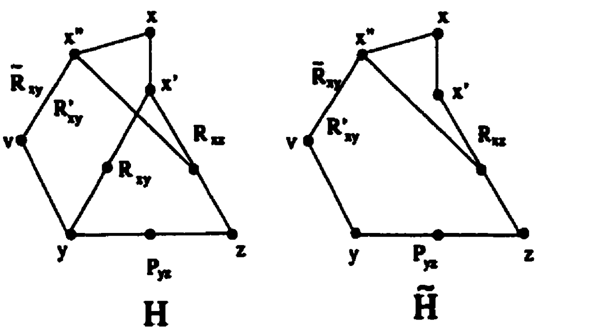
vu is not an edge, since otherwise $\tilde{H} \setminus \{m\}$ is (1,1,2) and therefore not an HDP graph. That is, v does not belong to a path in R_{xz}, i.e., $\tilde{H}$ is not (1,2,2).

Figure 5.2:

Proof: Denote by (α, β) a DP of H. By Lemma 5, α and β cannot both belong to $\mathcal{P}_{x,y}$, cannot both belong to $\mathcal{P}_{x,z}$, and cannot both belong to $\mathcal{P}_{y,z}$.

By Lemma 6 (2), α and β cannot belong to the union of vertices of $R_{x',y}$ and $R_{x',z}$, where $R_{x',y}$ and $R_{x',z}$ are any paths in $\mathcal{R}_{x',y}$ and $\mathcal{R}_{x',z}$ respectively.

If $\tilde{\mathcal{R}}_{x,y} \neq \emptyset$ and $\tilde{\mathcal{R}}_{x,y}$ has a long path, then it is not the case that one of α, β is equal to z and the other one is an internal vertex of a long path $\tilde{R}_{x,y} \in \tilde{\mathcal{R}}_{x,y}$ for the following reason. Assume to the contrary. Thus, w.l.o.g., $\alpha = z$ and $\beta \in \tilde{R}_{x,y} \setminus \{x, y\}$. Take the subgraph $\tilde{H}$ of H induced on $\tilde{R}_{x,y} \cup R_{x,z} \cup P_{y,z}$, for any path $R_{x,z} \in \mathcal{R}_{x,z}$ and any $P_{y,z} \in \mathcal{P}_{y,z}$ (such paths $R_{x,z}$ and $P_{y,z}$ exist by definition of H). Now, the claim is that $\tilde{H}$

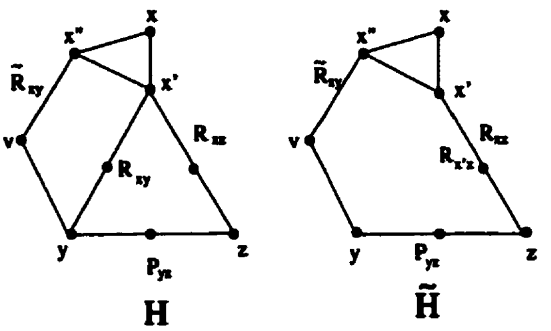


Since $\tilde{R}_{xy}$ shares a vertex (namely x") different from x and y with a path in P_{xz} this contradicts the definition of $\tilde{R}_{xy}$.

Figure 5.3:

is a two-long-sided (2,2,2) graph containing α, β, and thus, by Lemma 7 (a) and Claim 7, it is not the case that one of α, β is equal to z and the other one is an internal vertex of $\tilde{R}_{x,y}$. To see that $\tilde{H}$ is a (2,2,2) graph (and not (1,2,2)) various edges are inserted and it is noticed how such insertions contradict certain definitions. These insertions are illustrated in Figures 5.1, 5.3, 5.4, and 5.2. For example (see Figure 5.1), if v, a non-neighbor of x on $\tilde{R}_{x,y} \in \tilde{\mathcal{R}}_{x,y}$, is adjacent to x', then this contradicts the definition of $\tilde{R}_{x,y}$, since the path $\tilde{R}_{x,y} \in \tilde{\mathcal{R}}_{x,y}$ shares a vertex v different from x and y with the path $R'_{x,y} \in \mathcal{R}_{x,y}$ induced on $\{xx'\} \cup \{x'v\} \cup \{vy\}$. Figure 5.2 deals with the case when v is adjacent to a non-neighbor of x in $\mathcal{R}_{x,z}$, i.e., to a vertex in $\mathcal{R}_{x',z} \setminus \{x', z\}$. In Figures 5.3 and 5.4, x'', the neighbor of x in $\tilde{R}_{x,y}$, is adjacent to an internal vertex of $R_{x',z}$ and to x' respectively.

If $\tilde{\mathcal{R}}_{x,y} \neq \emptyset$ and it has a long path, then it is not the case that one of α, β is an internal vertex of some long path $\tilde{R}_{x,y} \in \tilde{\mathcal{R}}_{x,y}$ and the other one is an internal vertex of some path $R_{x,z} \in \mathcal{R}_{x,z}$ for the following reason. Assume to the contrary. Thus, w.l.o.g., $\alpha \in \tilde{R}_{x,y} \setminus \{x, y\}$ and $\beta \in R_{x,z} \setminus \{x, z\}$. Similar to the above, take the subgraph $\tilde{H}$ of H induced on $\tilde{R}_{x,y} \cup R_{x,z} \cup P_{y,z}$, for any $P_{y,z} \in \mathcal{P}_{y,z}$. Now, $\tilde{H}$ is a two-long-sided (2,2,2) graph containing α, β, and thus, by Lemma 6 (1) and Claim 7, it is not the case that one of α, β is an internal vertex of $\tilde{R}_{x,y}$ and the other one is an internal vertex of $R_{x,z}$,



x'x" edge is allowed and $\tilde{H}$ is a (2,2,2) graph, since vertex x" does not belong to a path in R_{xz}. Note that the path induced on {xx"}U{x"x'}U$R_{x'z}$ is not in the set R_{xz} since this is the set of induced paths, but xx' is an edge.

Figure 5.4:

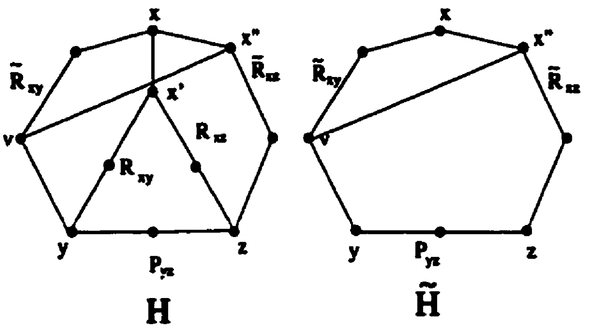
contradicting our assumption.

If both $\tilde{\mathcal{R}}_{x,y} \neq \emptyset$ and $\tilde{\mathcal{R}}_{x,z} \neq \emptyset$, and if both have long paths, then it is not the case that one of α, β is an internal vertex of some long path $\tilde{R}_{x,y} \in \tilde{\mathcal{R}}_{x,y}$ and the other one is an internal vertex of some long path $\tilde{R}_{x,z} \in \tilde{\mathcal{R}}_{x,z}$ for the following reason. Assume to the contrary. Thus, w.l.o.g., $\alpha \in \tilde{R}_{x,y} \setminus \{x, y\}$ and $\beta \in \tilde{R}_{x,z} \setminus \{x, z\}$. Similar to the above, take the subgraph $\tilde{H}$ of H induced on $\tilde{R}_{x,y} \cup \tilde{R}_{x,z} \cup P_{y,z}$, for any $P_{y,z} \in \mathcal{P}_{y,z}$. Now, $\tilde{H}$ is a two-long-sided (2,2,2) graph containing α, β, and thus, by Lemma 6 (1) and Claim 7, it is not the case that one of α, β is an internal vertex of $\tilde{R}_{x,y}$ and the other one is an internal vertex of $\tilde{R}_{x,z}$, contradicting our assumption; note that $\tilde{H}$ cannot be (1,2,2) as illustrated in Figure 5.5, where a non-neighbor v of x in $\tilde{R}_{x,y} \in \tilde{\mathcal{R}}_{x,y}$ is adjacent to the neighbor $\tilde{x}$ of x in $\tilde{R}_{x,z} \in \tilde{\mathcal{R}}_{x,z}$.

The only options for DP vertices α, β are that either one of them is in $N[x]$ and the other one is in M, or that one of them is in $N[z]$ and the other one is in W_1, or that one of them is in $N[y]$ and the other one is in W_2.

Examples showing that each of these three types of DPs can occur are given in Figure 5.6. DP vertices are shaded in these examples. □

The structure of (1,2,2) graphs that have one DP vertex in $N[z]$ and the other one in



vx" is not an edge, since otherwise x" would belong to the path in R_{xy} induced on {xx"}U{x"v}U{vy} contradicting the definition of $\tilde{R}_{xz}$.

Figure 5.5:

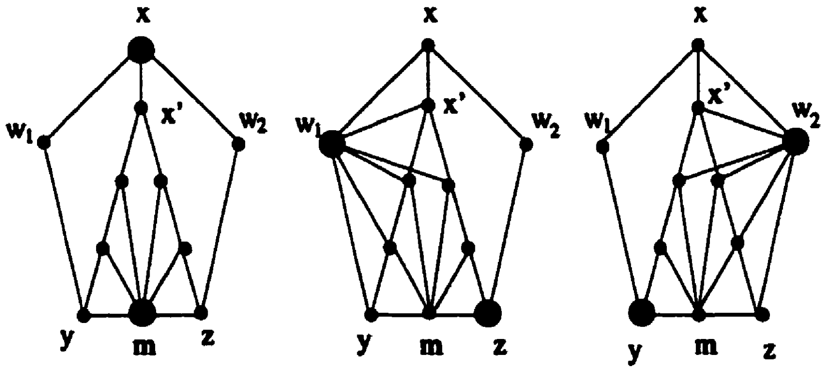


Figure 5.6:

W_1, or symmetrically, one DP vertex in $N[y]$ and the other one in W_2, will be examined in subsection 5.1.1. Here, only the structure of long-sided graphs with one DP vertex equal to x and the other one in M is presented in the following claim.

Claim 40 *Let (α, β) be a DP of a long-sided (1,2,2) graph H, and let $\{\alpha, \beta\} = \{x, m\}$, for some $m \in M$. If v is the vertex of distance i from x on a path $\tilde{R}$ in $\tilde{\mathcal{R}}_{x,y}$, for $i \geq 2$, then either $vm \in E$, or v is adjacent to every neighbor of x in $\mathcal{P}_{x,z}$; in addition, for any $P \in \tilde{\mathcal{R}}_{x,y} \setminus \tilde{R}$, v has to be adjacent to a vertex on the shortest path between x and m induced on $P \cup \{m\}$.*

Proof: Let v be the vertex of distance i from x on some path $\tilde{R} \in \tilde{\mathcal{R}}_{x,y}$, for $i \geq 2$. Assume to the contrary. Thus, $vm \notin E$ and v is not adjacent to the neighbor of x on some path $P \in \mathcal{P}_{x,z}$. Note that, by Claim 15, v cannot be adjacent to any non-neighbor of x on P. Therefore, the path from x to m induced on $P \cup \{zm\}$ misses v.

If for some $P \in \tilde{\mathcal{R}}_{x,y} \setminus \tilde{R}$, v is not adjacent to any vertex on the shortest x, m-path Q induced on $P \cup \{m\}$, then the x, m-path induced on Q misses v. □

5.1.1 A Long Side Has a Short Path

There are two graph structures to consider here. One structure happens when there exists a P_3 in $\mathcal{R}_{x',y} \cup \mathcal{R}_{x',z}$, and the other one happens when there exists a P_3 in $\tilde{\mathcal{R}}_{x,y} \cup \tilde{\mathcal{R}}_{x,z}$. All the properties that the first structure satisfies have already been described for long-sided (1,2,2) graphs. However, the second structure has some additional properties that will be described in this subsection.

The following notation will be used in this subsection. Denote by M the set of mid-vertices of all paths in $\mathcal{P}_{y,z}$ and by W_1 the set of mid-vertices of all P_3's in $\tilde{\mathcal{R}}_{x,y}$ in a long-sided (1,2,2) graph H where $W_1 \neq \emptyset$.

The following claim is one of the main results of this chapter. It describes the positions of DP vertices in long-sided (1,2,2) graphs with a long side having a short path.

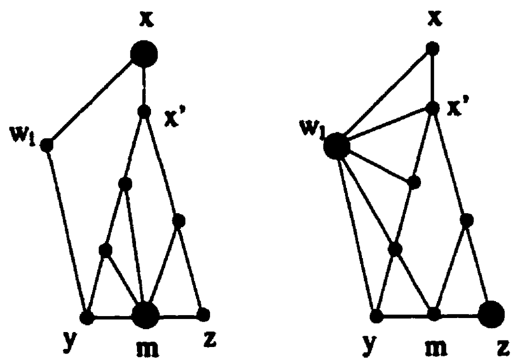


Figure 5.7:

Claim 41 *One DP vertex of H is in $N[x]$ and the other one is in M, or one DP vertex is in $N[z]$ and the other one is in W_1. Each of these two types of DPs can occur. (Note that the symmetric result would hold if there was a P_3 in $\tilde{\mathcal{R}}_{x,z}$.)*

Proof: Follows the proof of Claim 39.

Examples showing that each of these two types of DPs can occur are given in Figure 5.7. DP vertices are shaded in these examples. □

The structure of long-sided graphs when one DP vertex is x and the other one is in M was presented in Claim 40. The following claim will describe some of the structural properties of these graphs in which one DP vertex is in $N[x] \setminus \{x, x'\}$ and the other one is in M. The reader is reminded that other edges may be forced to ensure that (α, β) is a DP.

Claim 42 *Denote by (α, β) a DP of H. If one of the DP vertices of H is in $N[x] \setminus \{x, x'\}$ and the other one is in M, then:*

(1) If $\{\alpha, \beta\} = \{w, m\}$, for some $w \in W_1$ and some $m \in M$, then:

(a) All vertices in $\mathcal{P}_{x,z}$ must be adjacent to m, or to w.

(b) All vertices in long paths in $\mathcal{R}_{x',y}$ must be adjacent to m, or to w.

(c) All non-neighbors of y in $\tilde{\mathcal{R}}_{x,y}$ must be adjacent to m, or to w.

(2) If $\{\alpha,\beta\} = \{x_r, m\}$, where x_r is the neighbor of x on a long path $\tilde{R}_{x,y} \in \tilde{\mathcal{R}}_{x,y}$, then the following claims must be satisfied:

(a) All non-neighbors of x_r on $\tilde{R}_{x,y}$, if they exist, must be adjacent to m, or to all neighbors of x in $\mathcal{P}_{x,z}$. If $\tilde{P}_{x,y}$ is a short path in $\tilde{\mathcal{R}}_{x,y}$, then all non-neighbors of x_r and y on $\tilde{R}_{x,y}$, if they exist, that are not adjacent to the mid-vertex of $\tilde{P}_{x,y}$ must be adjacent to m.

(b) Neighbors of x' on long paths in $\mathcal{R}_{x',y} \cup \mathcal{R}_{x',z}$ must be adjacent to x_r, or to all $w \in W_1 \cup W_2$, or to m.

Proof: (1) (a) Let v be a vertex in $\mathcal{P}_{x,z}$ that is not adjacent to m and is not adjacent to w. Then the path between m and w induced on $\{my\} \cup \{yw\}$ misses v.

(b) By Claim 37, all non-neighbors of x' on long paths in $\mathcal{R}_{x',y}$ are adjacent to all $m \in M$. Now, prove the claim for neighbors of x' on long paths in $\mathcal{R}_{x',y}$ (note that the claim holds for x' by part (a) above). Let v be the neighbor of x' on a long path $R_{x',y} \in \mathcal{R}_{x'y}$ that is not adjacent to m, and is not adjacent to w. Then the path between m and w induced on $\{my\} \cup \{yw\}$ misses v; note that $vy \notin E$, since $vx' \in E$ and $R_{x',y}$ is a long path.

(c) Similarly, if v is a non-neighbor of y in $\tilde{\mathcal{R}}_{x,y}$ that is not adjacent to m, and is not adjacent to w, then the path between m and w induced on $\{my\} \cup \{yw\}$ misses v.

(2) Let $\{\alpha,\beta\} = \{x_r, m\}$, where x_r is the neighbor of x on a long path $\tilde{R}_{x,y} \in \tilde{\mathcal{R}}_{x,y}$.

(a) The first part is true, since otherwise the x_r, m-path induced on $\{x_r x\} \cup P_{x,z} \cup \{m\}$ would miss the non-neighbor v of x_r that is not adjacent to the neighbor of x on $P_{x,z} \in \mathcal{P}_{x,z}$ and is not adjacent to m; note that by Claim 15, v can only be adjacent to neighbors of x in $\mathcal{P}_{x,z}$.

Note that by applying Claim 17 to the subgraph $\tilde{H}$ of H induced on $\tilde{R}_{x,y} \cup \tilde{P}_{x,y} \cup R_{x,z} \cup P_{y,z}$, for any $R_{x,z} \in \mathcal{R}_{x,z}$ and $m \in P_{y,z} \in \mathcal{P}_{x,z}$, all non-neighbors of x_r on $\tilde{R}_{x,y}$ are adjacent to m. Note that since (1,2,2) graphs are being considered, the non-neighbor of x_r on $\tilde{R}_{x,y}$ can be non-adjacent to m if it is adjacent to x' (this can be proved by following the proof of Claim 17). Therefore, all non-neighbors of x_r on $\tilde{R}_{x,y}$ are adjacent to x' or to m.

The proof of the second part is as follows. Assume that a non-neighbor p of x_r and y on $\tilde{R}_{x,y}$ is not adjacent to m and is not adjacent to the mid-vertex of $\tilde{P}_{x,y}$. (By the first path of this claim, $px' \in E$.) Then the path from x_r to m induced on $\{x_r x\} \cup \tilde{P}_{x,y} \cup \{ym\}$ misses p.

(b) W.l.o.g. assume that a neighbor x'' of x' on a long path in $\mathcal{R}_{x',y}$ is not adjacent to x_r, w, and m, for some $w \in W_1$. Then the path from x_r to m induced on $\{x_r x\} \cup \{xw\} \cup \{wy\} \cup \{ym\}$ misses x''. □

The following claim describes some structure of these graphs forced by one DP vertex being in $N[z]$ and the other one in W_1.

Claim 43 *Let (α, β) be a DP of a long-sided (1,2,2) graph H defined as in the second paragraph of this subsection. If one DP vertex of H is in $N[z]$ and the other one is in W_1, then the following conditions must be satisfied:*

(1) Let $\{\alpha, \beta\} = \{z, w\}$ for some $w \in W_1$.

(a) x' and all of its neighbors on long paths in $\mathcal{R}_{x',y} \cup \mathcal{R}_{x',z}$ must be adjacent to w, or to all $m \in M$.

(b) If $\tilde{\mathcal{R}}_{x,z} \neq \emptyset$, then the neighbors of x' on paths in $\mathcal{R}_{x',y}$ must be adjacent to w, or to all neighbors of x in $\tilde{\mathcal{R}}_{x,z}$.

(2) Let $\{\alpha, \beta\} = \{z', w\}$, where z' is the neighbor of z on a path $R_{x',z} \in \mathcal{R}_{x',z}$, or on a path $\tilde{R}_{x,z} \in \tilde{\mathcal{R}}_{x,z}$, and $w \in W_1$. Then:

(a) If z' belongs to a path in $\mathcal{R}_{x',z}$, then all vertices $v \in N(x')$ on long paths in $\mathcal{R}_{x',y}$ must be adjacent to w, or universal to M.

(b) Let $z' \in R_{x',z} \in \mathcal{R}_{x',z}$. Then, all non-neighbors of x' on $\mathcal{R}_{x',y}$ must be adjacent to w. Also, all non-neighbors of z' on $R_{x',z}$, if they exist, must be adjacent to all $m \in M$, or to w.

(c) Let $z' \in \tilde{R}_{x,z} \in \tilde{\mathcal{R}}_{x,z}$. Then, all non-neighbors of x on $\tilde{\mathcal{R}}_{x,y}$ must be adjacent to w, or to all neighbors of x on $\mathcal{P}_{x,z}$.

Note that the case when $\{\alpha, \beta\} = \{m, w\}$, for some $m \in M$ and $w \in W_1$ is covered in Claim 42.

Proof: (1) Let $\{\alpha, \beta\} = \{z, w\}$ for some $w \in W_1$.

(a) Let v be either x' or the neighbor of x' on a long path in $\mathcal{R}_{x',y} \cup \mathcal{R}_{x',z}$ that is not adjacent to w, and is not adjacent to some $m \in M$. Then the path from w to z induced on $\{wy\} \cup \{ym\} \cup \{mz\}$ misses v.

(b) Let $\tilde{\mathcal{R}}_{x,z} \neq \emptyset$ and let v be the neighbor of x' on a path in $\mathcal{R}_{x',y}$ that is not adjacent to w, and is not adjacent to some neighbor x'' of x on $\tilde{R}_{x,z} \in \tilde{\mathcal{R}}_{x,z}$. Then the path between w and z induced on $\{wx\} \cup \tilde{R}_{x,z}$ misses v; note that by Claim 15, v cannot be adjacent to a non-neighbor of x on $\tilde{R}_{x,z}$.

(2) Let $\{\alpha, \beta\} = \{z', w\}$.

(a) Let z' belong to a path in $\mathcal{R}_{x',z}$. Let $v \in N(x')$ be on a long path in $\mathcal{R}_{x',y}$ that is non-adjacent to some $m \in M$ and to w. Then the path from z' to w induced on $\{z'z\} \cup \{zm\} \cup \{my\} \cup \{yw\}$ misses v; note that v is not adjacent to any vertex in $\mathcal{R}_{x',z} \setminus \{x'\}$ by Claim 33.

(b) Let $z' \in \mathcal{R}_{x',z}$. Let v be a non-neighbor of x' on $\mathcal{R}_{x',y}$ that is not adjacent to w. Then the path from z' to w induced on $R_{x,z'} \cup \{xw\}$, where $R_{x,z'}$ is the path between x and z' induced on $R_{x',z} \cup \{x'x\}$, misses v. (Note that, by Claim 33, there are no edges between vertices on a path in $\mathcal{R}_{x',y}$ and vertices on a path in $\mathcal{R}_{x',z}$; also, $vx \notin E$ since all paths in $\mathcal{R}_{x,z}$ are induced.)

Note that all non-neighbors of x' on $R_{x',z}$ are adjacent to all $m \in M$, by Claim 37. If a neighbor x'' of x' on $R_{x',z}$ that is a non-neighbor of z', is not adjacent to w, and is not adjacent to some $m \in M$, then the z', w-path induced on $\{z'z\} \cup \{zm\} \cup \{my\} \cup \{yw\}$ misses x''.

(c) Similarly, let $z' \in \tilde{R}_{x,z} \in \tilde{\mathcal{R}}_{x,z}$. Let v be a non-neighbor of x on $\tilde{\mathcal{R}}_{x,y}$ that is not adjacent to w and is not adjacent to a neighbor of x on $P_{x,z} \in \mathcal{P}_{x,z}$. Then the path from z' to w induced on $\{z'z\} \cup P_{x,z} \cup \{xw\}$ misses v. (Note that, by Claims 15 and 33, there are no edges between non-neighbors of x on a path in $\mathcal{P}_{x,y}$ and non-neighbors of x on a path in $\mathcal{P}_{x,z}$.) $\square$

5.1.2 A Long Side Has at Least Two Long Paths

Let H be a 2-long-sided, or a 1-long-sided (1,2,2) graph. There are three graph structures to consider here. The first one is when a long side, say $\mathcal{R}_{x',y}$, has at least two long paths. The second one is when in addition to the existence of a long path in $\mathcal{R}_{x',y}$, there also exists a long path in $\tilde{\mathcal{R}}_{x,y}$. The third one is when in addition to the existence of a long path in $\mathcal{R}_{x',y}$, there also exist at least two long paths in $\tilde{\mathcal{R}}_{x,y}$. Some aspects of these three structures will be described separately in this section.

The position of DPs in such graphs is explained in Claim 39 and it depends on whether there are short paths in $\tilde{\mathcal{R}}_{x,y}$ and $\tilde{\mathcal{R}}_{x,z}$, or not. The structure of these graphs depends on the position of its DP vertices and most of it is described in Claims 42 and 43. The only property of these graphs that is still unaddressed is the "interaction" between the long paths that belong to a long side. This will be addressed in the following three Facts, for each of the structures mentioned in the previous paragraph separately.

Fact 1 *For all $P, Q \in \mathcal{R}_{x',y}$ that are not P_3's, $\forall y_1 \in P \setminus N[x']$, $\forall y_2 \in Q \setminus N[x']$, $y_1 \in D(x', y_2)$, or $y_2 \in D(x', y_1)$.*

Proof: Similar to the proof of Claim 20. Assume to the contrary. Let $P, Q \in \mathcal{R}_{x',y}$ that are not P_3's, and let $y_1 \in P \setminus N[x']$, $y_2 \in Q \setminus N[x']$, such that $y_1 \notin D(x', y_2)$, and $y_2 \notin D(x', y_1)$. Let $P_{y,z}$ be any path in $\mathcal{P}_{y,z}$, let m be the mid-vertex of $P_{y,z}$, and let $P_{x',z}$ be any path in $\mathcal{R}_{x',z}$. Let $\check{H}$ be the subgraph of H induced on $P \cup Q \cup P_{x',z} \cup P_{y,z} \cup \{x\}$. Now, $\check{H} \setminus \{y, m\}$ does not have a DP any more, since $\{y_1, y_2, z\}$ is an AT that contradicts Claim 6, in particular $x'z \notin E$. (Note that no edges between y_1 and any vertex in $P_{x',z}$

exist, by Claim 33, and that the same holds for vertex y_2.) Therefore, for all $P, Q \in \mathcal{R}_{x',y}$ that are not P_3's, $\forall y_1 \in P \setminus N[x']$, $\forall y_2 \in Q \setminus N[x']$, $y_1 \in D(x', y_2)$, or $y_2 \in D(x', y_1)$.

By symmetry, for all $P', Q' \in \mathcal{R}_{x',z}$ that are not P_3's, $\forall y_1 \in P' \setminus N[x']$, $\forall y_2 \in Q' \setminus N[x']$, $y_1 \in D(x', y_2)$, or $y_2 \in D(x', y_1)$. □

Similar claims holds for long paths in $\mathcal{R}_{x',y}$ and $\tilde{\mathcal{R}}_{x,y}$.

Fact 2 *For all $P \in \mathcal{R}_{x',y}$ and $Q \in \tilde{\mathcal{R}}_{x,y}$ that are not P_3's, $\forall y_1 \in P \setminus N[x']$, $\forall y_2 \in Q \setminus N[x]$ such that $y_2 x' \notin E$, $y_1 \in D(x, y_2)$, or $y_2 \in D(x', y_1)$.*

Proof: Similar to the proof of Fact 1. □

Fact 3 *For all $P \in \tilde{\mathcal{R}}_{x,y}$ and $Q \in \tilde{\mathcal{R}}_{x,y}$ that are not P_3's, $\forall y_1 \in P \setminus N[x]$ such that $y_1 x' \notin E$, $\forall y_2 \in Q \setminus N[x]$ such that $y_2 x' \notin E$, $y_1 \in D(x, y_2)$, or $y_2 \in D(x, y_1)$.*

Proof: Similar to the proof of Fact 1. □

5.2 No-Long-Sided Graphs

Denote by M the set of mid-vertices of all paths in $\mathcal{P}_{y,z}$, by U_1 the set of mid-vertices of all paths in $\mathcal{R}_{x',y}$, and by U_2 the set of mid-vertices of all paths in $\mathcal{R}_{x',z}$. Also, denote by W_1 the set of mid-vertices of all short paths in $\tilde{\mathcal{R}}_{x,y}$, and by W_2 the set of mid-vertices of all short paths in $\tilde{\mathcal{R}}_{x,z}$. Note that W_1 and W_2 can be empty, and that $\tilde{\mathcal{R}}_{x,y}$ and $\tilde{\mathcal{R}}_{x,z}$ may have long paths.

The following claim is one of the main results of this chapter. It describes the positions of DP vertices in no-long-sided (1,2,2) graphs. The two claims following it determine some structural properties forced by different positions of DP vertices in these graphs.

Claim 44 *One DP vertex of a no-long-sided graph H is in $N[x]$ and the other one is in M, or one DP vertex of H is in $N[z]$ and the other one is in W_1, or one DP vertex of H is in $N[y]$ and the other one is in W_2. Each of these three types of DPs can occur.*

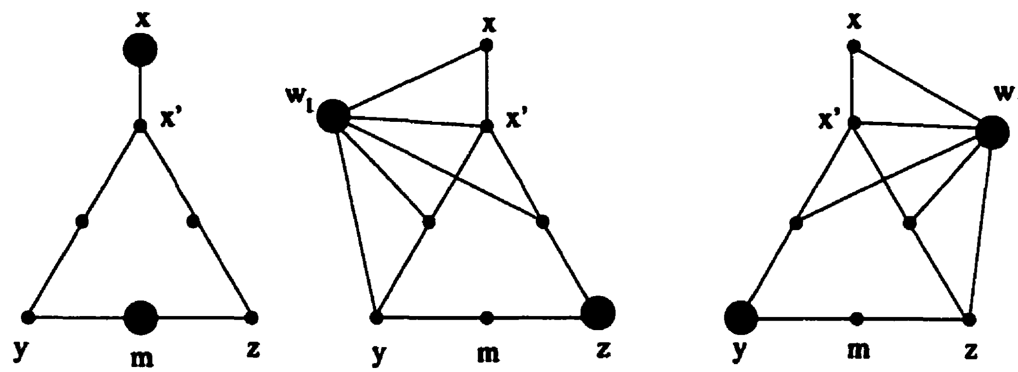


Figure 5.8:

Proof: Follows the same proof as Claim 39.

Examples showing that each of these three types of DPs can occur are given in Figure 5.8. DP vertices are shaded in these examples. □

Claim 45 *Let one DP vertex of a no-long-sided graph H be in $N[x]$ and the other one in M. Denote by (α,β) such a DP of H. Then:*

(1) If $\{\alpha,\beta\}=\{x,m\}$, for some $m\in M$, then all non-neighbors of x on a path P in $\mathcal{P}_{x,z}$ must be adjacent to m, or to all neighbors of x in $\mathcal{P}_{x,y}$.

(2) If $\{\alpha,\beta\}=\{x_p,m\}$, for some $m\in M$ and $x_p=P\cap N(x)$, where P is a long path in $\tilde{\mathcal{R}}_{x,z}$, then each vertex in U_1 is adjacent to m.

(3) If $\{\alpha,\beta\}=\{w_2,m\}$, for some $m\in M$ and some $w_2\in W_2$, then the internal vertices of all paths in $\mathcal{P}_{x,y}$, and the internal vertices of all paths in $\tilde{\mathcal{R}}_{x,z}$ that are not neighbors of z, must be adjacent to w_2, or to m.

Proof: (1) Otherwise, if there exists a vertex v that is a non-neighbor of x on a path P in $\mathcal{P}_{x,z}$, is not adjacent to m, and is not adjacent to some neighbor x'' of x in $P_{x,y}\in\mathcal{P}_{x,y}$, then the path from x to m induced on $P_{x,y}\cup\{ym\}$ does not hit v; note that v cannot be adjacent to any vertex in $\mathcal{P}_{x,y}$ other than the neighbor of x, by Claims 15 and 33.

(2) Let $\{\alpha,\beta\}=\{x_p,m\}$. Let $u_1\in U_1$ be non-adjacent to m. Then, the path from x_p to m induced on $\{mz\}\cup P\setminus\{x\}$ misses u_1; note that, by Claim 15, u_1 cannot be adjacent

to any vertex on P other than x_p, and also $u_1x_p \notin E$ since P is a long path in $\tilde{\mathcal{R}}_{x,z}$, i.e., if $u_1x_p \in E$, then P would share vertex x_p different from x and y with the path L induced on $\{xx_p\} \cup \{x_pu_1\} \cup \{u_1y\}$, $L \in \mathcal{R}_{x,y}$, which means that $P \in \mathcal{R}_{x,z}$ contradicting the assumption that $P \in \tilde{\mathcal{R}}_{x,z}$.

(3) Let $\{\alpha, \beta\} = \{w_2, m\}$ and w.l.o.g. let v be an internal vertex of a path in $\mathcal{P}_{x,y}$ that is not adjacent to w_2 and is not adjacent to m. Then the path from w_2 to m induced on $\{w_2z\} \cup \{zm\}$ misses v. □

Claim 46 *Let one DP vertex of a no-long-sided graph H be in $N[z]$ and the other one in W_1. Denote by (α, β) such a DP of H. Then:*

(1) Let $\{\alpha, \beta\} = \{z, w_1\}$, for some $w_1 \in W_1$. Then all internal vertices of $\mathcal{P}_{x,y}$ that are not neighbors of y, and all internal vertices of $\mathcal{P}_{x,z}$ that are not neighbors of z, must be adjacent to w_1, or universal to M. Also, all non-neighbors of x in $\mathcal{P}_{x,y}$ that are not adjacent to w_1 must be adjacent to all neighbors of x in $\mathcal{R}_{x,z}$; in addition, if a neighbor x'' of x on $\mathcal{R}_{x,y}$ is adjacent to some non-neighbor of x in $\mathcal{P}_{x,z}$, then all non-neighbors of x in $\mathcal{P}_{x,y}$ must be adjacent to x'', or to w_1.

(2) Let $\{\alpha, \beta\} = \{w_1, w_2\}$, for some $w_1 \in W_1$ and some $w_2 \in W_2$. Then, every $m \in M$, and every non-neighbor of x in $\mathcal{P}_{x,y} \cup \mathcal{P}_{x,z}$, must be adjacent to w_1. or to w_2.

(3) Let $\{\alpha, \beta\} = \{z_p, w_1\}$, where P is a long path in $\tilde{\mathcal{R}}_{x,z}$, and $z_p = P \cap N(z)$. Also, let $x_p = P \cap N(x)$. Every non-neighbor of x on $\mathcal{P}_{x,y}$ must be adjacent to w_1.

(4) Let $\{\alpha, \beta\} = \{w_1, u_2\}$, for some $w_1 \in W_1$ and some $u_2 \in U_2$. Let $u_2w_1 \notin E$. Then all non-neighbors of x on $\tilde{\mathcal{R}}_{x,y}$ must be adjacent to w_1, or to all $m \in M$.

Proof: (1) Let v be an internal vertex of $\mathcal{P}_{x,y}$ that is not a neighbor of y, or an internal vertex of $\mathcal{P}_{x,z}$ non-adjacent to z. If v is not adjacent to w_1, and is also not adjacent to some vertex $m \in M$, then the path from z to w_1 induced on $\{zm\} \cup \{my\} \cup \{yw_1\}$ does not hit v.

Now, let v_1 be some non-neighbor of x in $\mathcal{P}_{x,y}$ that is not adjacent to w_1, and is not adjacent to a neighbor v_2 of x in $\mathcal{R}_{x,z}$. Let $v_2 \in P \in \mathcal{P}_{x,z}$. Then the path from z to w_1

induced on $P \cup \{xw_1\}$ misses v_1. (Again, by Claim 15, the only vertex on P that can be adjacent to v_1 is v_2.) Note as before, that no non-neighbor of x in $\mathcal{P}_{x,y}$ is adjacent to a neighbor of x in $\tilde{\mathcal{R}}_{x,z}$ by definition of $\tilde{\mathcal{R}}_{x,z}$.

If a neighbor x'' of x on $\mathcal{R}_{x,y}$ is adjacent to some non-neighbor p of x in $P_{x,z} \in \mathcal{P}_{x,z}$, and if a non-neighbor v of x in $\mathcal{P}_{x,y}$ is not adjacent to x'' and is not adjacent to w_1, then the path from z to w_1 induced on $P_{z,p} \cup \{px''\} \cup \{x''x\} \cup \{xw_1\}$ misses v, where $P_{z,p}$ is the path between z and p induced on $P_{x,z}$. (In the case that $v \in \mathcal{R}_{x',y}$ and $p \in \mathcal{R}_{x',z}$, by Claim 33, $vp' \notin E, \forall p' \in P_{z,p}$. Otherwise, by Claim 15, the only vertex on $P_{x,z}$ that v can be adjacent to is the neighbor of x, i.e., again $vp' \notin E, \forall p' \in P_{z,p}$. Note again that a neighbor of x in $\tilde{\mathcal{R}}_{x,y}$ cannot be adjacent to a non-neighbor of x in $\mathcal{P}_{x,z}$ by definition of $\tilde{\mathcal{R}}_{x,y}$.)

(2) Let some $v \in M \cup \mathcal{P}_{x,y} \cup \mathcal{P}_{x,z} \setminus N[x]$ be non-adjacent to w_1, and non-adjacent to w_2. Then the path from w_1 to w_2 induced on $\{w_1x\} \cup \{xw_2\}$ misses v.

(3) If some vertex v that is a non-neighbor of x on $\mathcal{P}_{x,y}$ is not adjacent to w_1, then the path between z_p and w_1 induced on $P \cup \{xw_1\}$ misses v; note that by Claim 15, v cannot be adjacent to any vertex in P other than x_p, and also v cannot be adjacent to x_p since $P \in \tilde{\mathcal{R}}_{x,z}$.

(4) Note that by Claim 17, all vertices of distance i from x, $i \geq 3$, on $\tilde{\mathcal{R}}_{x,y}$ must be universal to M. Let v be the second neighbor of x in $\tilde{\mathcal{R}}_{x,y}$ that is not adjacent to w_1 and to some $m \in M$. Then the path between u_2 and w_1 induced on $\{u_2z\} \cup \{zm\} \cup \{my\} \cup \{yw_1\}$ misses v. □

Chapter 6

Concluding Remarks

This thesis first gave an overview of the hierarchy of graph classes in the neighborhood of HDP graphs, and known results about structural properties of AT-free and HDP graphs. Then it established some new structural properties of HDP and minimal HDP graphs. The positions of DP vertices in minimal HDP graphs were determined. Also, the structural properties of a minimal HDP graph that are forced by the position of the graph's DP vertices were examined.

This thesis did not discuss the complexity of determining whether a graph is HDP, which is an interesting topic for further research. The hereditary structure of HDP graphs would make any brute force algorithm for determining whether a graph is HDP run in exponential time. However, knowing the structure of HDP graphs might result in a new algorithm that runs in polynomial time.

In AT-free graphs, there exists a simple linear time Lexicographic Breadth First Search algorithm for finding a DP of a connected AT-free graph (Corneil, Olariu, and Stewart 1999). Therefore, another interesting topic for future work is to see if there exists a polynomial time algorithm that finds a DP in a connected HDP graph.

The full structure of HDP graphs remains unexplored. Results about HDP and minimal HDP graphs established in this thesis could be a building block for discovering more

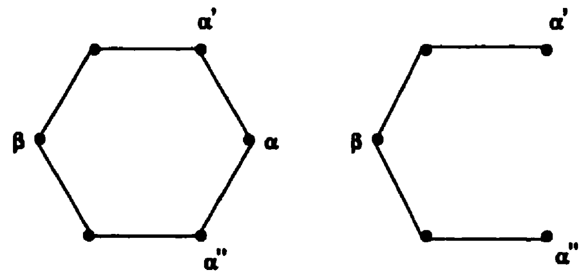


Figure 6.1: C_6 counterexample to generalizing Spine theorem to HDP∩AT graphs.

structure of HDP graphs. Also, it is not clear how to lift the structural properties of AT-free graphs to HDP graphs. To illustrate this, the next section considers the Spine property of AT-free graphs and tries to lift it to HDP graphs.

6.1 HDP Spine Property for $HDP \cap AT$ Graphs

Corneil, Olariu, and Stewart (1997) established an elegant property of AT-free graphs that allows for finding DP vertices of induced subgraphs of an AT-free graph that miss a DP vertex. The property is called the *Spine property*, and it has already been defined in Chapter 2. They also proved the Spine theorem (Theorem 14 from Chapter 2). It would be nice if the Spine theorem would generalize to all HDP graphs, i.e., to HDP graphs with asteroidal triples.

The first attempt to generalize the Spine theorem is motivated by C_6. Note that after a DP vertex α is removed from C_6, neither (α', β) nor (α'', β) are DPs of $C_6 \setminus \{\alpha\}$ (see Figure 6.1). This example suggests that perhaps a slightly different variation of the Spine property would work for HDP∩AT graphs. Therefore, the following conjecture seems at the first glance to be reasonable.

Possible Conjecture 1 *(The HDP Spine conjecture?) A graph G is HDP if and only if every connected induced subgraph H of G satisfies the following:*

for every nonadjacent dominating pair (α, β) in H, either:

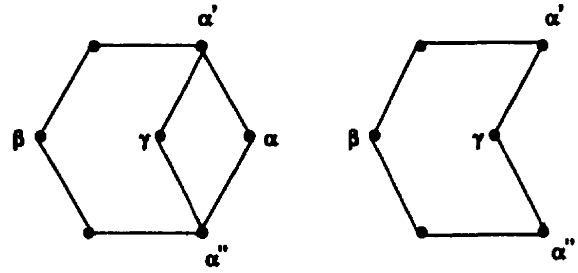


Figure 6.2: A counterexample to Possible Conjecture 1.

(i) $\exists \alpha' \in N(\alpha) \cap C_\beta$ such that (α', β) is a DP of C_β, where C_β denotes the connected component of $G \setminus \{\alpha\}$ containing β; or

(ii) $\exists \alpha', \alpha'' \in N(\alpha) \cap C_\beta$ such that (α', α'') is a DP of C_β.

It is easy to see that the $\Leftarrow$ direction of the Possible Conjecture 1 is true, since if every connected induced subgraph of G satisfies this property, that means that every connected induced subgraph of G has a DP, which, by definition, means that G is HDP. However, the other direction (i.e., $\Rightarrow$) does not hold. To see this, consider the counterexample shown in Figure 6.2. It can be seen that neither (i) nor (ii) of Possible Conjecture 1 hold for $G \setminus \alpha$ in Figure 6.2. But, the set of conditions can be extended to handle this counterexample as follows.

Possible Conjecture 2 *(The HDP Spine conjecture?) A graph G is HDP if and only if every connected induced subgraph H of G either satisfies one of the conditions of Possible Conjecture 1, or the following:*

(iii) for every nonadjacent dominating pair (α, β) in H, there exists a vertex γ universal to $N(\alpha)$ such that (β, γ) is a DP of the connected component of $G \setminus \{\alpha\}$ containing β.

Unfortunately, even though this new Possible Conjecture 2 covers the example presented in Figure 6.2, it again fails to satisfy the $\Rightarrow$ direction. This can be seen by the counterexample in Figure 6.3. In this counterexample, when α is removed, no condition

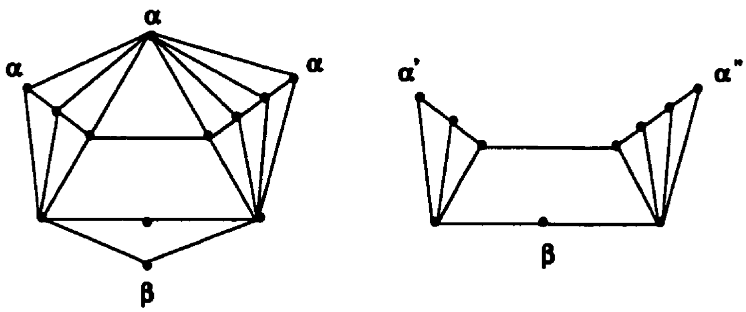


Figure 6.3: A counterexample to Conjecture 2.

from Possible Conjecture 2 is satisfied in $G \setminus \{\alpha\}$. Therefore, Possible Conjecture 2 could be further extended to deal with the example in Figure 6.3 as follows.

Conjecture 1 *(The HDP Spine conjecture) A graph G is HDP if and only if every connected induced subgraph H of G satisfies:*

for every nonadjacent dominating pair (α, β) in H, either:

(i) $\exists \alpha' \in N(\alpha) \cap C_\beta$ such that (α', β) is a DP of C_β, where C_β denotes the connected component of $G \setminus \{\alpha\}$ containing β; or

(ii) $\exists \alpha', \alpha'' \in N(\alpha) \cap C_\beta$ such that (α', α'') is a DP of C_β; or

(iii) there exists a vertex γ universal to $N(\alpha)$ such that (β, γ) is a DP of the connected component of $G \setminus \{\alpha\}$ containing β; or

(iv) $\exists \beta', \beta'' \in N(\beta)$ such that (β', β'') is a DP of the connected component of $G \setminus \{\alpha\}$ containing β.

Whether Conjecture 1 holds or not remains unexplored.

These examples illustrate that it is hard to lift the Spine property and the Spine theorem from AT-free to HDP graphs. It seems that the position of the DP vertices in a subgraph of an HDP graph G induced on $V(G) \setminus \{\alpha\}$ is not determined by the position of the DP vertices α and β in G, as was the case for AT-free graphs. Chapters 3, 4, and 5 suggest that perhaps the position of DP vertices in HDP graphs is determined by the position of AT vertices. It might also be true that the position of AT vertices in an HDP graph is determined by the position of DP vertices in the graph. These are interesting topics for further research.

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